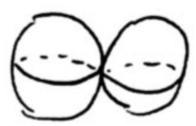
Perverse Sheaf & Fundamental Lemmas

§ 1 Perverse Sheaf

Intersection homology : reconstruct Poincaré duality over singular spaces

Ex: $xy=0$



$H_0 = \mathbb{C}$

$H_1 = 0$

$H_2 = \mathbb{C}^2$

Require Geometric chains to be transversal to the singular point.

1-chains & 0-chains need to avoid the singlar point.

so $IH_0 = \mathbb{C}^2 \quad IH_1 = 0 \quad IH_2 = \mathbb{C}^2$

Hard Lefschetz: $f: X \to Y$ family of proj var

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \\ & \mathbb{P}^N \times Y \arrow[u] \end{tikzcd}$$

Degeneration of the Leray spectral sequence

$$H^i(X, \mathbb{C}) = \bigoplus_{j+k=i} H^j(Y, H^k(X_y))$$

($H^k(X_y)$ are semisimple local systems (i.e. $\pi_1(Y)$-rep))

If X is smooth

f is proper not necessarily smooth

then $H^i(X, \mathbb{C}) = \bigoplus_{\alpha} IH^{i-d_\alpha}(\overline{Y}_\alpha, \mathcal{L}_\alpha)$

where $(Y_\alpha, \mathcal{L}_\alpha, d_\alpha)$

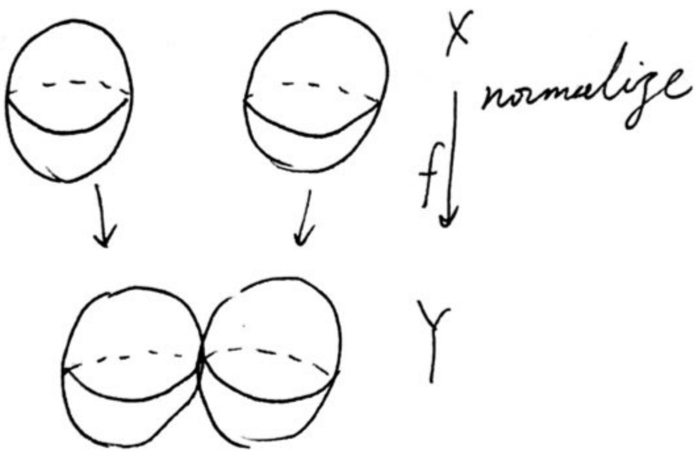
smooth locally closed subset of Y

local system on Y_α, semisimple.

some integer

The sum is finite.

Ex: ①



$H^0(X) = \mathbb{C}^2$ $\Rightarrow$ the singularity point does n't appear

$H^1(X) = 0$

$H^2(X) = \mathbb{C}^2$

If you can normalize Y, say X

then $IH^\bullet(Y) = H^\bullet(X)$

② $\mathbb{P}^1 = GL(2)/B$

$0 \to B \to GL(2) \to \mathbb{P}^1 \to 0$

$B \curvearrowright \mathbb{A}^1 = \{ \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \}$

↪ vector bundle on $\mathbb{P}^1$, $\tilde{\mathcal{N}}$ (total space)

$\mathcal{N}$ = nilpotent cone in $sl(2)$

$$\begin{array}{ccc} \tilde{\mathcal{N}} & = \{ (\mathfrak{b}, & u \in \text{rad}\, \mathfrak{b}) \} \\ \downarrow & \downarrow & \\ \mathcal{N} & u & \end{array}$$

$\begin{bmatrix} x & y \\ z & -x \end{bmatrix}$, $x^2 + yz = 0$ 0 = vertex, $\mathcal{N} - 0$

(similar to blow-up, maybe it is)

$H^0(\mathcal{N}) = \mathbb{C}$ $H^2(\mathcal{N}) = \mathbb{C}$

vertex $\mathcal{N} - 0$ both appear as Y_α

$$\begin{array}{cc} \text{vertex} & \mathcal{N}-0 \\ \downarrow & \downarrow \\ H^2 & H^0 \end{array}$$

Perverse Sheaf

$$IH^{\cdot}(\bar{Y}_\alpha, \mathcal{L}_\alpha) = H^{\cdot}_c(\bar{Y}_\alpha, \underbrace{IC(\mathcal{L}_\alpha)}_{\text{IC-sheaf}})$$

↑

em exp of ~~[illegible]~~ perverse sheaf

Derived Category : $\mathcal{D}^b_c(X) \leftarrow$ primary obj

Abelian category $\leftarrow$ secondary

<u>t-structure</u> : $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0} \subset \mathcal{D}$

$$\rightsquigarrow \mathcal{D} \to \mathcal{A} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$$

<u>Gluing t-structures</u>:

$$U \underset{\text{open}}{\xhookrightarrow{j}} X \underset{\text{closed}}{\xhookleftarrow{i}} Z \quad (U^c = Z,\ Z^c = U \text{ in } X)$$

$$\mathcal{D}_Z \xrightarrow{i_*} \mathcal{D}_X \xrightarrow{j^*} \mathcal{D}_U$$

$\Uparrow$ glue

t-structure

Λ: Stratification of X

$$X = \coprod X_\lambda$$

$$\{\mathcal{F} : i^*\mathcal{F} \in \mathcal{D}_Z^{\leq 0},\ j^*\mathcal{F} \in \mathcal{D}_U^{\leq 0}\}$$

$$\|$$

$$\{\mathcal{F} : i^!\mathcal{F} \in \mathcal{D}_Z^{\geq 0},\ j^!\mathcal{F} \in \mathcal{D}_U^{\geq 0}\}$$

$\mathcal{D}^b_{loc}(X_\lambda)$ t-structure: $\mathcal{D}^{\leq 0} : \{H^i(\mathcal{F}) = 0,\ i > 0\}$

$\mathcal{D}^{\geq 0} : \{H^i(\mathcal{F}) = 0,\ i < 0\}$

$\mathcal{A} = Loc(X_\lambda)$

$$\mathcal{D}^{\leq 0}: \{H^i(\mathcal{F})=0,\ i>-\dim(X_\lambda)\}$$

$$\mathcal{D}^{\geq 0}: \{H^i(\mathcal{F})=0,\ i<-\dim(X_\lambda)\}$$

$$\mathcal{A} = \mathcal{L}oc(X_\lambda)[\dim X_\lambda]$$

$$\left({}^p\mathcal{D}^{\leq 0}_\Lambda,\ {}^p\mathcal{D}^{\geq 0}_\Lambda\right) \ \text{t-structure on } \mathcal{D}^b_\Lambda(X)$$

perverse

$$\mathcal{A} = \mathrm{Perv}_\Lambda(X)$$

Standard Def

$${}^p\mathcal{D}^{\leq 0} = \{\mathcal{F} \mid \dim \operatorname{supp} \cancel{\mathcal{F}(i\mathcal{F})}\ H^i(\mathcal{F}) \leq i\}$$

full subcat

$${}^p\mathcal{D}^{\geq 0} = \text{verdier dual of } {}^p\mathcal{D}^{\leq 0}$$

$\mathrm{Perv}(X) =$ Artinian, Noetherian

simple local system

$$\text{simple obj} = IC(\bar{Y}_\alpha, \mathcal{L}_\alpha)$$

$$= j_{!*}(\mathcal{L}_\alpha),\quad j: Y_\alpha \hookrightarrow \bar{Y}_\alpha$$

$$j_{!*} = \mathrm{Im}({}^pH^0 j_! \to {}^pH^0 j_*)$$