Rep of Affine Lie Algebra

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$\mathfrak{g}$ a simple Lie alg

$L\mathfrak{g} = \{$ regular maps $\mathbb{C}^* \to \mathfrak{g}\}$

with bracket

$f_1, f_2 \in L\mathfrak{g}$, $[f_1, f_2](t) = [f_1(t), f_2(t)]$

$L\mathfrak{g} \cong \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] =: \mathfrak{g}[t, t^{-1}]$

$[X \otimes t^n, Y \otimes t^m] = [X, Y] \otimes t^{n+m}$

Kac - Moody Presentation

$$A^{(1)}_{n-1} : A^{(1)}_3 = \begin{array}{ccc} & \begin{array}{ccccc} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \Lambda_0 \end{array} & \\ & \begin{pmatrix} 2 & -1 & & -1 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ -1 & & -1 & 2 \end{pmatrix} \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} \alpha_0^\vee \\ \alpha_1^\vee \\ \alpha_2^\vee \\ \alpha_3^\vee \end{array} \\ & \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \end{array} & d \end{array}$$

$\mathfrak{g}(A)$ is generated by $e_0, \cdots, e_{n-1}$

$f_0, \cdots, f_{n-1}$

$\mathfrak{h} = \text{span}\{\alpha_0^\vee, \cdots, \alpha_{n-1}^\vee, d\}$

w. relations $\mathfrak{h}$ is commutative alg

$[h, e_i] = \alpha_i(h) e_i$

$[h, f_i] = -\alpha_i(h) f_i$

$[e_i, f_j] = \delta_{ij} \alpha_i^\vee$

$[e_i, [e_i, e_j]] = 0$ if $|i-j| = 1$

$[e_i, e_j] = 0$ if $|i-j| > 1$

$$\mathfrak{g}(A) = \mathfrak{h} \oplus \left(\bigoplus_{\beta \in \mathfrak{h}^*} \mathfrak{g}_\beta\right)$$

where $\mathfrak{g}_\beta = \{X \in \mathfrak{gl}(A) \mid [H, X] = \beta(H) \cdot X, \ \forall H \in \mathfrak{h}\}$

<u>Prop</u>: $L\mathfrak{sl}_n \approx \mathfrak{g}(A^{(1)}_{n-1})$

if $i \neq 0$, $e_i = E_{i,i+1}$

$f_i = E_{i+1,i}$

$\alpha_i^\vee = E_{ii} - E_{i+1,i+1}$

Define $e_0 = E_{n,1} \otimes t$

$f_0 = E_{1,n} \otimes t^{-1}$

$[e_0, f_0] := [E_{n,1}, E_{1,n}] \otimes 1 + K$

So $\mathfrak{g}'(A) = L\mathfrak{sl}_n \oplus \mathbb{C}K$

K is central

$[X \otimes t^n, Y \otimes t^m] = [X, Y] \otimes t^{n+m} + n\,\delta_{n,-m}(X, Y) K$

$\mathfrak{g}(A) = \mathfrak{g}'(A) \oplus \mathbb{C}d$

$[d, K] = 0$

$[d, X \otimes t^n] = n\, X \otimes t^n$

So indeed, $L\mathfrak{sl}_n \oplus \mathbb{C}K \oplus \mathbb{C}d \simeq \mathfrak{g}(A^{(1)}_{n-1})$

Roots of $\hat{\mathfrak{sl}}_n$ ($= L\mathfrak{sl}_n$

Let Φ = roots of $\mathfrak{sl}_n$, if $\beta \in \Phi$, $X \in (\mathfrak{sl}_n)_\beta$, i.e. $[H, X] = \beta(H) X$

real root $\rightarrow$ $X \otimes t^n \in (\hat{\mathfrak{sl}}_n)_{\beta + n\delta}$ where $\delta = \alpha_0 + \cdots + \alpha_{n-1}$

imagine root $\rightarrow$ $\alpha_i^\vee \otimes t^n \in (\hat{\mathfrak{sl}}_n)_{n\delta}$ multi $= n-1$

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2 Rep theory of $\hat{\mathfrak{sl}}_n$

Assume: (1) weight space decomp $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$

(2) Integrable , $\forall v$, $\exists N$ s.t.

$e_i^N v = 0$

$f_i^N v = 0$

V irred

$Kv = c\cdot v \quad \forall v \in V$ integrable $\Rightarrow c$ is an integer

↳ called level

positive level $c > 0 \to$ highest w.t. repr (category $\mathcal{O}$)

$c < 0$

$c = 0 \leftarrow$ new phenomena

since K kills everything, throw K (and d as well)

$L\mathfrak{sl}_n$, $V_1, \dots, V_n$ repr for $\mathfrak{sl}_n$

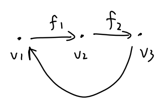
$a_1, \dots, a_n \in \mathbb{C}^*$

$$(X \otimes t^n)(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} a_i^n \, v_1 \otimes \cdots \otimes X v_i \otimes \cdots \otimes v_n$$

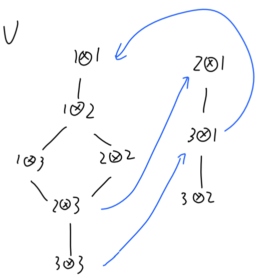
if all V_i are distinct rep, $V_1 \otimes \cdots \otimes V_n$ is irred repr

$\mathfrak{sl}_3$

$V = \mathbb{C}^3$



$V \otimes V$



$\boxed{\text{crystal basis}}$