Matrix Semi-invariants

Monday, February 1, 2016 4:11 PM

$SL_n \times SL_n$ on $\mathrm{Mat}_{n,n}^m$

$$(A,B)\cdot(X_1,\dots,X_m) \longmapsto (AX_1B^{-1},\dots,AX_mB^{-1})$$

$$\mathbb{C}^n \bullet \underset{\vdots}{\rightrightarrows} \bullet\, \mathbb{C}^n$$

$$R(n,m) = \mathbb{C}(\mathrm{Mat}_{n,n}^m)^{SL_n\times SL_n} \subset \mathbb{C}(\mathrm{Mat}_{n,n}^m)$$

$\searrow$ Hilbert : f.g. graded

Basic Q: find generators of $R(n,m)$

then minimal generators, syzygies ...

$$\beta(R(n,m)) = \min\{D \mid R(n,m)_{\le D} \text{ generates } R(n,m)\}$$

$\underline{m=1}$, $SL_n \times SL_n \circlearrowleft \mathrm{Mat}_{n,n}$

$$AXB^{-1} = \begin{pmatrix} \lambda & & & & \\ & \ddots & & & \\ & & \lambda & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \quad \begin{pmatrix} \lambda & & & \\ & \ddots & & \\ & & \lambda_0 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} \in \overline{\begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}}$$

so λ is the only invariant

$R(n,1) = \mathbb{C}[\det(X_1)]$ so $\beta(R(n,1)) = n$

$\underline{m=2}$, coeff of t^n in $\det(t_1X_1 + t_2X_2)$ $\rightarrow$ generates $R(n,2)$

$R(n,m)_d = 0$ unless d is a multiple of n

so only consider $R(n,m)_{dn}$

$$R(n,m)_n = \det(t_1X_1 + \dots + t_mX_m)$$

$$R(n,m)_{2n} = \det\left(\begin{array}{c|c} \alpha_1X_1 + \dots + \alpha_mX_m & \beta_1X_1 + \dots + \beta_mX_m \\ \hline \gamma_1X_1 + \dots + \gamma_mX_m & \delta_1X_1 + \dots + \delta_mX_m \end{array}\right)$$

$R(n,m)$ is Cohen-Macaulay (Hochster - Roberts)

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hscop $\longrightarrow f_1 \cdots f_r$ homg inv $\quad \deg f_i = d_i$

$R(n,m)$ is finite over $k[f_1, \cdots, f_r]$

C-M $\longrightarrow$ this is actually free module. $(g_1, \cdots, g_s)$ $\deg g_i = e_i$

the Hilb series $= \dfrac{t_1^{d_1} \cdots t_r^{d_r}}{(1-t_1^{e_1}) \cdots (1-t_s^{e_s})}$

$\Rightarrow \deg(\text{Hilb}) < 0$ so deg of f_i is bounded by deg of g_i

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$\mathrm{Mat}_{n,n}^m \cong \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^m$

$\mathbb{C}[\mathrm{Mat}_{n,n}^m] = \mathrm{Sym}^d(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^m)$

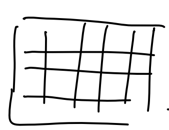
Formula: $S^\lambda(V \otimes W) = \bigoplus \left(S^\mu(V) \otimes S^\nu(W)\right)^{a_{\lambda\mu\nu}}$ $\leftarrow$ Kronecker coeff

$\mathrm{Sym}^d(V \otimes W) = \bigoplus S^\lambda(V) \otimes S^\lambda(W)$

$\mathrm{Sym}^d(V \otimes W \otimes Z) = \bigoplus S^\lambda(V \otimes W) \otimes S^\lambda Z$

$= \bigoplus \left(S^\mu(V) \otimes S^\nu(W) \otimes S^\lambda(Z)\right)^{a_{\lambda\mu\nu}}$

Q: find such λ, $S^\lambda(V)^{\mathfrak{sl}(V)} \neq 0$

so $\lambda =$  $\}n$

$R(n,m)_{nd} = \bigoplus S^{d^n}\mathbb{C}^n \otimes S^{d^n}\mathbb{C}^n \otimes S^\lambda$ $\quad a_{d^n, d^n, \lambda}$

As GL_n repr

$$R(n,m)_{nd} = \bigoplus_{\lambda \vdash nd} S^{\lambda}(\mathbb{C}^m)^{a_{\lambda,d,n}}$$

Going to prove $\beta(R(n,m)) \geq n^2$ if $m \gg 0$

$T_\lambda \otimes T_\mu = \bigoplus T_\nu^{a_{\lambda\mu\nu}}$ (Schur-Weyl duality)

if $\mu = (1^k)$ $\quad T_\lambda \otimes T_{(1^k)} = T_{\lambda'}$

so $\quad T_{d^n} \otimes T_{1^{nd}} = T_{n^d}$

so $\quad a_{d^n, 1^{nd}, d^n} = 0$ unless $d = n$

$\Rightarrow a_{d^n, 1^{nd}, d^n} = a_{d^n, d^n, 1^{nd}} = 0$ unless $d = n$

$n = 3$, $\quad T_{(1,1,1)} \otimes T_{(1,1,1)} = T_{(3)}$

$R(3,m)_3 = S^{(3)}\mathbb{C}^m$

$T_{(2,2,2)} \otimes T_{(2,2,2)} = T_6 + T_{3,3} + T_{4,2} + T_{3,1,1,1}$

$R(3,m)_6 = S^6\mathbb{C}^m + S^{4,2}\mathbb{C}^m + \underbrace{S^{3,3}\mathbb{C}^m + S^{3,1,1,1}\mathbb{C}^m}_{\text{cannot come from } R(3,m)_3}$

$\mathrm{Sym}^2(S^3\mathbb{C}) = S^6\mathbb{C}^m + S^{4,2}\mathbb{C}^m$

... Therefore 1^{n^2} only appears in $R(n,m)_{n^2}$

so $\quad \beta(R(n,m)) \geq n^2$

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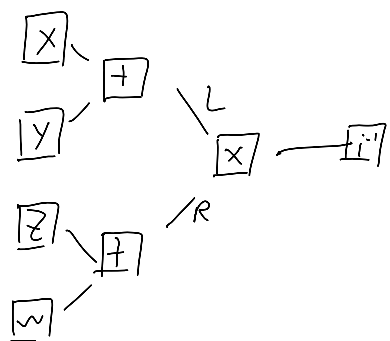
for 1^{n^2} $\quad X_1, \dots X_{n^2}$, let $X_i^c = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \\ a_{12} \\ \vdots \\ a \end{pmatrix}$

for $X_1, \dots X_{n^2}$, let $X_i = \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \\ \vdots \end{pmatrix}$

Then $\det(X_1^c \cdots X_{n^2}^c)$ is the invariant.

Application: Arithmetic circuit



$((x+y)(z+w))^{-1}$

Q: is this noncomm rational expression 0?

- RP time to detect this
- over $\mathbb{Q}$, deterministic poly time algorithm.