Local Cohomology of Powers

Jonathan Montaño
(Joint with Hailong Dao)

(R, m, K) Noetherian local of dim d

or $R = K[x_1, \dots, x_d]$ graded

Ex H^i_I may not be finitely generated if $i > 0$

i.e. $R = K[x]$, then $H^1_{(x)}(R) = R(x)/R := \cdots \oplus x^{-3} \oplus x^{-2} \oplus x^{-1}$

In char $k = 0$

<u>Thm</u> (Kodaira's vanishing, Huneke - Smith '97)

S graded normal domain

S_p regular $\forall\, p \in \operatorname{Spec}(S) \setminus \{m\}$

then $H^i_m(S)_{<0} = 0 \quad \forall i < \dim S$

Recently generalized

<u>Thm</u> (BBLSZ, '16)

R/I graded (normal) domain

locally complete intersection ring

(in $\operatorname{Spec}(R/I) \setminus \{m\}$)

Then $H^i_m(R/I^n)_{<0} = 0 \quad \forall\, i \leq \operatorname{codim}(\operatorname{sing} R)$

Goal: Study $\{ \lambda (H^i_m (R/I^n)) \}_{n \geq 1}$

The case $i=0$:

$$\ell = \lim_{n\to\infty} \frac{\lambda(H^0_m(R/I^n))}{n^d} \text{ exists}$$

for • R graded (CHST. '05)

• R analytically unramified local (Cutkosky '14)

The limit ℓ could be irrational (CHST)

it could be rational if I is

• monomial (HPV '08, Jeffrey '13)

• determinantal ideals (Jeff - M - '15)

The case $i > 0$:

Assuming $\lambda(H^i_m(R/I^n)) < \infty \quad n >> 0$

Big Question: Does $\displaystyle\lim_{n\to\infty} \frac{\lambda(H^i_m(R/I^n))}{n^d}$ exist?

First Step: Is $\displaystyle\limsup_{n\to\infty} \frac{\lambda(H^i_m(R/I^n))}{n^d} < \infty$?

Thm (Dao - M '17)

R graded, then $\forall \alpha \in \mathbb{Z}$

$$\limsup_{n\to\infty} \frac{\lambda(H^i_m(R/I^n)_{\geq -\alpha n})}{n^d} < \infty$$

<u>Cor</u> If R/I is locally complete intersection, char $k=0$

then $\displaystyle\limsup_{n\to\infty} \frac{\lambda(H_m^i(R/I^n))}{n^d} < \infty \quad \forall\, i < \dim(R/I)$

via <u>R. Lazarsfeld</u> R/I locally comp intersec

$$\Rightarrow \exists\, \alpha \text{ s.t. } H_m^i(R/I^n)_{<-\alpha n} = 0$$

$$\forall\, i < \dim(R/I),\ n \gg 0$$

we obtain the same conclusion for

- monomial ideals (Takayama, '05)
- determinantal ideals (Raicu, '16)

~~Notation~~ <u>Natural Question</u>

Assuming $\lambda(H_m^i(R/I^n)) < \infty$ for $n \gg 0$

Does there $\exists\, \alpha \in \mathbb{Z}$ s.t $\underline{\text{indeg}}\,(H_m^i(R/I^n)) \geq \alpha n \ \forall n \gg 0$?

initial deg $(M) = \min\{i \mid M_i \neq 0\}$

<u>Remark</u>: If $\lambda(H_m^i(R/I^n)) < \infty$, $\forall\, i < \dim R/I$, $n \gg 0$, $I = \sqrt{I}$

then I is locally complete intersection.

Some notation

$$E \subseteq F \cong R^n$$

$$\mathcal{R}(E) := \bigoplus_{n \geq 0} E^n \subset \operatorname{sym}(F) = R[T_1, .., T_n]$$

$\uparrow$

Rees algebra of E

Thm (Dao - M, '17)

Assume E is locally free in $\operatorname{Spec} R \setminus \{m\}$ (Vector bundle) and F/E is torsion

Then $\exists \alpha$ s.t. $\operatorname{indeg}(H_m^i(E^n)) \geq \alpha n$, $i < \dim R$

Ingredients

1) Theorem (Trung - Wang, '05, Dao - M)

M is f.g. over R $\Rightarrow$ Regularity $(E_M^n) = an + b$ for some $a, b \in \mathbb{Z}$, $n >> 0$

2) Thm (Horrock's '64, Dao - M)

R is CM with ω canonical module of R

M is locally maximum CM

$$H_m^i(M) \cong H_m^{d-i+1}(\operatorname{Hom}_R(M, \omega))^\vee$$

$(-)^\vee$ Matlis Dual.