Differential Operators & Symbolic Powers

<u>Recall:</u> If $I \subseteq R$ is a radical ideal, then

$$I^{(n)} = \{ f \in R \mid \exists\, w \notin \bigcup_{P \in \mathrm{Min}(I)} P \text{ s.t. } wf \in I^n \}$$

(R domain)

$$= I^n W^{-1}R \cap R \quad \text{where} \quad W = R - \bigcup_{P \in \mathrm{Min}(I)} P$$

There's a cool characterization of $I^{(n)}$ in polynomial rings over $\mathbb{C}$

$R = \mathbb{C}[\underline{x}]$

$$D_R^i := \bigoplus_{|\underline{\alpha}| \le i} R \frac{\partial^{\underline{\alpha}}}{\partial \underline{x}^{\underline{\alpha}}}$$

e.g. $$x_1^2 \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3} - x_1 x_2 x_3^2 \frac{\partial^2}{\partial x_1^2} + x_2^3 \frac{\partial}{\partial x_2} \in D_R^2$$

<u>Zariski-Nagata Theorem</u>

R, D_R^i as above, $I \subseteq R$ radical.

Then $I^{(n)} = \{ f \in R \mid \delta(f) \in I \ \forall\, \delta \in D_R^{n-1} \}$

<u>Ex:</u> $I = (xy, xz, yz)$

Use the theorem above to test if $xyz \in I^{(2)}$

so $\iff \delta(xyz) \in I \quad \forall\, \delta \in D_R^1$

but $\delta \in D_R^1 \rightsquigarrow \delta = f_1 + f_2 \frac{\partial}{\partial x} + f_3 \frac{\partial}{\partial y} + f_4 \frac{\partial}{\partial z} \qquad f_i \in R$

$$\delta(xyz) = f_1 xyz + f_2 yz + f_3 xz + f_4 xy \in I$$

$$\Rightarrow xyz \in I^{(2)}$$

(Easy to check with definition too!)

Want to discuss one containment, following DDS+

First, want to ~~and~~ understand D_R^i in more generality.

Grothendieck's definition of differential operators

$\mathbb{C}$-linear operators
$$\left\{\begin{aligned} &\cdot\ D_R^0 = \mathrm{Hom}_R(R,R) \\ &\cdot\ D_R^i = \{\delta \in \mathrm{Hom}_{\mathbb{C}}(R,R) \mid \delta\circ f - f\circ\delta \in D_R^{i-1}\ \forall f \in D_R^0\} \\ &\cdot\ D_R = \bigcup_{i\in\mathbb{N}} D_R^i \subseteq \mathrm{Hom}_{\mathbb{C}}(R,R)\end{aligned}\right.$$

If R is an A-algebra, then

- $D_{R/A}^0 = \mathrm{Hom}_R(R,R)$
- $D_{R/A}^i = \{\delta \in \mathrm{Hom}_A(R,R) \mid \delta\circ f - f\circ\delta \in D_{R/A}^{i-1}\ \forall f \in D_{R/A}^0\}$
- $D_{R/A} = \bigcup_{i\in\mathbb{N}} D_{R/A}^i \subseteq \mathrm{Hom}_A(R,R)$

Is this def reasonable?

i.e. Does $\frac{\partial}{\partial x} \in D^1_{R/\mathbb{C}}$?

Does $\frac{\partial}{\partial x} \circ f - f \circ \frac{\partial}{\partial x} \in D^0_{R/\mathbb{C}}$?

Yes: $\frac{\partial}{\partial x}(fg) - f \cdot \left(\frac{\partial}{\partial x}(g)\right) = \frac{\partial f}{\partial x} g + \frac{\partial g}{\partial x} f - \frac{\partial g}{\partial x} f$

$$= \frac{\partial f}{\partial x} g$$

graded ($\frac{\partial}{\partial x}$ has degree $= -1$)

__Ex__ 1) $R_1 = \mathbb{C}[\underline{x}] \rightsquigarrow D^i_{R/\mathbb{C}}$ agrees with our original definition

2) $R_2 = \mathbb{F}_p[\underline{x}] \rightsquigarrow D^i_{R/\mathbb{F}_p} = \bigoplus_{|\alpha| \le i} R \frac{1}{\alpha_1!} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{1}{\alpha_d!} \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}$

3) $S = (R_1)^{(2)}$ = second veronse subring of R_1 (subring of R_1 generated by monomials of even degree)

$\rightsquigarrow D_S = (D_{R_1})^{(2)}$

__Thm__ (Zariski, DDS+):

$A \subseteq R$ subring, $I \subseteq R$ radical

$f \in I^{(n)} \Rightarrow \delta(f) \in I \quad \forall \delta \in D^{n-1}_{R/A}$

Def A, R, I as above

$$I^{\langle n\rangle_A} := \{ f \in R \mid \delta(f) \in I \ \forall \delta \in D^{n-1}_{R/A} \}$$

↖ "differential powers of I"

(The thm is saying $I^{(n)} \subseteq I^{\langle n\rangle_A}$)

Outline of proof:

1) $I^{\langle n\rangle_A}$ is an ideal

2) $I^n \subseteq I^{\langle n\rangle_A}$

3) If P is prime, then $P^{\langle n\rangle_A}$ is P-primary

Pf: (1): $f, g \in I^{\langle n\rangle}$, $r \in R$, $\delta \in D_R^{n-1}$

$$\delta(f+g) = \delta(f) + \delta(g) \in I \Rightarrow f+g \in I^{\langle n\rangle}$$

$$\delta(rf) = \underset{\substack{\cap \\ D_R^{n-2}}}{[\delta, r]}(f) + r\delta(f) \in I^{\langle n\rangle}$$

(2): $I^n \subseteq I^{\langle n\rangle}$ by induction on n.

$n=1$, $I^1 = I^{\langle 1\rangle}$ by def

$n \Rightarrow n+1$: If $f \in I$, $g \in I^n$, $\delta \in D_R^n$

$$\delta(f \cdot g) = \underbrace{\underbrace{[\delta, f]}_{D_R^{n-1}} \underbrace{(g)}_{I^n}}_{I^{\langle n\rangle}} + \underbrace{f}_{I}\delta(g) \in I$$

2) P prime, WTS $P^{\langle n\rangle}$ is P-primary, by induction

$n \Rightarrow n+1$: $r \notin P$, $f \in P$, $rf \in P^{\langle n+1\rangle}$, $\delta \in D_R^{n}$

want: $\delta(f) \in P$

Known: $rf \in P^{\langle n+1\rangle} \subseteq P^{\langle n\rangle}$ → P-primary by induction

$\Rightarrow f \in P^{\langle n\rangle}$

$$\underbrace{\delta(rf)}_{\in P} = \underbrace{[\delta, r](f)}_{\in P} + r\,\delta(f) \quad \Rightarrow r\,\delta(f) \in P$$

$$\Rightarrow \delta(f) \in P$$