A Zariski-Nagata Theorem in Mixed Characteristic

I. Introduction

ZN Thm: $R = \mathbb{C}[\underline{x}]$ $P \subseteq R$ prime

$$P^{(n)} = \bigcap_{\substack{m \supseteq P \\ \text{maximal}}} m^n$$

$$\overset{②}{=} \{ f \in R \mid \delta(f) \in P \ \forall\, \delta \in D_R^{n-1} \}$$

where $D_R^{n-1} \subseteq R\langle \frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_d} \rangle$

diff operators of order $\leq n-1$

Goal: Give generalization of ② to primes in $\mathbb{Z}_p[x]_G$

Along the way, gives different proof of ZN and multival some ingredients

II. Differential Operators

Def. (Grothendieck) $A \subseteq R$ subring

$$D^0_{R/A} = \mathrm{Hom}_R(R,R)$$

$$D^i_{R/A} = \{ \delta \in \mathrm{Hom}_A(R,R) \mid [\delta, f] \in D^{i-1}_{R/A} \ \forall f \in D^0_{R/A} \}$$

$$D_{R/A} = \bigcup_{i \in \mathbb{N}} D^i_{R/A} \subseteq \mathrm{Hom}_A(R,R)$$

Ex 1) $R = \mathbb{C}[\underline{x}] \rightsquigarrow D^i_{R/\mathbb{C}} = \bigoplus_{|\alpha|\le i} R \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$

2) $R = \mathbb{F}_p[\underline{x}] \rightsquigarrow D^i_{R/\mathbb{F}_p} = \bigoplus_{|\alpha|\le i} R \frac{1}{\alpha_1!}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{1}{\alpha_d!}\frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$

the same

$R = A[\underline{x}] \rightsquigarrow D^i_{R/A} = \bigoplus_{|\alpha|\le i} R \frac{1}{\alpha_1!}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots$

3) If R is finitely generated over A

$W \subseteq R$ multiplicatively closed

$D^i_{W^{-1}R/A} \cong W^{-1} D^i_{R/A}$

Extend diff operators on R to $W^{-1}R$ by quotient rule. $\delta(f/g) = \dfrac{\delta(f) - [\delta, g](f/g)}{g}$

III. Differential Powers

Def (DDSGHNB): $I \subseteq R$ an ideal, $A \subseteq R$ a subring

$$I^{\langle n \rangle_A} := \{ f \in R \mid \delta(f) \in I \ \forall \delta \in D^{n-1}_{R/A} \}$$

diff powers

Restatement of ZN:

If $R = K[\underline{x}]$, K perfect field

$\mathfrak{p} \subseteq R$ prime, then $\mathfrak{p}^{(n)} = \mathfrak{p}^{\langle n \rangle_K}$

Remarks: 1) ZN fails for $\mathbb{F}_p(t)[x]$ and $I = (t - x^p)$

2) ZN fails for all nonsmooth $\mathbb{C}$-algebras

In fact, if R is f.g./K perfect,

$P^{(n)} = P^{\langle n \rangle K}$ for some P and some $n > 1$

then R_P is smooth.

3) If $R = \mathbb{Z}[\underline{x}]$, $P = (2)$

any $\mathbb{Z}$-linear map $R \to R$ takes P to P

Hence any $\delta \in D^{n-1}_{R/\mathbb{Z}}$

so $P \subseteq P^{\langle n \rangle} \subseteq P \rightsquigarrow P^{\langle n \rangle \mathbb{Z}} = P \quad \forall n$

but $P \neq P^{(n)}$

<u>Need</u> maps that decrease 2-adic order.

IV. p-derivations

<u>Def</u> (Joyal, Buium): A <u>p-derivation</u> on a ring R is a map $R \xrightarrow{\delta} R$ such that

- $\delta(1) = 0$
- $\delta(x+y) = \delta(x) + \delta(y) + C_p(x,y) \qquad \forall x, y \in R$
- $\delta(xy) = x^p \delta(y) + y^p \delta(x) + p\,\delta(x)\delta(y)$

where $C_p(x,y) = \dfrac{x^p + y^p - (x+y)^p}{p} \in \mathbb{Z}[x,y]$

These are not linear!

If R is p-torsion-free, δ is a p-derivation

iff $\delta(x) = \dfrac{\phi(x) - x^p}{p}$ for a ring homo $R \xrightarrow{\Phi} R$

s.t.
$$\begin{tikzcd} R \arrow[r, "\Phi"] \arrow[d] & R \arrow[d] \\ R/p \arrow[r, "\mathrm{Frob}"] & R/p \end{tikzcd}$$
commutes

<u>Ex</u>: • $R = \mathbb{Z}_p \rightsquigarrow \delta(n) = \dfrac{n - n^p}{p}$ is a p-derivation

$$\delta(p) = 1 - p^{p-1} \notin (p)$$

$$\delta(p^a) = p^{a-1}(1 - p^{a(p-1)}) \in (p^{a-1}) \setminus (p^a)$$

~~Same for $\mathbb{Z}p$.~~

• $R = \mathbb{Z}_p[\underline{x}] \rightsquigarrow \delta(f)(\underline{x}) = \dfrac{f(x_1^p, \dots, x_d^p) - f(x_1, \dots, x_d)^p}{p}$

is a p-derivation

Can localize a p-derivation to a prime containing p

⚠ Not every regular local ring has a p-derivation

e.g. $\mathbb{Z}_p[x,y]/(p-xy)$

V. Main Result

<u>Def.</u>: Let δ be a p-derivation on a ring R

The <u>mixed differential powers</u> of an ideal I

are $I^{\langle n\rangle_{mix}} := \{ f \in R \mid \delta^a \partial(f) \in I \quad \forall\, a+b \leq n-1,\ \partial \in D^b_{R/A} \}$

<u>Thm.</u> (DDS + ⋯)

Let R be essentially smooth over $\mathbb{Z}_p$ (ie. $\mathbb{Z}_p[x]_Q$)

and suppose R has a p-derivation

Let $P \in R$ be prime!

1) If $P \cap \mathbb{Z}_p = (0)$, then $P^{(n)} = P^{\langle n\rangle_{\mathbb{Z}_p}}$

2) If $P \cap \mathbb{Z}_p = (p)$, then $P^{(n)} = P^{\langle n\rangle_{mix}}$