Singularities associated with pencils of quadratics avoiding large linear spaces

Stillman Given a polynomial ring in N variables over a field $K[x_1, \cdots, x_N]$ Assume $K = \overline{K}$

if one has polynomials $F_1, \cdots, F_n$, homogeneous of degrees $d_1, \cdots d_n$

is there a bound on the projective dimension of R/I

where $I = (F_1, \cdots, F_n)$ indep of N

For four quadrics, the bound is 6

For three cubics, $[5, 36]$

There is a bound if the degrees ≤ 4

if char $K \neq 2, 3$

in case of quadrics, all char are ok.

For quadrics, a bound $\sim 3 \cdot n^2 \cdot 2^{n-1}$

In all these cases, we get bounded on projective dimension

by trapping $F_1, \cdots, F_n \in K[G_1, \cdots, G_{\underbrace{B(n, d_1, \cdots, d_n)}_{\text{indep of } N}}]$

and G_i form a reg seq

flat, free module

$K[x_1, \cdots, x_N]$

& have degree $\leq \max\{d_i\}$

A form F of degree ≥ 2

has a k-collapse if it is the ideal generated

by k forms of strictly lower degree

If $\deg F = 2$ or 3, these forms can be taken to be linear

$\equiv$ $V(F)$ contains an algebraic set $V(G_1, \cdots, G_k)$

$F = \sum_{i=1}^{k} G_i H_i \quad \deg G_i < \deg F$

defined by k forms of $\deg < \deg(F)$

If $\deg F$ is 2 or 3, this says $V(F)$ contains a linear space of codim $\leq k$

Set $R = K[x_1, \cdots, x_N]$, $K = \overline{K}$, $V = KF_1 + \cdots + KF_n$

Call V k-guarded if no $F \subset V - \{0\}$ has k-collapse.

$F \overset{\text{linear equivalent}}{\sim}$ $x_1 x_2 + \cdots + x_{2k-1} x_{2k}$ even

$x_1 x_2 + \cdots + x_{2k-1} x_{2k} + x_{2k+1}^2$ odd

$\mathrm{Rad}(\underset{\underset{\frac{\partial F}{\partial x_i}}{\shortparallel}}{\mathrm{Span}\{F_i\}}, F) \cap R_1$ = span of all the x_i

F is k-guarded $\Leftrightarrow$ rank $F \geq 2k+1$

There is a function a of degree n such that

$a(n)$-guarded $\Rightarrow F_1, \cdots, F_n$ is reg seq

By taking $a(n)$ to be a larger function,

we can make all the quotients $R/(F_1, \cdots, F_n)$ nice properties

① reduced

② domain

③ UFD

④ R_k

What is the smallest choice of $a_k(n)$ that makes the singular locus where codim $\geq k+1$

F_1	2-guarded	$\Rightarrow$ UFD
F_1, F_2	3-guarded	$\Rightarrow$ UFD
F_1, F_2, F_3	12-guarded	$\Rightarrow$ UFD
$F_1, \cdots, F_n$	$\lceil \frac{3}{4}n^2 + \frac{7}{4}n \rceil$ - guarded	$\Rightarrow$ UFD