Arboreal Representations

Trevor Hyde

Groups acting on trees

G finite group, X is a G-set

X breaks into transitive G-sets

X is a ~~transitive~~ transitive G-set with a chosen "base pt" x

same thing as $H \subseteq G$

Change x to $y = gx$, $G_y = gG_xg^{-1}$

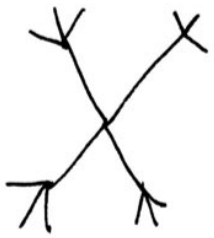
What about G-maps?

$f: X \to Y$ $\quad f(gx) = g(f(x))$

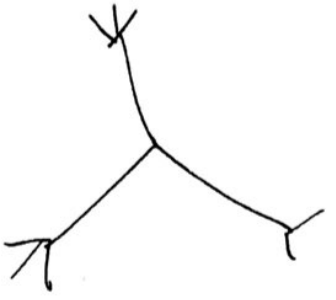
$x \mapsto y$ (preserve base points) $\quad (\iff G_x \subseteq G_y$

A <u>tree</u> is a graph without cycles:

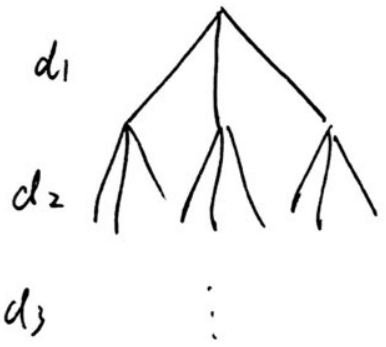
<u>Def</u> A tree T is regular if $Aut(T)$ acts transitively on leaves



regular



not regular



Rees product

$(((\cdots) \wr S_{d_3}) \wr S_{d_2}) \wr S_{d_1}$

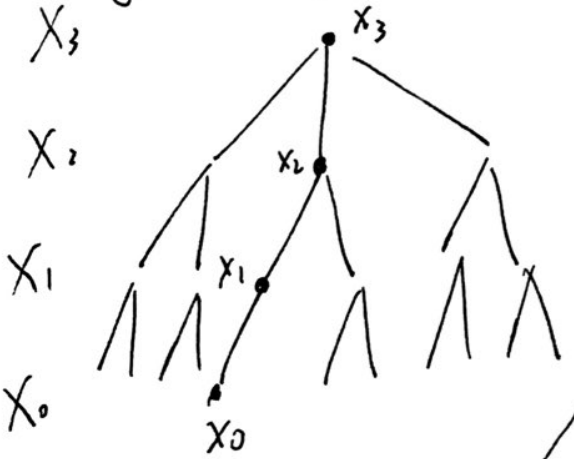
Totally useless.

Def: A G-tree is a tree with G action

Def: A regular G-tree is a regular tree with a transitive G action on the leaves.

Goal: Find a nice classification of regular G-trees and their morphisms in the spirit of G-sets.

Say a regular G-tree



Choose a "base" leaf

$G_{x_0}, G_{x_1}, G_{x_2}, G_{x_3} = G$

$\Leftrightarrow G_{x_0} \subseteq G_{x_1} \subseteq G_{x_2} \subseteq G_{x_3} = G$

Conversely, given $H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_m = G$ have a regular G-tree

Regular G-tree T with chosen base leaf x_0

$\Updownarrow$

$H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_m = G$

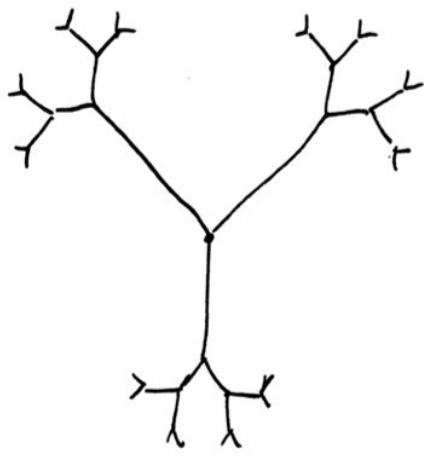
maps of G-trees with base leaf

$\Updownarrow$

inclusions of towers of subgroups.

