# An Intro to tensor Cat with examples

Monday, January 25, 2016 4:11 PM

1. What is a tensor $\mathcal{C}$?

* objects $X, Y, Z$
* $\mathcal{C}(X,Y)$ : vector space of homs
* $\mathcal{C}(Y,Z) \otimes_k \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$
* $-\otimes- : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$

$$\begin{matrix} X & X' \\ f\downarrow & g\downarrow \\ Y & Y' \end{matrix} \quad \to \quad \begin{matrix} X\otimes X' \\ \downarrow f\otimes g \\ Y\otimes Y' \end{matrix}$$

* The associator : a natural isom

(A1) $a : (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z)$

(A2) pentagon Axiom

$$(X\otimes Y)\otimes(Z\otimes W)$$

$$((X\otimes Y)\otimes Z)\otimes W \qquad\qquad X\otimes(Y\otimes(Z\otimes W))$$

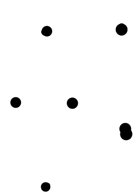
$$(X\otimes(Y\otimes Z))\otimes W - X\otimes((Y\otimes Z)\otimes W)$$

$\mathcal{C}$ is semisimple if every object is a direct sum of simple objects

**Example 1** $G$-rep $= \{ G\text{-rep f.d. vector space} \}$ $G$ finite

$g$ acts $X\otimes Y$ by $g(x\otimes y) = gx \otimes gy$

**Example 2** $G$ a finite group $\mathcal{C}$ = vector bundle / $G$



$A = \mathrm{Func}(G, \mathbb{C})$

$A$ is a Hopf alg

simple object : $V_g = \begin{cases} \mathbb{C} & \text{over } g \\ 0 & \text{over } h \neq g \end{cases}$

$V_g \otimes V_h = V_{gh}$

$\omega(g,h,i) : V_{(gh)i} \xrightarrow{\sim} V_{g(hi)}$

$\omega : G\times G\times G \to \mathbb{C}^\times$

$\omega(g_1 g_2, g_3, g_4) \cdot \omega(g_1, g_2, g_3 g_4)$

$= \omega(g_1, g_2, g_3)\,\omega(g_1, g_2 g_3, g_4)\,\omega(g_2\, g_3\, g_4)$

i.e. $\omega$ is a 3-cocycle on $BG$

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$\mathcal{C}$ a semisimple tensor category

- Grothendieck ring, basis $\lambda, \mu, \nu \in \hat{\mathcal{C}} \leftarrow$ simple obj

  Formal addition

$$V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus a_{\lambda,\mu}^\nu}$$

6j symbols $\quad H_{\lambda,\mu}^\nu = \mathcal{C}(V_\nu, V_\lambda \otimes V_\mu)$ so $V_\nu^{\otimes a_{\lambda\mu}^\nu} = H_{\lambda,\mu}^\nu \otimes V_\nu$

$$(V_\alpha \otimes V_\beta) \otimes V_\gamma = \bigoplus_\nu \Big( \bigoplus_\lambda H_{\alpha\beta}^\lambda \otimes H_{\lambda\gamma}^\nu \Big) \otimes V_\nu$$

$$V_\alpha \otimes (V_\beta \otimes V_\gamma) = \bigoplus_\nu \Big( \bigoplus_\lambda H_{\beta\gamma}^\lambda \otimes H_{\alpha\lambda}^\nu \Big) \oplus V_\nu$$

$$(\Phi_{\alpha\beta\gamma}^\nu)_j^i : \bigoplus_\lambda H_{\alpha\beta}^\lambda \otimes H_{\lambda\gamma}^\nu \longrightarrow \bigoplus_\lambda H_{\beta\gamma}^\lambda \otimes H_{\alpha\lambda}^\nu$$

$\uparrow$ 6j-symbol

Pentagon axiom translates into a cubic relation among 6j symbols

<u>Thm</u> two semisimple tensors cats are tensor equivalent iff they have the same Grothendieck group and the same 6j symbols

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$$M(Gr) = \left\{ \begin{matrix} \text{space of} \\ \text{possible} \\ \text{6j symbols} \end{matrix} \right\} \Big/ GL(H)$$

$$\pi_0\, M(\mathbb{Z}[G]) = H^3(BG, \mathbb{C}^*)$$

<u>Tannaka duality</u>: Finite group determined up to iso by $Rep(G)$ as a tensor category.

$D_4, Q_8 \longrightarrow$ have the same Grothendieck group not iso $\Rightarrow$ 6j symbols are not different

Are $SL_2$ and $U_q(\underline{SL_2})$ tensor equivalent? No

Computing 6j symbols is using $\underline{\text{webs}}$

$$SL_2 = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \middle| \, ad - bc = 1 \right\} \qquad V = \mathbb{C}\{x, y\}$$

$$\cup : \mathbb{C} \longrightarrow V^{\otimes 2} \quad \begin{bmatrix} 0 \\ i \\ -i \\ 0 \end{bmatrix}$$

$$\cap : V^{\otimes 2} \longrightarrow \mathbb{C} \quad [0 \; i \; -i \; 0]$$

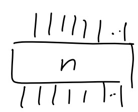
$$| : V \longrightarrow V \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X : V^{\otimes 2} \longrightarrow V^{\otimes 2} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x \otimes y \longmapsto y \otimes x$$

$$X = || + \substack{\cup \\ \cap}$$

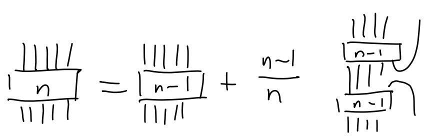
$$O = -2$$

$V^{\otimes n} \supset Sym^n(V)$  $: V^{\otimes n} \longrightarrow V^{\otimes n}$

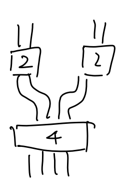
 $= \frac{1}{2}(|| + X)$

$= \frac{1}{2}(2|| + \substack{\cup \\ \cap})$

$= || + \frac{1}{2} \substack{\cup \\ \cap}$



## The classical + Quatum 6j Symbols

$H^{22}_4$ 

$H^{33}$ 

$W^{33}_4$