What is the Hodge Decomposition?

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Thm Let E, F be smooth vector bundle of rank k over a smooth oriented Riemannian manifold compact M. Assume E and F have metrics.

The let $P: C^\infty(E) \to C^\infty(F)$ be an elliptic partial differential operator.

C^∞ sections

Then $\mathrm{Ker}(P) \subset C^\infty(E)$ is finite-dim

and $C^\infty(E) = \mathrm{Ker}(P) \oplus \mathrm{Im}(P^*)$

Formal adjoint

orthogonal w.r.t. $L_2(E)$

① What is a PDO?
② What is an elliptic PDO?
③ What is the formal adjoint?

ex $\Delta: \Omega^k(M) \to \Omega^k(M)$

$\Delta = d\delta + \delta d = (d+\delta)^2$ where $\delta = (-1)^k *^{-1} d *$
$= (-1)^k (-1)^{(k-1)(n-k-1)} * d *$
$= - * d * \quad$ (if n even)

$*$: Hodge Star
$\Omega^k(M) \to \Omega^{n-k}(M)$
locally, $\omega_1, \dots, \omega_n$ is an oriented basis of $\Omega^1(U)$, then $*(\omega_{i_1} \wedge \dots \wedge \omega_{i_k}) = \omega_{j_1} \wedge \dots \wedge \omega_{j_{n-k}}$
where $\omega_{i_1} \wedge \dots \wedge \omega_{i_k} \wedge \omega_{j_1} \wedge \dots \wedge \omega_{j_{n-k}}$ is an orientation form.

Def A map A from $C^\infty(E)$ to $C^\infty(F)$ is a PDO if locally it can be written as a matrix of partial derivatives $\begin{bmatrix} \sum a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} & \\ & \end{bmatrix}$

Remark The highest term is not changed

Def $\sigma_P \in \Gamma(\mathrm{Hom}(\pi^*(E), \pi^*(F)))$ where $\pi: \mathring{T}^*M \to M$
is defined by $\sigma_P(\xi)\alpha = P\left(\frac{f^k}{k!}\alpha\right)\pi(\xi)$ where $df(\pi(\xi)) = \xi$
$f(\pi(\xi)) = 0$

Symbol of P

In local coordinates, the symbol of $f = \sum a_\alpha \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$
$= \sum a_\alpha \xi^{\alpha_1} \cdots \xi^{\alpha_n}$

Def If $\sigma_P(\xi) \in \Gamma(\mathrm{Hom}(E,F))$ is an isomorphism for every nonzero $\xi \in T^*M$
$(\xi \in \mathring{T}^*M)$

then we say that P is elliptic.

ex compute the symbol in local coordinates : $\sigma_d(\xi)\alpha = \xi\wedge\alpha$

and $\sigma_\delta(\xi_x)\xi \lrcorner \alpha = -\alpha(\xi, \cdot\,\cdot)$

$\sigma_\Delta(\xi_x)\alpha = (\xi\wedge - \xi\lrcorner)^2\alpha = -\|\xi\|^2\alpha$

$\sigma_\Delta(\xi) = -\|\xi\|^2 Id$ is invertible whenever $\xi_x \neq 0$

so Δ is elliptic

Def Formal Adjoint

$P^*: C^\infty(F) \to C^\infty(E)$ is a PDO defined by

$$\langle P\alpha, \beta\rangle_{L^2} = \langle \alpha, P^*\beta\rangle_{L^2} \quad \forall \alpha\in C^\infty(E),\ \beta\in C^\infty(F)$$

$$\langle\alpha,\beta\rangle_{L^2} = \int_M \alpha\wedge *\beta = \int\langle\alpha,\beta\rangle\, dvol.$$

Remark This is constructed locally by integration by parts (Stokes)

Ex Claim : $\delta = d^*$

want $\displaystyle\int d\alpha\wedge *\beta = \int \alpha\wedge *\delta\beta = \int\alpha\wedge *(-1)^k *^{-1} d*\beta$

$= \int \alpha\wedge(-1)^k d(*\beta)$ $[k = \deg\alpha]$

Note $d(\alpha\wedge *\beta) = d\alpha\wedge *\beta + (-1)^{k-1}\alpha\wedge d(*\beta)$

$= \int d\alpha\wedge *\beta$

Remark $(P\circ Q)^* = Q^*\circ P^*$

$$\begin{aligned}\Delta^* &= (dd^* + d^*d)^*\\ &= dd^* + d^*d\\ &= \Delta\end{aligned}$$

formally self-adjoint

Cor $\Omega^k(M) = Ker(\Delta)\oplus Im(\Delta) = \underbrace{Ker(\Delta)\oplus Im(d)}_{Ker\, d}\oplus Im(d^*)$

$Ker\, d^*$

Punchline Each $\alpha\in H^*_{dR}(M)$ can be uniquely represented by an element of the $Ker(\Delta)$

Conversely every $\alpha\in Ker(\Delta)$ is closed & hence represents $\in H_{dR}$

Basic Application: $H^*_{dR}(M)$ is finite dim