10/22/2015

What is a multiplier ideal?

History: · Esnault, Kawamata, Viehweg
· Nadel, Siu

<u>Classical Problem</u> $S \subset \mathbb{P}^n$ finite
$A \subset \mathbb{P}^n$ hypersurface of degree d, with $\mathrm{mult}_x A \geq k$, $\forall x \in S$
[Think d & k large]

Will show: can ~~fit~~ find hypersurface ~~such that~~ together S with degree $\leq \lfloor \frac{dn}{k} \rfloor$ $(-(n-1))$
↑ open question

Complex Analysis Land.

Fix $t \in \mathbb{Q}_{\geq 0}$, f holomorphic in a ~~nbd~~ neighbourhood of $0 \in \mathbb{C}^n$

$$J_0^{an}(f,t) = \{ h \in \mathcal{O}_{\mathbb{C}^n,0} : \frac{|h|}{|f|^t} \in L^2_{loc}(\mathbb{C}^n, 0) \}$$

if $f \in \mathcal{O}(\mathbb{C}^n)$

$$J^{an}(f,t) = \{ h \in \mathcal{O}(\mathbb{C}^n) : \frac{|h|}{|f|^t} \in L^2_{loc}(\mathbb{C}^n) \}$$

"multiplier ideal"

<u>Rmk</u>: If $t=0$, $J^{an}(f,0) = (1)$

<u>Def</u>: $\mathrm{lct}(f) = \sup\{ t : J^{an}(f,t) = (1) \}$
$= \sup\{ t : \frac{1}{|f|^t} \in L^2_{loc}(\mathbb{C}^n) \}$

For $t \geq \mathrm{lct}(f)$, ???

<u>Example</u>: $f = z_1^{a_1} \cdots z_n^{a_n}$ $\quad \frac{1}{|f|^t} \in L^2_{loc}(\mathbb{C}^n) \Leftrightarrow \frac{1}{|z_i|^{ta_i}} \in L^2_{loc}(\mathbb{C}^n)$ $\forall i$

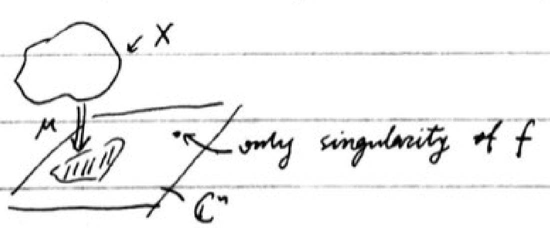
$\Leftrightarrow ta_i < 1 \quad \forall i$

$\Leftrightarrow t < \min\{ \frac{1}{a_i} \}$

<u>"Change of Coordinates"</u>

want $\mu : X \to \mathbb{C}^n$ holomorphic s.t. $\frac{|h|}{|f|^t} \in L^2_{loc}(\mathbb{C}^n) \Leftrightarrow \frac{|\mu^* h|}{|\mu^* f|^t} |\det \mathrm{Jac}(\mu)| \in L^2_{loc}(X)$

<u>Problem</u>.



Want: μ birational + proper

In fact, look at: fix $D = \{f = 0\}$

a log resolution of D is a holomorphic $\mu: X \to \mathbb{C}^n$ st.

- μ birational and proper
- $\mu^* D + \mathrm{Jac}(\mu)$ has single normal crossing support

Jacobian divisor

- $\mathrm{Zeros}(\det \mathrm{Jac}(\mu))$ [Hironaka $\Rightarrow$ they exist]
- $K_{X/\mathbb{C}^n} = K_X - \mu^* K_{\mathbb{C}^n}$

How does this help? Pick $P \in X$, $\exists$ coordinates $z_1, \cdots, z_n$

$$\det \mathrm{Jac}(\mu) = z_1^{a_1} \cdots z_n^{a_n}$$

$$\mu^* f = z_1^{b_1} \cdots z_n^{b_n}$$

$$\mu^* h = z_1^{c_1} \cdots z_n^{c_n} \cdot h' \quad \text{where } h' \text{ has no } z_i\text{-factors}$$

$$h \in J^{an}(f,t) \iff \frac{|\mu^* h|}{|\mu^* f|^t} |\det \mathrm{Jac}(\mu)| \in L^2_{loc}(X)$$

$$\iff \int_K |z_1|^{2a_1 + 2c_1 - 2tb_1} \cdots |z_n|^{2a_n + 2c_n - 2tb_n} |h'|^2 \, d\mathrm{vol}$$

