"Life Hacking" in Asymptotic Commutative Algebra

<u>Thesis</u>: The asymptotics of objects associated to a graded system of ideals $I_\bullet$ can be rigged by "ALGECOM" info.

<u>Def</u>: A system $I_\bullet = \{I_c\}_{c=1}^{\infty}$ is <u>graded</u>

$$\text{if } I_a \cdot I_b \subseteq I_{a+b} \quad \forall a, b \geq 1$$

<u>Ex</u>: Fix an ideal $I \subseteq R$

① Regular powers: $I_\bullet = \{I^c\}$

i.e. $I = (x,y) \subseteq \mathbb{R}[x,y] \Rightarrow I^2 = (x^2, xy, y^2)$

② Symbolic powers: $I_\bullet = \{I^{(c)}\}$

i.e. If I is prime, then $I^{(c)} = (I^c R_I)^{\text{contraction}}$

<u>Ex</u>: $P = (x,y)$ $\quad U = \mathbb{C}[x,y,z]/(xy, xz, yz)$

$$V = \mathbb{C}[x,y,z]/(y^2 - xz)$$

$\Rightarrow$ in U: $P^{(c)} = P$

in V: $P^2 = (x^2, xy, \underset{\substack{\| \\ xz}}{y^2}) \subsetneq (x)V = P^{(2)}$

$$P^2 \neq P^{(2)}$$

i.e. if I is radical, then

$$I^{(c)} = \{f \in R : uf \in I^c \text{ for some } u \in \bigcap_{\substack{P \text{ associated} \\ \text{prime of } I}} R - P\}$$

<u>Q</u>: When does $I^{(c)} = I^c \ \forall\, c \geq 1$

<u>A</u>: When ....

AL. I = maximal ideal <u>or</u> generated by an R-regular sequence

↑ complete intersection ideal

GE: $R = \mathbb{C}[V = \text{smooth affine variety}]$, I = radical ideal s.t.

Z = zero locus of I in V

is smooth

CDM: I = radical square-free monomial ideal

$\updownarrow$

bipartite graph G

(Symbolic)

③ Generic Initial Systems

Fix a homogeneous ideal I in $S(n) = K[x_1, \cdots, x_n]$ (char $= 0$)

& "$<$" = monomial order

in revlex.

<u>Review</u>: Given $T \in \mathbb{N}^n$, define

$$X^T := \prod_{i=1}^{n} x_i^{T_i} \in S(n) \quad \& \quad |T| = \sum_{i=1}^{n} |T_i|$$

<u>generic initial ideals</u>:

$g(x_i) = \sum g_{ij} x_j$

Each $g = (g_{ij}) \in GL_n(K)$ acts on $S(n)$ by coordinate change.

it sends homogenous degree d poly to homog deg d poly.

<u>Thm</u> (Galligo - Green) For any homogenoous ideal I in $S(n)$

There's a Zariski open subset $\mathcal{U} \subset GL_n(K)$ s.t. $\mathrm{gin}(I) = \mathrm{In}(g(I))$ is constant ~~$\forall g \in \mathcal{U}$~~ & Borel-fixed $\forall g \in \mathcal{U}$

Common properties of $\mathrm{gin}(I)$ & I

→ Same Hilbert function

→ Same depth

→ (If I supported on zero-dim scheme) same length

The generic initial ~~ideal~~ system of I

$$J_\bullet = \{ J_i = \mathrm{gin}(I^i) \}_{i=1}^{\infty}$$

The symbolic

$$J_\bullet' = \{ J_i' = \mathrm{gin}(I^{(i)}) \}_{i=1}^{\infty}$$

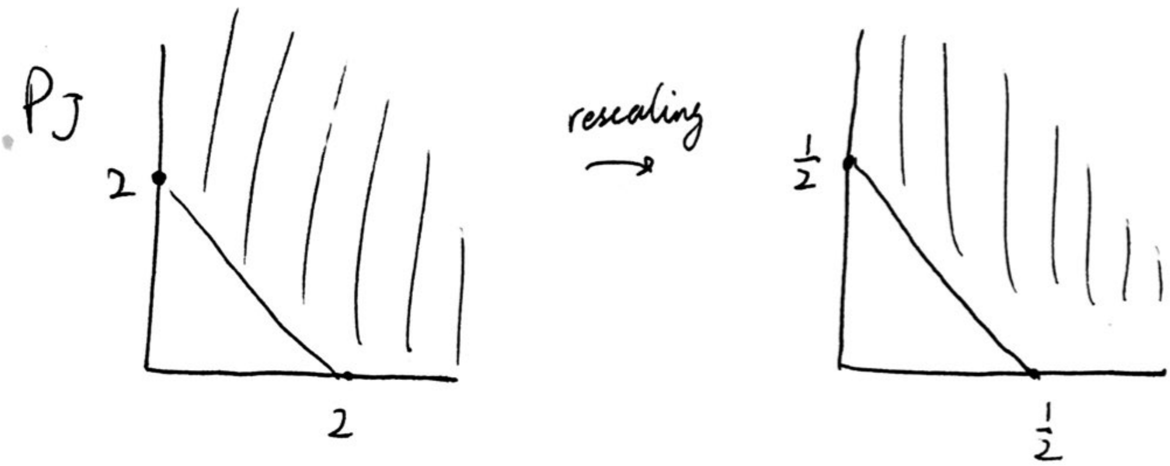
both are graded systems

$J_\bullet$ = graded system of monomial ideals in $S(n)$

Def: The Newton polyhedron P_J of ~~$J_\bullet$~~ J monomial ideal in $S(n)$

is $P_J = \mathrm{Conv}\left(\{ T \in (\mathbb{Z}_{\geq 0})^n : x^T \in J \}\right) \subseteq \mathbb{R}^n_{\geq 0}$

Ex: $n=2$, $J = (x^2, xy, y^2)$



$Q_J := \overline{(\mathbb{R}^n_{\geq 0} - P_J)}$ closure of the complement of P_J in the first Octant.

<u>Fact</u>: $\frac{1}{c} \cdot P_{J_c} \subseteq \frac{1}{c+1} P_{J_{c+1}} \quad \forall c \geq 1$

$$\Rightarrow \Delta(J_\bullet) = \bigcup_{c=1}^{\infty} \frac{1}{c} P_{J_c}$$

$$\Gamma(J_\bullet) = \overline{(\mathbb{R}^n_{\geq 0} - \Delta(J_\bullet))}$$

$$= \bigcap_{c=1}^{\infty} \frac{1}{c} Q_{J_c}$$