Abelian Variety

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Abelian variety A is a connected complete group variety /k ← field

Facts: abelian varieties are smooth, proj, comm

Ex elliptic curves

$\overset{1}{=}$ pair (E, O) where E is (sm+proj) genus curve$/k$ & $O \in E(k)$

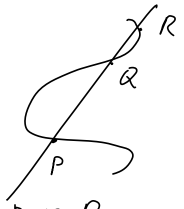
$\overset{2}{=}$ plane curve $y^2 = x^3 + ax + b \subset \mathbb{P}^2_k$

Group law: (1) $E(k) \to \mathrm{Pic}^0(E)$

$x \longmapsto [x] - [O]$

pullback to group structure on $E(k)$

(2)



$P+Q+R=0$

Prop An abelian variety, $\dim A = 1 \Rightarrow \text{genus}(A) = 1$

Pf: trivializes Ω^1 $\underset{\overset{\|}{\text{genus}}}{h^0(A,\Omega^1)} = h^0(A,\mathcal{O}) = 1$

Q: How do abelian varieties help us study curves of higher genus?

- every elliptic curve embeds in its Jacobian
- the Jacobian is the universal abelian variety with this property
- Jacobian + [extra structure] completely determines the curve.

Construction $/\mathbb{C}$

X curve of genus $g/\mathbb{C}$

$$H_1(X,\mathbb{Z}) \hookrightarrow H^0(X,\Omega^1)^*$$

$$\gamma \longmapsto \left(\omega \mapsto \int_\gamma \omega\right)$$

$$\rightsquigarrow \mathrm{Jac}(X) = H^0(X,\Omega^1)^* / H_1(X,\mathbb{Z})$$

Rmk. this is a complex torus

$$H^0(X,\Omega^1)^* \cong \mathbb{C}^g \cong \mathbb{R}^{2g}$$

$$H_1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g} \qquad \rightsquigarrow Jac(X) \cong (S^1)^{2g}$$

$X \hookrightarrow Jac(X)$ fix $x_0 \in X$

$$x \mapsto \int_{x_0}^{x}$$

Alternative description

$$Cl^0(X) \to Jac(X)$$

$$\sum n_x[x] \longmapsto \sum n_x \int_{x_0}^{x}$$

Abel-Jacobi:

$$\begin{tikzcd} \mathrm{Div}^0(X)/\mathrm{PDiv}(X) \arrow[r, "\sim"] \arrow[d, equal] & Jac(X) \\ \mathrm{Pic}^0(X) & \end{tikzcd}$$

Abelian varieties/$\mathbb{C}$

X abelian varieties $/\mathbb{C}$ $\leftrightsquigarrow$ $X(\mathbb{C})$ compact connected complex Lie gp

① X is commutative

Sketch: $X \xrightarrow{C_x} C$ $\quad V = T_eX \rightsquigarrow X \to End(V) \cong \mathbb{C}^n$

$y \longmapsto xyx^{-1} \qquad x \longmapsto (dC_x)_e$

$\rightsquigarrow (dC_x)_e = 1_V$

so $C_x(\exp y) = \exp(d(C_x)_e y) = \exp(y)$

but exp is surj

② $M = \ker(\exp)$ is a lattice in V

③ $X \cong V/M$ complex torus

④ $\underset{\text{n torsion part}}{X[n]} \cong \frac{1}{n}M/M \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$

Line bundles on complex torus

$X = V/M$ complex torus

$\mathrm{Pic}(X) = \{$ isomorphism classes of line bundles on $X\}$ group under $\otimes$

$\cup$

$\mathrm{Pic}^0(X) = \{$ line bundle which are top trivial $\}$

$NS(X) = \mathrm{Pic}(X) / \mathrm{Pic}^0(X)$ Néron-Severi group

<u>Def</u> a Riemann form on V is a Hermitian form H s.t.

$E = \mathrm{Im}\, H$ is $\mathbb{Z}$-valued on M

$\mathcal{R} = \{$Riemann forms$\}$ group under "+"

$$\mathcal{P} = \{ (H, \alpha) : H \in \mathcal{R},\ \alpha : M \to U(1) \ \text{s.t.}\ \alpha(x+y) = e^{i\pi E(x,y)} \alpha(x)\alpha(y) \}$$

$\cup$

$\mathcal{P}^0 = \{ (H, \alpha) : H = 0 \} = \mathrm{Hom}(M, U(1))$

<u>Thm</u>: (Appell-Humbert) $\exists$ nat isom's $\mathcal{P} \xrightarrow{\sim} \mathrm{Pic}(X)$

$(H, \alpha) \longmapsto L(H, \alpha)$

$\mathcal{P}^0 \xrightarrow{\sim} \mathrm{Pic}^0(X)$

$\mathcal{R} \xrightarrow{\sim} NS(X)$

↙ pos def $\longleftrightarrow$ (very) ample

$V \xrightarrow{\pi} X$, $L \in \mathrm{Pic}\, X$ $\pi^* L$ is trivial

<u>but</u> commutes with M-action

and the quotient is L

Given $(H, \alpha) \in \mathcal{P}$, define $M \curvearrowright V \times \mathbb{C}$

↖ trivial line bundle

$$\underset{\overset{\cap}{M}}{\lambda} \cdot (v, z) = \left(v + \lambda,\ \alpha(\lambda) e^{\pi H(v,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)}\right)$$

$\rightsquigarrow$ set $L(H, \alpha) =$ quotient

<u>Rmk</u>: $(H, \alpha) \in \mathcal{P}$

$E = \mathrm{im}\, H : \Lambda^2 M \to \mathbb{Z}$

i.e. $E \in \mathrm{Hom}(\Lambda^2 M, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$

under this isom

$E \longmapsto c_1(L(H, \alpha))$

$L(H, \alpha)$ is top triv $\Leftrightarrow c_1 = 0 \Leftrightarrow E = 0 \Leftrightarrow H = 0$

$\Leftrightarrow (H, \alpha) \in \mathcal{P}^0$