# Examples of blowing up in algebraic geometry

Ming Zheng

Blowing up : $(0,0) \hookrightarrow \mathbb{A}^2$ over $k$

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] \arrow[d] & Bl_{(0,0)}\mathbb{A}^2 \arrow[d, "\pi"] & \subset \mathbb{A}^2 \times \mathbb{P}^1 \\ (0,0) \arrow[r, hook] & \mathbb{A}^2 & \end{tikzcd}$$

$$((x,y), [X:Y])$$

$$Bl_{(0,0)}\mathbb{A}^2 = \{ (x,y), [X:Y] \mid xY - Xy = 0 \}$$

$\mathbb{P}^1 = U_1 \cup U_2 \qquad U_i \cong \mathbb{A}^1$

$U_1 \to \{Y \neq 0\}$, $U_2 \to \{X \neq 0\}$

$\mathbb{A}^2 \times \mathbb{P}^1 = V_1 \cup V_2 \qquad V_i \cong \mathbb{A}^3$

$V_1$ parametrized by $(x,y,s) \in \mathbb{A}^3$

$V_2$ parametrized by $(x,y,t) \in \mathbb{A}^3$

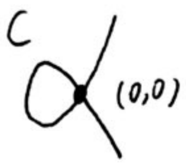
In $V_1$, $\{x - ys = 0\} \subset \mathbb{A}^3$, $\cong \mathbb{A}^2$ smooth

In $V_2$, $\{xt - y = 0\} \subset \mathbb{A}^3$, $\cong \mathbb{A}^2$ smooth

$$Bl_{(0,0)} \setminus \pi^{-1}\{(0,0)\} \simeq \mathbb{A}^2 \setminus \{(0,0)\}$$

## Application: resolve singularity on plane curves

Ex 1: $y^2 = x^3 + x^2$




we want $\overset{\text{smooth}}{\tilde{C}} \xrightarrow[\pi]{} C$

$$\tilde{C} \setminus \pi^{-1}(0) \overset{\text{iso}}{\simeq} C \setminus \{0\}$$

$$\begin{tikzcd}
\pi^{-1}(C\setminus\{0\}) \arrow[r] \arrow[d] & Bl_{(0,0)}\mathbb{A}^2 \arrow[d, "\pi"] \\
C\setminus\{0\} \arrow[r, hook] & \mathbb{A}^2
\end{tikzcd}$$

Claim: we can take $\tilde{C}$ to be

$$\overline{\pi^{-1}(C\setminus\{0\})} \subset Bl_{(0,0)}\mathbb{A}^2$$

In $V_1$: $\quad 0 = x^3 + x^2 - y^2 \xrightarrow{x=ys} y^3s^3 + y^2s^2 - y^2 = 0$