$$\iff a_i + c_i - tb_i > -1 \quad \forall i$$

$$\iff a_i + c_i - \lfloor tb_i \rfloor \geq 0 \quad \forall i$$

Consider $D_i = \{z_i = 0\}$ $\qquad K_{X/\mathbb{C}^n} = \sum a_i D_i$

$$\mu^* D = \sum b_j D_j$$

$$h \in J^{an}(f,t) \iff \mathrm{div}(\mu^* h) + K_{X/\mathbb{C}^n} - \lfloor t \cdot \mu^* D \rfloor$$

$$\mathrel{\|} \quad \mathrm{div}(h') + \sum_j \underbrace{(a_j + c_j - \lfloor tb_j \rfloor)\, D_j}_{\text{effective divisor}}$$

$$\iff \mu^* h \text{ is a section of } \mathcal{O}_X(K_{X/\mathbb{C}^n} - \lfloor t \mu^* D \rfloor)$$

$$\iff h \text{ is a section of } \mu_* \mathcal{O}_X(K_{X/\mathbb{C}^n} - \lfloor t \cdot \mu^* D \rfloor)$$

Def. $J^{alg}(f,t) = \mu_* \mathcal{O}_X(K_{X/\mathbb{C}^n} - \lfloor t \mu^* D \rfloor)$

$$= \{ h \in \mathbb{C}[z_1, \cdots, z_n] : \frac{|h|}{|f|^t} \in L^2_{loc}(X) \}$$

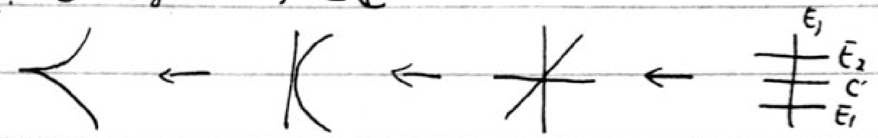
Def. X smooth irreducible variety $/\mathbb{C}$, D effective $\mathbb{Q}$-divisor

pick $\mu: X' \to X$ of D set $J(D) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor \mu^* D \rfloor)$

Ex: D is simple normal crossing divisor on X,

$$\mathcal{J}(D) = \mathcal{O}_X(-\lfloor D \rfloor)$$

Ex: $C = \{y^2 - x^3 = 0\} \subset \mathbb{C}^2$



$K_{X/\mathbb{C}^2} = E_1 + 2E_2 + 4E_3$

$\mu^* C = 2E_1 + 3E_2 + 6E_3 + C'$

$\mathcal{J}(t \cdot C) = \mu_* \mathcal{O}_X\big((1 - \lfloor 2t \rfloor)E_1 + (2 - \lfloor 3t \rfloor)E_2 + (4 - \lfloor 6t \rfloor)E_3 + (-\lfloor t \rfloor)C'\big)$

- For $0 \le t < \frac{5}{6}$, all coefficients are $\ge 0 \Rightarrow \mathcal{J}(t \cdot C) = (1)$
- For $\frac{5}{6} \le t < 1$, $\mathcal{J}(t \cdot C) = \mu_* \mathcal{O}_X(-E_3) = (x, y)$
- For $1 \le t < 1 + \frac{5}{6}$, $\mathcal{J}(t \cdot C) = (f) \cdot \mathcal{J}((t-1)C) = (f) \cdot (1) = (f)$

Def: $\mathfrak{a} \subset \mathcal{O}_X$ ideal sheaf, $t \in \mathbb{Q}_{\ge 0}$

Pick a log resolution $\mu: X' \to X$ of $\mathfrak{a}$

[i.e. $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$, F effective and $F + Exc(\mu)$ has SNC support]

$$\mathcal{J}(\mathfrak{a}^t) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor t \cdot F \rfloor)$$

Ex: $\mathfrak{a} \subset \mathbb{C}[z_1, \cdots, z_n]$ monomial ideal

$\langle z^m = z_1^{m_1} \cdots z_n^{m_n} \rangle$

$\wr$

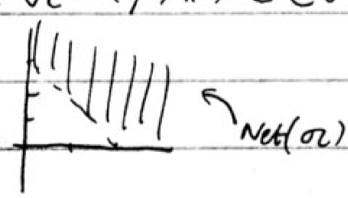
$m \in \mathbb{Z}^n$

$\wr$

$Newt(\mathfrak{a}) \subset \mathbb{R}^n$

Thm (Howald) $\mathcal{J}(\mathfrak{a}^t) = \langle z^m : m + (1, \cdots, 1) \in Int(t \cdot Newt(\mathfrak{a})) \rangle$

Ex: $\mathfrak{a} = (y^2, x^3) \subset \mathbb{C}[x, y]$



$\mathcal{J}(\mathfrak{a}) = \langle x, y \rangle$