A case of Eisenbud-Green-Harris Conjecture

(joint work with Mel) Sema

Set up • $R = K[x_1, \cdots, x_n]$ K field
with standard grading

• I homogeneous ideals

• lexicographic order $x_1 > x_2 > \cdots > x_n$

Def: A monomial ideal L is called a lex ideal if, for all degrees d,

L_d = { homogeneous component of L in degree d }

is generated by the initial lex - ~~segm~~ segment in degree d.

Hilbert function of R/I, I is homogeneous ideal

is $H(R/I, i) := \dim_K [R/I]_i$ for all $i > 0$

$H(R/I, i) \geq 0$

$H(R/I, 0) = 1$

Similarly, one can define $H(I_i) = \dim_K I_i$ to be the Hilbert function of I.

O-sequences of R/I: $\{e_i = H(R/I, i)\}_{i\geq 0}$

Macaulay, 1927: Hilbert function of any homogeneous ideal I in $R = K[x_1, \dots, x_n]$ can be attained by a lexicographic ideal.

Clements - Lindström: Replace R by $R/(x_1^{a_1}, \dots, x_n^{a_n})$ for $2 \leq a_1 \leq \dots \leq a_n$

$\forall\, I \supset (x_1^{a_1}, \dots, x_n^{a_n}) \Rightarrow \exists\, L = (x_1^{a_1}, \dots, x_n^{a_n}) + J^{lex}$

lex plus powers

s.t. $H(R/I, i) = H(R/L, i)$ $\forall i \geq 0$

('92, '93)

EGH: Given $2 \leq a_1 \leq \dots \leq a_n$, let I be a homogeneous ideal containing a regular sequence $f_1, \dots, f_n$ with $\deg f_i = a_i$ then there exists a lex plus power ideal

$$L = (x_1^{a_1}, \dots, x_n^{a_n}) + J^{lex}$$

s.t. $H(R/I, i) = H(R/L, i)$ $\forall i \geq 0$

$EGH_{\underline{a},n}$ is the EGH for $\underline{a} = (a_1, \dots, a_n)$

Known:
- $n = 2$ (Richert)
- (04') $I = (\underbrace{f_1, \dots, f_n}_{\text{regular seq}}, g)$ a.c.i (Francisco)
- (12') Largest case: Degree bounds $a_i > \sum_{j=1}^{i-1} (a_j - 1)$ $\forall i \geq 2$ (Caviglia - Maclagan)

- $\underline{\underline{n=3}}$: when $\underline{a} = (2, a, a)$ $a \geq 2$ (Cooper)

  $\underline{a} = (3, a, a)$ $a \geq 3$

  $\underline{a} = (3, a, a+1)$

- If I is generated by generic quadratic forms

  char 0 (Herzog-Popescu)

  Any char (Croshorov)

For quadratic case ($\Rightarrow a_1 = a_2 = \cdots = 2$)

Richact claims $2 \leq n \leq 5$, but no proof

$\underline{\underline{n=4}}$: Chen gives a proof.

$EGH_{\underline{a},n}(d)$: Given $\underline{a} \in \mathbb{N}^n$, $I \supset (\overbrace{f_1, \cdots, f_n}^{\text{reg. seq.}})$

then $\exists$ a l.p.p L s.t. $H(R/I, d) = H(R/L, d)$

$H(R/I, d+1) = H(R/L, d+1)$

<u>Lem</u>: Let $s = \sum_{i=1}^{n} (a_i - 1)$, then for any degree $0 \leq d \leq s-1$

$EGH_{\underline{a},n}(d)$ holds $\Leftrightarrow$ $EGH_{\underline{a},n}(s-1-d)$ holds

Furthermore, $EGH_{\underline{a},n}$ holds iff $EGH_{\underline{a},n}$ holds for all degree d.

<u>Idea of Chen</u>: • $n=4$, $a_1=a_2=\cdots=2$;

$s=4$; $EGH_{2,4}(0) \iff EGH_{2,4}(3)$

$EGH_{2,4}(1) \iff EGH_{2,4}(2)$

$\downarrow$ Chen showed this!

• $n=5$ $EGH_{2,5}(0) \iff EGH_{2,5}(4)$

$EGH_{2,5}(1) \iff EGH_{2,5}(3)$

<u>Q</u>: Is $EGH_{2,5}(2)$ true?

$\iff$ $I \supset \underbrace{(f_1, \cdots f_n)}_{\text{quadratic form}}$ is there a l.p.p $L=(x_1^2,\cdots,x_n^2)+J^{lex}$

s.t. $\dim I_2 = \dim L_2$

$\dim I_3 = \dim L_3$

$\iff$ · — — — ————

— — —

$\dim I_3 \geq \dim L_3$

Set $I = (\underbrace{f_1, \cdots f_n}_{\text{r.s.}}, \underbrace{g, h}_{\text{quadratic forms}})$ $\xrightarrow{}$ $\dim = n+2$ $\Rightarrow$ defect 2 ideal generated by quadratic forms

$L = (x_1^2, \cdots, x_n^2, x_1x_2, x_1x_3) \rightsquigarrow \dim L_2 = n+2$

Can we show $\dim_K I_3 \geq \dim_K L_3 = n^2+2n-5$?

<u>Chen</u>: If $\dim_K((f_1\cdots f_n)_3 \cap gR_1) = 2$, then $\boxed{\text{Yes}}$

<u>Thm</u> (Sema, Mel) Given Homogeneous defect 2 ideal I generated by quadratic forms

$$\dim_K I_3 \geq n^2+2n-5$$

<u>Cor</u> $EGH_{2,n}$ for defect 2-quadratic ideals when $n=5,6$

<u>Cor</u> $EGH_{2,5}$ is true for all hom defect 2

$$I=(f_1,\dots,f_5,g,h) \quad 2\leq \deg g\leq \deg h\leq 4$$

<u>Next possible cases:</u>

<u>Defect 3 - defect 4 quadratics</u>

<u>When $n=5$;</u>

$$L=(x_1^2,\dots,x_5^2,x_1x_2,x_1x_3,x_1x_4)$$

$$\dim_K L_2=8$$

$$\dim_K L_3=31$$

<u>WTS</u>: I: defect 3 quadratic $\Rightarrow \dim I_3\geq 31$

$$L'=(x_1^2,\dots,x_5^2,x_1x_2,x_1x_3,x_1x_4,x_1x_5)$$

$$\dots\to \dim_K L_3'=31$$

<u>Q</u>: Is there a homogeneous defect 3 quadratic ideal $I\subseteq K[x_1,\dots,x_5]$ with Q-sequences of $R/I=\{1,5,7,5,*\}$