More about Quivers

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(Q_0, Q_1, h, t)

Repn $\longrightarrow V(x)$ for each $x \in Q_0$

$$V(ta) \xrightarrow{V(a)} V(ha)$$

Morphism V, W

$$\begin{tikzcd} V(x) \arrow[d, "\varphi_x"] \\ W(x) \end{tikzcd}$$

natural transformation

irred repr — no sub repr

indec repr — no direct repr

- A_2 quiver (1-Kronecker quiver)

$$\begin{tikzcd} V = & K \arrow[r, "{[1]}"] & K \\ S_2 = & 0 \arrow[r] \arrow[u] & K \arrow[u, "{[1]}"'] \end{tikzcd}$$

$S_1 : K \longrightarrow 0$

$0 \to S_2 \to V \to S_1 \to 0$

but $V \neq S_1 \oplus S_2$

$$\begin{tikzcd} 0 \arrow[r, shift left] & K \arrow[l, shift left] \arrow[r] \arrow[dr] & 0 \arrow[r] & 0 \\ & & 0 \arrow[ur] & \end{tikzcd}$$

irred repr (1st)

$$\begin{tikzcd} K \arrow[d] & K \arrow[l] \\ K \arrow[r] & K \arrow[r] & K \arrow[ul] \end{tikzcd}$$

irred repr

Quivers with no cycles

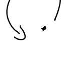
1st type irreducible repr is the only types.

What about indecomposable?

$$V = \underbrace{V_1}^{\oplus n_1} \oplus \cdots \oplus \underbrace{V_k}^{\oplus n_k}$$

each is indecomposable

Eg. ① Jordan Quiver



$$\begin{tikzcd} K^n \arrow[loop above, "A"] \arrow[d, "\varphi"] \\ K^n \arrow[loop below, "B"] \end{tikzcd}$$

$B\varphi = \varphi A$

$\Updownarrow$

$B = \varphi A \varphi^{-1}$

isom classes $\longleftrightarrow$ Jordan form

indecomposable $\longleftrightarrow$ Jordan blocks

② A_2 quiver

$$K^n \xrightarrow{[\ ]} K^m$$

$$\sim \begin{array}{c|c} & \overbrace{\qquad}^{r} \quad \overbrace{\quad}^{s} \\ r \{ & \left(\begin{array}{c|c} 1 \ddots 1 & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}$$

② A_2 quiver

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} K^n \arrow[r, "{[\ ]}"] & K^m \end{tikzcd} \sim \begin{array}{l} r\{ \\ t\{ \end{array}\left(\begin{array}{c|c} 1 \ddots 1 & 0 \\ \hline 0 & 0 \end{array}\right)$$

change of basis on both sides

$$= (K \xrightarrow{[1]} K)^{\oplus r} \oplus (0 \to K)^{\oplus t} \oplus (K \to 0)^{\oplus s}$$

roots in the A_2 root system

$\varepsilon_1, \varepsilon_2, \varepsilon_1+\varepsilon_2$

Which quivers have finitely many decomposables?

Rule out: ① No loops

② No multiple arrows: $\bullet \overset{1}{\underset{\lambda}{\rightrightarrows}} \bullet$

λ changes gives diff nonisom reprs

so its 1-parameter $\mathbb{P}^1$-worth indecomp reprs

③ No $\rightarrow \overset{\downarrow}{\underset{\uparrow}{\cdot}} \leftarrow$ $\tilde{D}_4$

$$\begin{tikzcd} & K \arrow[d, "b"] & \\ K \arrow[r, "a"] & K^2 & K \arrow[l, "c"'] \\ & K \arrow[u, "d"] & \end{tikzcd}$$

for distinct subspaces of K^2

so 4 pts of $\mathbb{P}^1$

3 pts $\longrightarrow (0, 1, \infty)$

1 pt $\longrightarrow$ some pt

$\uparrow$

$\mathbb{P}^1$-worth of reprs

④ No cycles

⋮

Rule out all affine types

only A, D, E left

can show A, D, E explicitly

$\alpha: Q_0 \to \mathbb{Z}_{\geq 0}$ dimension vector

$$K^{\alpha(ta)} \xrightarrow{[\ ]} K^{\alpha(ha)}$$

$$\begin{tikzcd} \mathrm{Rep}(Q,\alpha) = \bigoplus_{a \in Q_1} \mathrm{Mat}_{\alpha(ha),\alpha(ta)} \\ GL(Q,\alpha) = \bigoplus_{x \in Q_0} GL(\alpha(x)) \arrow[u, bend left] \\ \mathbb{C}^* \arrow[u] \end{tikzcd}$$

$\mathbb{C}^*$ acts trivially

orbits $\longleftrightarrow$ isom classes of α-dim reprs

Suppose Q of finite type $\Rightarrow$ finitely many orbits

$\exists\, V \in \mathrm{Rep}(Q,\alpha)$ s.t. O_V is Zariski dense in $\mathrm{Rep}(Q,\alpha)$

$$\dim \mathrm{Stab}_{GL(\alpha(x))} V + \dim O_V = \dim GL(Q,\alpha)$$

$$\overset{\geq 1}{} \qquad \overset{\shortparallel}{\dim \mathrm{Rep}(Q,\alpha)}$$

$$\Rightarrow \dim GL(Q,\alpha) - \dim \mathrm{Rep}(Q,\alpha) \geq 1$$

$$\Rightarrow q(\alpha) = \sum_{x\in Q_0} (\alpha(x))^2 - \sum_{a\in Q_1} \alpha(ha)\alpha(ta) \geq 1$$

$\uparrow$ quadratic form

Q f. type $\Rightarrow$ quadratic form is pos-defn.

$\downarrow$ classify this

classify Rep systems.

$\nearrow$ semi-def

Q tame type $\longrightarrow$ almost finite

other dim vectors $\cup$ dim vector α

$\downarrow$

have only 1 indec $\quad n\alpha \longrightarrow 1$ parameter of indecp

E.g. $\quad \bullet \rightrightarrows \bullet$

$$K \underset{\lambda}{\overset{1}{\rightrightarrows}} K, \quad K^2 \underset{\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}}{\overset{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}{\rightrightarrows}} K^2, \quad K^3 \underset{\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}}{\overset{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}}{\rightrightarrows}} K^3$$

$$K^2 \underset{\begin{pmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}}{\overset{\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}}{\rightrightarrows}} K^3$$

Q wild $\longrightarrow$ indef quadratic

$\exists$ dim vector, for which # of parameters becomes arbitrarily large.

Q wild if you can embed

$$\mathrm{Rep}(\circlearrowleft \bullet \circlearrowright) \hookrightarrow \mathrm{Rep}(Q)$$

$$A \circlearrowleft K^n \circlearrowright B \qquad K^n \underset{B}{\overset{I}{\overset{\xrightarrow{A}}{\rightrightarrows}}} K^n$$

Kac: $Q \longrightarrow$ quadratic form

$\downarrow$

root system

$\downarrow$

$$\Phi^+_{re} \cup \Phi^+_{im} = \left\{ \alpha \;\middle|\; \begin{array}{l} \mathrm{supp}(\alpha) \text{ is connected} \\ q(\alpha) \leq 0 \end{array} \right\}$$

Φ^+_{re} — 1 decomp

$\Phi^+_{im} \longrightarrow$ # of parameters $= 1 - q(\alpha)$