Gröbner Basis in AG

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I. Term orders

$R = K[x_1, \cdots, x_n]$

Each monomial $x^\alpha \in R$ is associated to $\alpha \in (\mathbb{Z}_{\geq 0})^n$

Fixing an order on $(\mathbb{Z}_{\geq 0})^n$ will give an order on the monomial in R

Def A term order is

(i) total order

(ii) If $\alpha > \beta$, then $\forall \gamma \in (\mathbb{Z}_{\geq 0})^n$, $\alpha + \gamma \geq \beta + \gamma$

(iii) well-ordering

Ex: LEX, Revlex

II. Division Algorithm

III Grobner basis:

Def: $in_>(I) = \{ in_>(f) \mid f \in I \}$

Def: $G = \{g_1, \cdots, g_t\}$ is a Gröbner basis for $I = \langle G \rangle$

if $in_>(I) = \langle in_>(g_i) \rangle$

Def the remainder r is called the <u>normal form</u> of f

IV. Applications

Studying R/I

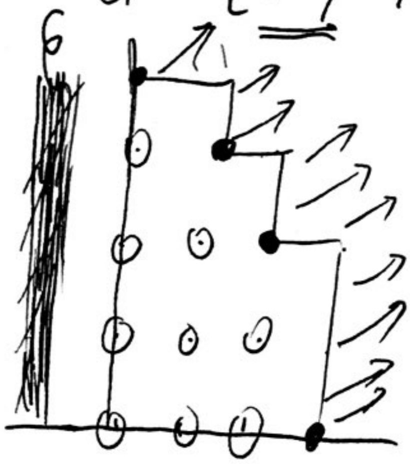
① If G is a Grobner basis, $\overline{f}^{G}$ is the normal form of $f \notin I$

② $\overline{f}^{G} \notin I$ and $\overline{f}^{G} = k$-linear comb of monomials NOT in $in_>(I)$

③ $\{ x^\alpha \mid \cancel{\notin}\ x^\alpha \notin in_> I \}$ is a linear ind set in R/I

Ex $I = \langle xy^2 - x^2, x^3y^2 - y \rangle$

$G = \{ \underline{\underline{x^3y^2}} - y, \underline{\underline{x^4}} - y^2, \underline{\underline{x^2y^3}} - x^2, \underline{\underline{y^4}} - xy \}$



$R/I = \text{span} \{ 1, x, x^2, y, xy, x^2y, x^3y, y^2, y^2x, xy^3 \}$

Hilbert Series is

$1 + 2t + 3t^2 + 4t^3 + 2t^4$

Flat families

$R \to R[t]$

$g \longmapsto t^b g(t^{-\lambda(x_1)} x_1, \dots, t^{-\lambda(x_d)} x_d)$

where $\lambda : \mathbb{Z}^m \to \mathbb{Z}$

get $\tilde{I}$ from I via the map

then $R[t]/(t, \tilde{I}) \cong R/\mathrm{in}_\lambda(I)$

<u>Thm</u>: $R[t]/\tilde{I}$ is a <u>flat</u> $K[t]$-module

whose fiber $\begin{cases} \text{over } 0 \text{ is } R/\mathrm{in}_\lambda(I) \\ \text{over } t \neq 0 \text{ is } R/I \end{cases}$