Density results on 2-part of class groups

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$K = \mathbb{Q}(\sqrt{D})$ disc D

$\mathcal{O}_K = \mathbb{Z}[\frac{D+\sqrt{D}}{2}] \to Cl_K = I_K / P_K$

$Cl_K^+ = I_K / P_K^+ \leftarrow (\alpha) \quad \sigma(\alpha) > 0$ for $\sigma : K \hookrightarrow \mathbb{R}$

$Cl_K \leftarrow$ finite abelian group

ℓ prime, $rk_{\ell^k} Cl_K = \dim_{\mathbb{F}_\ell} Cl_K^{\ell^{k-1}} / Cl_K^{\ell^k}$

- $\sum_{0<D<X} 3^{rk_3 Cl(D)} \sim \frac{4}{3} \sum_{0<D<X} 1$ (1971)
- Cohen-Lenstra (1984) $\mathrm{Prob}(G) \sim \frac{1}{\# \mathrm{Aut}(G)}$ ← probability of G being a class group of some imaginary field
- (Gauss) $rk_2 Cl(D)^+ = \#\{p \mid D\} - 1$
- Gerth (1987) $Cl(D)^2$ behaves randomly in the sense of C-L
- $\sum_{0<D<X} 2^{rk_4 Cl(D)}$ Fouvry - Klüners
- 8-ranks in families of the form $\{\mathbb{Q}\sqrt{dp}\}$ d fixed, p prime

  Stevenhagen (1989)

  governing field: $\exists$ normal $M_d / \mathbb{Q}$

  $rk_8 Cl(dp)$ is determined by the conjugacy class $\left(\frac{p}{M_d/\mathbb{Q}}\right)$

- Thm $\frac{\#\{p<X \,;\, p \equiv -1 \pmod 4,\ rk_{16} Cl(-8p) = 1\}}{\#\{p<X\}} = \frac{1}{16}$

$$\frac{1}{2}(1+\chi_4(a)) = \begin{cases} 1 & \text{if } a \equiv 1 \pmod 4 \\ 0 & \text{if } a \equiv -1 \pmod 4 \end{cases}$$

	$d=-4$	$d=-8$
	$p \equiv 1\ (4)$	$p \equiv -1\ (4)$
$rk_8 = 1$	$p \equiv 1\ (8)$	$p \equiv -1\ (8)$

($x^2 - Dy^2 = 1$ always have solution

$x^2 - Dy^2 = -1$ has sol $\Leftrightarrow$ $Cl(D) = Cl(D)^+$)

$\left(\begin{array}{ll} x^2 - Dy^2 = 1 & \text{always have solution} \\ x^2 - Dy^2 = -1 & \text{has sol} \iff Cl(D) = Cl(D)^+ \end{array} \right)$

Conj: $x^2 - 2py^2 = -1$

$$\lim_{X \to \infty} \frac{\#\{p < X : \text{has a sol.}\}}{\#\{p < X ; p \equiv 1 \ (4)\}} = \frac{2}{3}$$

result $\frac{1}{2} < \lim < \frac{3}{4}$

$Cl_K \xrightarrow{\sim} Gal(H_K/K)$ H_K: Hilbert field : maximal field ext of K

$[p] \longmapsto \left(\frac{p}{H_K/K}\right)$