Finite Torsors over strongly F-regular singularities

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§1 Questions

§2 Answers & Corollaries

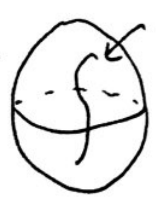
§3 On the proof

Set up. (R, m, k) Henselian complete local normal domain over k

$k = \bar{k}$

$X = \mathrm{Spec}(R) \quad Z = \mathrm{Spec}(R/I) \quad \mathrm{ht}\, I \geq 2$

$U = X \setminus Z$

$X \to$ 

$$\begin{array}{c} V \\ \downarrow h \\ U \end{array}$$

Q1: To what extend are there G-torsors $\nexists$ that do not come from restriction a G-torsor $\begin{array}{c} V \\ \downarrow \\ X \end{array}$?

* G-torsors for any group scheme G

$$\begin{tikzcd} V \arrow[d, "h"', "\text{f.flat}"] & V \times G \arrow[l, "\alpha"'] \arrow[d] \\ U \arrow[r, phantom, "\lrcorner"] & V \arrow[l, "h"] \end{tikzcd}$$

Remark: $H^1(U_{ft}, G)$ classifies G-torsors

Q2: To what extend $\rho_X(G) : H^1(X, G) \to H^1(U, G)$ is non surjective?

Q3: To what extend are there $R \subseteq S$ finite local extensions such that
G acts on S
$\mathcal{O}(G)$ coacts on S
makes $R = S^G$

Coaction:

$$\begin{tikzcd} S \arrow[r, "\alpha^\#"] & S \otimes \mathcal{O}(G) \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

$$\rightsquigarrow \quad S \otimes_R S \xrightarrow{\varphi} S \otimes \mathcal{O}(G)$$

and φ_p is an isom $\forall p \notin Z$?

Ex: $S = k[x, y]$

$G = \mu_n \qquad \mathcal{O}(G) = k[t]/(t^n - 1)$

$$\mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$$
$$t \mapsto t \otimes t$$

$$\begin{aligned} S &\xrightarrow{\alpha^\#} S \otimes \mathcal{O}(G) \\ x &\longmapsto x \otimes t \\ y &\longmapsto y \otimes t \end{aligned}$$

so

$$\begin{tikzcd} S \arrow[r] & S \otimes \mathcal{O}(G) \\ k[x^n, x^{n-1}y, \dots, y^n] \arrow[u] \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S \arrow[u] \end{tikzcd}$$

then $S \otimes_R S \xrightarrow{\varphi} S \otimes \mathcal{O}(G)$

is an isom away from $(x^n, x^{n-1}y, \dots, y^n)$

§2 Answers

A "nice" answer for us looks like

$\exists X^*$ of same type and a finite morphism

$$X^* \xrightarrow{g} X$$

s.t. $P_{X^*}(G)$ is onto $\forall G$ or at least for a large class of these?

<u>Thm A</u> (Xu 2014)

X is KLT/$\mathbb{C}$

$\exists X^*$ KLT and $X^* \xrightarrow{g} X$ geometrically Galois

s.t. $P_{X^*}(U)$ is onto for all G.

$$H^1(X^*_{ft}, G) \to H^2(U^*_{ft}, G)$$

"0" in char 0 — thm asserts this is 0

<u>Thm B</u> (–, Schwede, Tucker 2016)

X is strongly F-regular (SFR) / k of char $p>0$

$\exists X^*$ SFR singularity + geometrically Galois cover

$$g : X^* \to X$$

s.t. $P_{X^*}(U)$ is onto for all G étale

<u>and</u> $\deg g \leq 1/S(R)$ and $p \nmid \deg g$

Theorem C (Bhatt, Gabber, Olsson, 2017)

Theorem B "$\Rightarrow$" Theorem A

by reduction or mod p

Question: Is it possible to extend thm B to cases beyond the étale case?

Remark: G group scheme $/k$

$$0 \to \underset{\text{connected}}{G^0} \to G \to \pi_0(G) \to 0$$

$\pi_0(G)$ ← étale = discrete group

Thm D (—, 2017) X SFR,

$\exists$ finite chain:

$$\begin{array}{ccccccccc} X & \xleftarrow{h_0} & X_1 & \xleftarrow{h_1} & X_2 & \leftarrow \cdots \leftarrow & X_t & = X^* \\ \cup\!\mathrm{l} & & \cup\!\mathrm{l} & & \cup\!\mathrm{l} & & \cup\!\mathrm{l} & \\ U & \leftarrow & U_1 & \leftarrow & U_2 & \leftarrow \cdots \leftarrow & U^* & \end{array}$$

s.t. 1) X_i is SFR $\forall\, i$

2) h_i is a G_i-torsor over U_i with G_i linearly-reductive

3) $\deg(X \leftarrow X^*) \leq 1/S(R)$

4) $P_*(G)$ is onto for all G s.t. G^0 is either triangulizable or nilpotent

Ex $S = k[X_1, \cdots, X_n]$

$R = S^{(p^2)}$ Veronese subring

$R \subseteq S^{(p)} \subseteq S$

Thm E (–, 2017)

$P_X(G)$ is onto for all G unipotent

$\hookrightarrow G \subseteq \mathbb{U}_n$