Some instances of "Life Hacking" II

Review Notation

$S(n) = K[x_1, \dots x_n]$ $\quad K$ char $= 0$

$J_\circ$ = graded system of monomial ideals

$J_a J_b \subseteq J_{a+b}$

Two regions of $\mathbb{R}^n_{\geq 0}$

$$\Delta(J_\circ) = \bigcup_{c=1}^{\infty} \frac{1}{c} P_{J_c}$$

$$\Gamma(J_\circ) = \mathbb{R}^n_{\geq 0} - \Delta(J_\circ) = \bigcap_{c=1}^{\infty} \frac{1}{c} Q_{J_c}$$

$$P_{J_c} = \mathrm{Conv}(\{ T \in \mathbb{Z}^n_{\geq 0} : X^T \in J \}) \subseteq \mathbb{R}^n_{\geq 0}$$

$$Q_J = \mathbb{R}^n_{\geq 0} - P_J$$

Section 4 after proof of Thm 1.1 (1202.1317)

For a complete intersection "$\cap$" ideal $I \subseteq S(n)$

of type $(d_1 \leq \dots \leq d_r)$

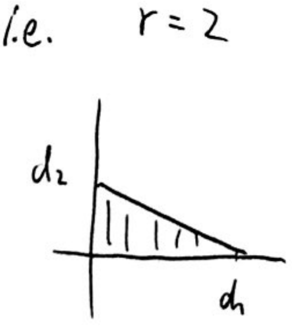
then $\Gamma(\{ gin(I^i) \}_{i=1}^{\infty}) = \{ (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n_{\geq 0} : 1 \geq \frac{\lambda_1}{d_1} + \dots + \frac{\lambda_r}{d_r} \}$

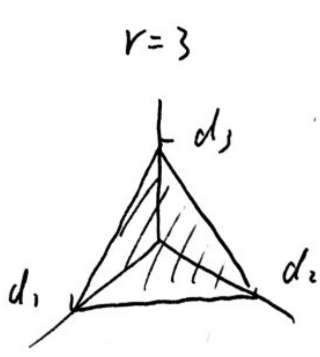
When $n = r$, this is the simplex with vertices v_0 = origin

& $v_i = d_i \cdot e_i$ for $1 \leq i \leq r$

$\uparrow$ i^{th} unit vector

i.e. $\quad r = 2$ $\qquad\qquad\qquad r = 3$





If $J_\bullet$ = graded system of zero-dimensional monomial ideals in $S(n)$

then $\mathrm{vol}(J_\bullet) := \limsup_{m\to\infty} \dfrac{n! \cdot \mathrm{length}(R/J_m)}{m^n}$

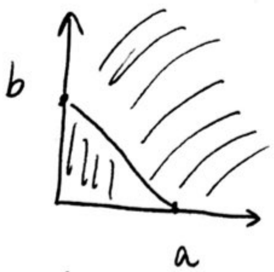
$$= n! \cdot \mathrm{vol}_{\mathbb{R}^n}(\Gamma(J_\bullet))$$

(M. Mustata, 2002)

<u>Thm (1.1) (1210.1622)</u>

Suppose $I \subseteq K[x,y,z]$ is the ideal of forms vanishing at r points in $\mathbb{P}^2$ in general position

(i.e. $\forall d \geq 1$, no $\binom{d+2}{2}$ points lie on a curve of deg d)

Then  $\Delta(\{\mathrm{gin}(I^{(i)})\}_{i=1}^{\infty})$

s.t. $a \cdot b = r$

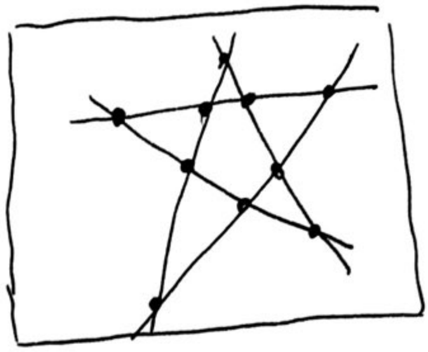
<u>Star Configuration (1401.4736)</u>

Fix an integer $s \geq n \geq 1$ & general Hyperplane $H_1, \ldots, H_s$ in $\mathbb{P}^n$. Each n of the H_i intersects in a single point (Bezout thm) Let Z be the collection of these $\binom{s}{n}$ intersection points.

This is a star configuration.

i.e. $n=2$

$s=5$



in $\mathbb{P}^2$

Let $I = I_Z \subseteq S(n+1)$ be the homogeneous coordinates of a reduced scheme Z

The symbolic powers

$$I^{(c)} = (I^c)^{sat} := I^c : m^\infty$$

$$= \bigcup_{n=1}^{\infty} I^c : m^n$$

<u>Fact (Green)</u>: For $J \subseteq S(n+1)$ homogeneous + saturated, no minimal generators of $\mathrm{gin}(J)$ contains the variable

$\Rightarrow \mathrm{gin}(I^{(c)}) \subseteq S(n) \quad \forall c \geq 1$ & x_{n+1}

$\Delta(\{\mathrm{gin}(I^{(c)})\}_{c=1}^{\infty})$ & $\Gamma(\cdots)$ lies in $\mathbb{R}^n_{\geq 0}$

<u>Thm 1.1</u> (1401.4736) For $I = I_Z$,

$\Gamma(I)$ is the simplex in $\mathbb{R}^n_{\geq 0}$ with vertices

v_0 = origin & $v_c = \frac{s-(i-1)}{n-(i-1)} \cdot e_i \quad (1 \leq i \leq n)$

Thus if $s=n$ → standard n-simplex