Isotropic Flags

Friday, March 11, 2016 5:09 PM

$X = Fl(n)$ fix $\mathbb{C}^n$

$V_\cdot = \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n\}$ V_i has dim i

Concretely : $B_- \backslash GL(n)$

$\uparrow$ lower triangular matrices

downward row operations

$$A = \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad A_1 = \langle *, 1, 0, 0 \rangle \quad \vdots$$

$A \longleftrightarrow A_\cdot$ span of top i rows

<u>Notice</u>: rep each flag with echelon matrix

$w = 2413$

locus of all flags with permutation w is called Schubert Cell. $\mathring{X}_w$

Closure X_w is called "Schubert variety" X_w

<u>Facts</u>: $\mathring{X}_w \cong \mathbb{A}^{l(w)}$ Length of w. $l(w)$ is the number of inversions

2413 so $l(w) = 3$

Stratification!

<u>Fact</u>: "affine stratification" $\rightsquigarrow$ generate Chow ring

---

Another way: Fix flag... have $e_1, \cdots, e_n$ standard basis

$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \cdots$

$\mathring{X}_w = \{ L_\cdot \in Fl(n) : \dim(L_p \cap E_q) = \#\{ i \le p : w(i) \le q \} \}$

$E_q = \langle e_1, \cdots, e_q \rangle$

To get X_w, turn "=" into "$\geq$"

$Fl(n) = B_- \backslash GL(n) \longrightarrow A_n$

symbolic algebraic

$F\ell(n)$ $\longrightarrow A_n$

parabolic subgp   algebraic gp

Can generalize to "Classical groups"

Symplectic gps $\longrightarrow C_n$

Orthogonal gps $\longrightarrow B_n, D_n$

Type C_n - Symplectic form

Skew-sym form $\langle v,w\rangle = -\langle w,v\rangle$

nondegenerate - nothing nonzero that's $\perp$ to everything

Symplectic gp = matrices fix $\langle\ ,\ \rangle$

Define $\langle \cdot,\cdot \rangle$ on $\mathbb{C}^{2n}$

$\langle e_i, e_{2n+1-i}\rangle = 1 \quad i \le n$

$\langle e_i, e_{2n+1-i}\rangle = -1 \quad i > n$

$$\left(\begin{array}{c|c} 0 & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} \\ \hline \begin{matrix} & & -1 \\ & \iddots & \\ -1 & & \end{matrix} & 0 \end{array}\right)$$

Subspace V of $\mathbb{C}^{2n}$ is <u>isotropic</u> w.r.t. $\langle\ ,\ \rangle$

if $\langle u,v\rangle = 0$ for $u,v \in V$

For $V \subseteq \mathbb{C}^{2n}$ dim k

$\dim(V^{\perp}) = 2n-k$

$\Rightarrow$ All maximal <u>isotropic</u> subspaces have dim n.

Maximal isotropic flag is $\underbrace{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n}_{V_\bullet}$ all iso

$B_- \backslash Sp(2n)$

<u>Note</u>: can extend $V_\bullet$ to complete flag uniquely by setting $V_{2n+1-i} = V_i^{\perp}$

Weyl group of type C_n

$W_n^C = \{ w \in S_{2n} : w(i) + w(2n+1-i) = 2n+1 \}$

Ex: $w = 4321$ $4+1=5$, $3+2=5$

$n=2$ $\begin{bmatrix} t & a & b & c & 1 \\ & t & d & 1 & 0 \end{bmatrix}$ rows are $\perp$ inner product is 0

$\Rightarrow t = b - cd$

$3n$ $[* \quad * \ 1]$

$\Rightarrow t = 6 - ccl$

so $\left[\begin{array}{cc:cc} * & * & * & 1 \\ \bullet & * & 1 & 0 \end{array}\right]$

Can always write matrix charts $\bullet$'s "opposite" pivot 1's in higher rows

once fix $\bullet$'s, $*$'s give affine coords

$$\ell(w) = \#\{ i < j \le n : w(i) > w(j) \}$$
$$+ \# \{ i < j \le n : w(i) + w(j) > 2n+1 \}$$

get $*$ in row i

"opposite" $w(j)$