Injective dimension of F-finite F-modules and holonomic D modules

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(· joint work with Wenliang Zhang)

All rings are commutative & Noetherian with 1.

<u>Background on injective dimension of local cohomology</u>:

If R is regular containing a field,

then injdim $H^i_I(R) \leq$ dim supp $H^i_I(R)$ $\forall i$, $\forall I$

char $p>0$ Huneke - Sharp '93

char $=0$ Lyubeznik '93

↳ first showed if $R = K[[x_1, \cdots, x_n]]$ (char $K=0$)

and M is any $\underbrace{D(R,K)}$ -module

ring of diff operators on $R = R\langle \frac{\partial}{\partial x_i}, \cdots \rangle$

showed any D-module M

inj dim $\underset{M}{\cancel{H^i_I(M)}}$ $\leq$ dim supp $\underset{M}{\cancel{H^i_I(M)}}$

Lyubeznik deduced the bound of local cohomology over any regular ring containing a field.

Lyubeznik '97 strengthened Huneke - Sharp

over char p reg ring: inj dim$_R$ $M \leq$ dim supp$_R$ M for any <u>F-module</u> M

$F(M) \cong M$

Can we say anything about the lower bound for $\mathrm{inj\,dim}_R M$?

No without some finiteness hypothesis

$E_R(R/p)$ inj dim 0, dim Supp 0. dim R

$p \in \mathrm{Spec}\, R$

We'll consider F-finite F-modules (resp. holonomic $D(R,K)$-modules)

smallest possible "D-module" dimension

$M \xrightarrow{\sim} F(M)$ is F-finite

if $\exists$ f.g. submod $M' \subset M$

and R-linear map $M' \to F(M')$

s.t.
$$\varinjlim \begin{pmatrix} M' & \to & F(M') & \to & F^2(M') & \to & \\ \downarrow & & \downarrow & & \downarrow & & \cdots \\ F(M') & \to & F^2(M') & \to & F^3(M') & \to & \end{pmatrix}$$
is the structure iso $M \xrightarrow{\sim} F(M)$

local cohomology modules $H^i_I(R)$ are F-finite (resp. holonomic)

<u>A remark about localization</u>:

If R is char p reg ring, M is an F-finite F-module

$W \subset R$ is multiplicatively subset, then

$W^{-1}M$ is an $F_{W^{-1}R}$ - finite $F_{W^{-1}R}$-module

$(\mathrm{inj\,dim}_{W^{-1}R} W^{-1}M \leq \dim \mathrm{Supp}_{W^{-1}R} W^{-1}M)$

On the other hand, if $R = K[[X_1, \cdots, X_n]]$ and M is a holonomic $D(R,K)$-module, ~~then~~ and $W \subset R$ as before

then $W^{-1}M$ is a $D(W^{-1}R, K)$-module.

① Don't have a theory of holonomic $D(W^{+}R, K)$-module

② $W^{+}R$ isn't a power series ring, so Lyubeznik's theorem doesn't apply

That is, we don't know whether $\operatorname{inj\,dim}_{W^{+}R} W^{+}M \leq \dim \operatorname{Supp}_{W^{+}R} W^{+}M$

However, two piece of good news:

① If $\mathfrak{p} \in \operatorname{Spec} R$, $\operatorname{inj\,dim}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq \dim \operatorname{Supp}_{R_\mathfrak{p}}(M_\mathfrak{p})$ for any D-module M

② M holonomic $\Rightarrow$ $W^{+}M$ is a finite length $D(W^{+}R, K)$-module

<u>Main Theorem</u>:

If M is either ① an F-finite F-module over a regular char p ring R

or ② a holonomic D-module over a power series ring

Then $\operatorname{inj\,dim}_R M \geq \dim \operatorname{Supp}_R M - 1$

(Cor. There are only two possible cases)

<u>E.g.</u> $R = k[[x_1, \dots, x_n]]$, $\mathfrak{p}$ $ht = n-1$

$E(R/\mathfrak{p})$ is a holonomic $D(R, K)$-module

$0 = \operatorname{inj\,dim} < \dim \operatorname{Supp} \leq 1$

Strategy: Study the last term of the minimal injective resolution of M

Key proposition:
- (Char $p \neq 0$)
- regular domain
- all max ideals have the same height

(1) If M is an F-finite F-module with $t = \text{inj dim}_R M$, then

$$\mu^t(P, M) > 0 \Rightarrow E_R(R/p) \text{ is F-finite}$$

(2) $E(R/p)$ is F-finite iff $\text{ht} P \geq n-1$ and P is contained in only finitely many distinct maximal ideals

How the main theorem is proved?

M F-finite F-module over a char p regular ring R

STOA (suffices to assume) (R, m) local.

Pick $f \in m$ not in any minimal ass primes of M

Now enough to show $\text{inj dim}_{R_f} M_f = \dim \text{Supp}_{R_f} M_f$

More generally, we show if $q \in \text{Supp}(M)$ and $f \notin$ any minimal prime of M_q

$$\text{inj dim}_{(R_q)_f} (M_q)_f = \dim \text{Supp} (M_q)_f$$

done by induction on the dim of support of M_q

If (R, m) ~~a~~ is a catenary local domain of dim ≥ 2 and $f \in m$, then

(1) any non-max prime of R is contained in infinitely many distinct max ideals

(2) All maximal ideals of R_f are of ht $\dim R - 1$.