Syzyges of curve and property N_p

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§1 Generalities

X sm proj variety

L very ample

$X \hookrightarrow \mathbb{P}(H^0(X, L)^*)$

$S = \mathrm{Sym}\, H^0(X, L)$

R = homogeneous coord ring of X (as an S-alg)

<u>Def</u> L is said to have

ⓐ N_0 when L is normally generated

$$R = \bigoplus_{i \geq 0} H^0(X, L^i)$$

ⓑ N_p for $p \geq 1$ if N_0 and

$$0 \to F_s \to \cdots \to F_0 \to R \to 0$$

$F_0 = S$ and $F_i = S(-i-1)^{\alpha_i}$

for $1 \leq i \leq p \ll s$

$$0 \to I_X \to S \to R \to 0$$

$$S(-2)^{\oplus \alpha_1}$$

N_1 : " ideal generated by quadratics"

$Q_1, \cdots, Q_k$ $\quad \sum \alpha_i Q_i = 0$ $\quad \alpha_i$: linear form

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X proj, L v.a., $S = \mathrm{Sym}\, H^0(X, L)$

$F \in \mathrm{Coh}(X)$

module $B = \bigoplus_{q \in \mathbb{Z}} H^0(X, F \otimes L^q) = \bigoplus_{q \in \mathbb{Z}} B_q$

$$\Lambda^{p+1} H^0(X, L) \otimes B_{q-1} \xrightarrow{d_{p+1, q-1}} \Lambda^p H^0(X, L) \otimes B_q \xrightarrow{d_{p,q}} \Lambda^{p-1} H^0(X, L) \otimes B_{q+1}$$

$$d_{p+1, q-1}(v_1 \wedge \cdots \wedge v_{p+1} \otimes s) = \sum (-1)^i v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_{p+1} \otimes (v_i \cdot s)$$

$$K_{p,q}(X, F, L) = \frac{\mathrm{Ker}(d_{p,q})}{\mathrm{Im}(d_{p+1,q-1})}$$

Koszul cohomology

① $K_{p,q}(X, L) := K_{p,q}(X, \mathcal{O}_X, L)$

② $K_{p,q}(X, L) := \mathrm{Tor}_p^S(R, S/S_+)_{p+q}$

③ $F_p = \bigoplus_q \underbrace{K_{p,q}(X, L)}_{\text{generators in deg } p+q \text{ for } F_p} \otimes S(-p-q)$

If prop N_p,

$$K_{\underset{\text{order}}{n},\,\underset{\text{weight}}{q}}(X,\mathcal{L}) = 0 \quad \text{for } n \le p,\ q \ge 2$$

X curve of genus g

$\mathcal{L}$ line bundle on X with $\deg(\mathcal{L}) = 2g+1+p$

$(X, \mathcal{L})$ satisfies property N_p.

<u>Def</u> (kernel bundle)

$$0 \to M_{\mathcal{L}} \to H^0(X,\mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L} \to 0$$

<u>Thm</u> $(X, \mathcal{L})$ $H^1(\mathcal{L}) = 0$ (non special)

$\mathcal{L}$ satisfies property N_p iff

$$H^1(X, \Lambda^{p+1} M_{\mathcal{L}} \otimes \mathcal{L}^k) = 0 \quad \text{for all } k \ge 1$$

Pf: assuming satisfying N_{p-1}

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & \Lambda^{p+1} M_{\mathcal{L}} \otimes \mathcal{L}^{k-1} \arrow[r] & \Lambda^{p+1} H(\mathcal{L}) \otimes \mathcal{L}^{k-1} \arrow[r] \arrow[dr] & \Lambda^{p} M_{\mathcal{L}} \otimes \mathcal{L}^{k} \arrow[r] \arrow[d] & 0 \\
 & & & \Lambda^{p} H^0(\mathcal{L}) \otimes \mathcal{L}^{k} \arrow[d] \arrow[dr] & \\
 & & 0 \arrow[r] & \Lambda^{p-1} M_{\mathcal{L}} \otimes \mathcal{L}^{k+1} \arrow[r] \arrow[d] & \Lambda^{p-1} H^0(\mathcal{L}) \otimes \mathcal{L}^{k+1} \\
 & & & 0 &
\end{tikzcd}$$

<u>Lem</u> $x_1, \dots, x_n \in X$ distinct s.t. $D = \sum x_i$, $\mathcal{L}(-D)$ is global gen

$$h^1(\mathcal{L}) = h^1(\mathcal{L}(-D))$$

Then $0 \to M_{\mathcal{L}(-D)} \to M_{\mathcal{L}} \to \bigoplus_{i=1}^{n} \mathcal{O}(-x_i) \to 0$