Call a G-tree maximal if it corresponds to a maximal tower of subgroups.

i.e. $1 \subsetneq C_2 \subsetneq D_2 \subsetneq D_4 \subsetneq S_4$



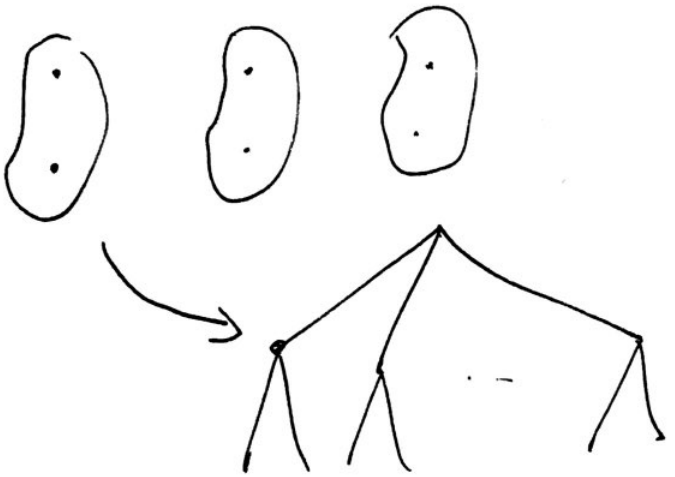
1. Primitive Group Action

Classical Def: A G-set X is called primitive

if 1. X is transitive

2. G preserves no nontrivial partitions

("blocks", or ~~equivalently~~ equivalence relations)



X is primitive $\Leftrightarrow$ ~~only~~ any regular G-tree with X as its leaves looks like

$\Leftrightarrow$ $H \subseteq G$

↖ maximal subgroup.

2. Galois Theory:

$$\begin{array}{ll} K_m & H = Gal(\bar{k}/k) \quad k = K_0 \\ \vdots & \cup | \\ & \vdots \\ K_2 & H_{m-2} \\ | & \cup | \\ K_1 & H_{m-1} \\ | & \cup | \\ K_0 & H_m \end{array}$$

First Guess: Make a tree with fields as nodes!

↖ Wrong

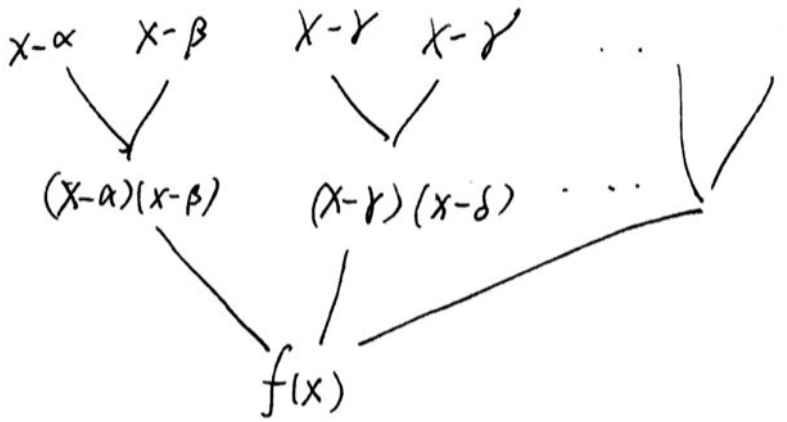
Second Guess: Make leaves embeddings $K_i \hookrightarrow \bar{k}$

↖ right guess

K_m/k Choose primitive element α

then $K_m \cong k[x]/(f(x)) \hookrightarrow \bar{k}$

embedding = choosing a root of $f(x)$

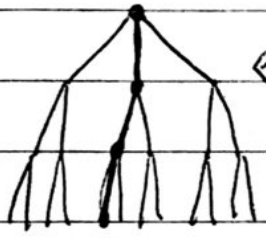


ARBOREAL REPRESENTATIONS

TREVOR HYDE

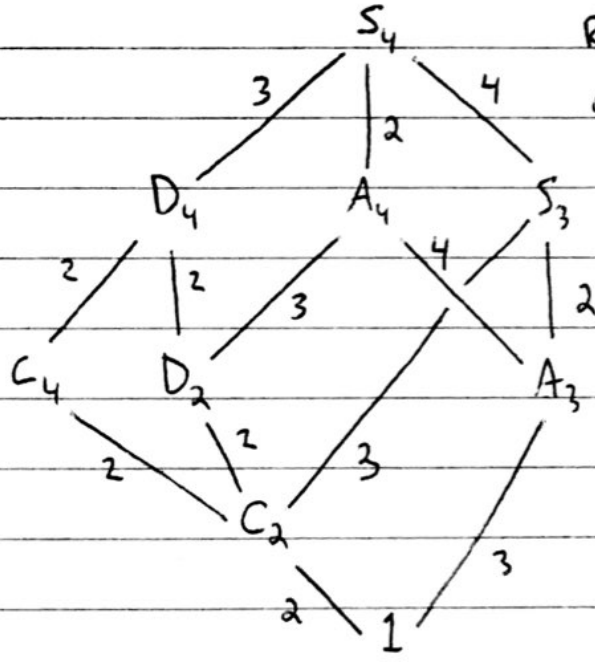
Def A *regular* G-tree T is a tree on which G acts transitively on the leaves.

CHARACTERIZATION OF REGULAR G-TREES



$\leftrightsquigarrow H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_m = G$

Ex. $G = S_4$



Representatives of subgroups of S_4. Edges labelled by index.

Examples of all max'l regular S_4-trees.