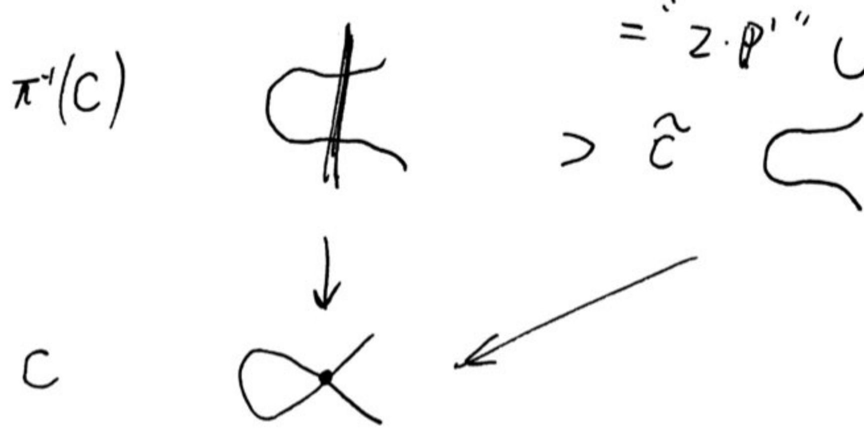
$$\Rightarrow y^2(ys^3 + s^2 - 1) = 0$$

$$= \text{“}2\cdot\mathbb{P}^1\text{”} \cup \text{“}\tilde{C}\cap V_1\text{”}$$

In $V_2$: $\quad x^3 + x^2 - y^2 = 0 \xrightarrow{y=xt} x^3 + x^2 - x^2t^2 = 0$

$$\Rightarrow x^2(x + 1 - t^2) = 0$$

$$= \text{“}2\cdot\mathbb{P}^1\text{”} \cup \text{“}\tilde{C}\cap V_2\text{”}$$



**Ex 2** $\quad y^2 = x^3$

In $V_1$ $\quad y^2 - x^3 = 0 \xrightarrow{x = ys} y^2 - y^3 s^3 = 0$

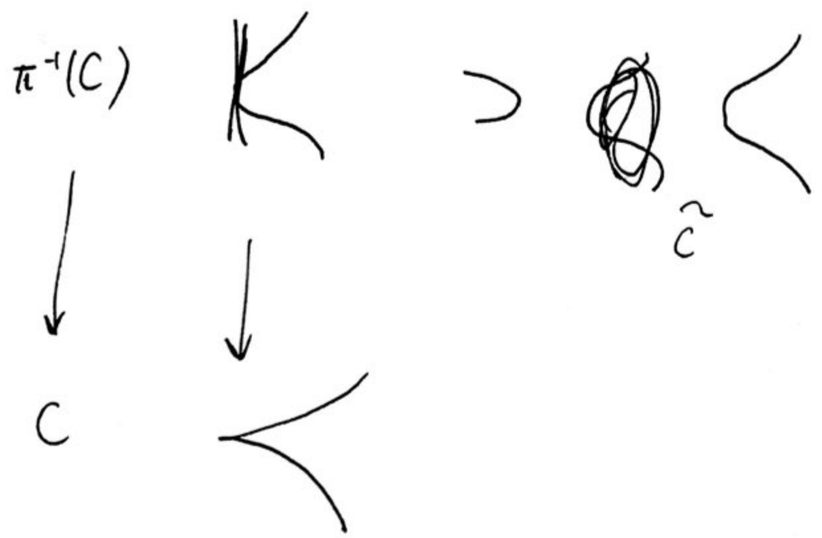
$$\Rightarrow y^2(1 - ys^3) = 0$$

$$= \text{"}2\cdot\mathbb{P}^1\text{"} \cup \text{"}\tilde{C} \cap V_1\text{"}$$

In $V_2$ $\quad y^2 - x^3 = 0 \xrightarrow{y = xt} x^2t^2 - x^3 = 0$

$$\Rightarrow x^2(t^2 - x) = 0$$

$$= \text{"}2\cdot\mathbb{P}^1\text{"} \cup \text{"}\tilde{C} \cap V_2\text{"}$$




**Ex 3** $\quad y^2 = x^{n+1} \rightarrow A_n$ singularity

In $V_1$ : $y^2 - x^{n+1} = 0 \xrightarrow{x = ys} y^2 - y^{n+1}s^{n+1} = 0$

$$\Rightarrow y^2(1 - y^{n-1}s^{n+1}) = 0$$

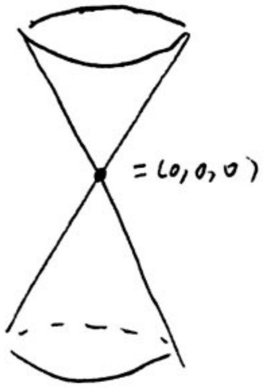
In $V_2$ : $y^2 - x^{n+1} = 0 \xrightarrow{y = xt} x^2t^2 - x^{n+1} = 0$

$$\Rightarrow x^2(t^2 - x^{n-2}) = 0$$

the blowed up curve has singularity of type $A_{n-1}$

Application to resolve surface singularities

<u>Ex</u>: $S = \{z^2 = x^2 + y^2\} \subset \mathbb{A}^3$




we want $\widetilde{S}$ as before

$$\begin{array}{ccc} \pi^{-1}(S \setminus \{0\}) & \longrightarrow & Bl_0\mathbb{A}^3 \subset \mathbb{A}^3 \times \mathbb{P}^2 \\ \downarrow & & \downarrow \pi \\ S \setminus \{0\} & \hookrightarrow & \mathbb{A}^3 \end{array}$$

$$\widetilde{S} = \overline{\pi^{-1}(S \setminus \{0\})} \subseteq Bl_0\mathbb{A}^3$$

---

<u>Def</u> ~~$Bl_0\mathbb{A}$~~ $Bl_0\mathbb{A}^n$ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$

$$(x_1, \cdots, x_n) \qquad [X_1 : \cdots : X_n]$$

then $Bl_0\mathbb{A}^n = \{ x_i X_j - x_j X_i = 0 \mid 1 \le i, j \le n \}$

~~$Bl_0\mathbb{A}^3 = \{$~~ $\mathbb{A}^n \times \mathbb{P}^{n-1}$ covered by $n$ copies $\mathbb{A}^n \times \mathbb{A}^{n-1}$

$Bl_0\mathbb{A}^3 \subseteq \mathbb{A}^3 \times \mathbb{P}^2 \rightarrow$ covered by 3 charts

<u>Chart 1</u>: $\{(x,y,z), [s:t:1] \mid sy - xt = 0,\ sz - x = 0,\ tz - y = 0\}$

$\cong \{(z,s,t)\} = \mathbb{A}^3$

<u>Chart 2</u>: $\{(x, t, w) \mid y = tx,\ z = wx\}$

<u>Chart 3</u>: $\{(y, s, w) \mid x = sy,\ z = wy\}$

for $S = \{-z^2+x^2+y^2=0\}$

In Chart I: $s^2z^2+t^2z^2-z^2=0$

$$\Rightarrow (s^2+t^2-1)z^2=0$$

In Chart II: $x^2+t^2x^2-w^2x^2=0$

$$\Rightarrow (1+t^2-w^2)x^2=0$$

In Chart III: $s^2y^2+y^2-w^2y^2=0$

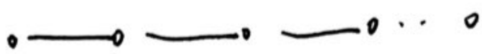
$$\Rightarrow (s^2+1-w^2)y^2=0$$

In general, $A_n$ singularity $z^2=y^2+x^{n+1} \subset \mathbb{A}^3$

we have to blow up a lot of times

to get a smooth surface

$$\pi: \tilde{S} \to S$$



Dykin Diagram

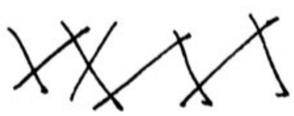
$D_n$ $z^2=x^2y+y^{n-1}$

$E_6$ $z^2=x^3+y^4$

$E_7$ $z^2=x^3+xy^3$

$E_8$ $z^2=x^3+y^5$

$\tilde{S}$



$\downarrow$

$S$ o