10/15/2015

[Kollar, which powers of ~~hat~~ holomorphic functions are ~~integ~~ integrable]

Consinder $f \in \mathbb{C}[z_1, \cdots, z_n]$, $f(0)=0$

study the singularity of $\{f=0\}$ at $0 \in \mathbb{A}^n$

$$\mathrm{ord}_0(f) = \max\{c \mid f \in m_0^c\}$$

Note: $\mathrm{ord}_0(f) = 1 \Leftrightarrow f$ is smooth at the origin

(ord_0 is a discrete valuation of $\mathbb{C}[z_1,\cdots,z_n]$)

limitations  $y^2-x^3=0$ $\quad \mathrm{ord}_0 = 2$ $\quad$ worse singularities

 $y^2-x^2-x^3=0$ $\quad \mathrm{ord}_0 = 2$

log Canonical ~~thr~~ threshold of f at 0

$$C_0(f) = \sup\{c \mid \frac{1}{|f|^c} \text{ is } L^2 \text{ in a nbhd of } 0\}$$

Note: $\lambda \in \mathbb{C}^*$, $C_0(\lambda f) = C_0(f)$

This measure singularities of $\{f=0\}$

Example: (dim 1)

- $C_0(z)$: when is $\frac{1}{|z|^{2c}}$ integrable in nbhd of 0

$$\Leftrightarrow \int_0^\varepsilon \int_0^{2\pi} \frac{1}{\rho^{2c}} \rho \, d\rho \, d\theta < \infty \Leftrightarrow \int_0^\varepsilon 2\pi \rho^{1-2c} d\rho < \infty$$

$$\Leftrightarrow 2c-1<1 \Leftrightarrow c<1$$

$$\Rightarrow C_0(z) = 1$$

- $C_0(z^m)$: $C_0(z^m) = \frac{1}{m}$
- $C_0(y^2-x^3) = c$ $\quad \frac{1}{|y^2-x^3|^{2c}}$

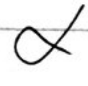
change of coordinates: $\begin{cases} x = uv^2 \\ y = uv^3 \end{cases}$ then $\frac{1}{|y^2-x^3|^{2c}} \longleftrightarrow \frac{|uv^4|^2}{|u^2v^6 - u^3v^6|^{2c}}$

$$\frac{1}{|u|^{4c-2}} \cdot \frac{1}{|v|^{12c-8}} \cdot \frac{1}{|1-u|^{2c}} \text{ integrable} \Leftrightarrow \begin{cases} 4c-2-1<1 \\ 12c-8-1<1 \end{cases}$$

$$\Rightarrow \begin{matrix} c<1 \\ c<5/6 \end{matrix} \quad \Rightarrow c = \frac{5}{6}$$

- Simple normal crossing :

$$C_0(z_1^{a_1}\cdots z_n^{a_n}) = \min\{1/a_i\}$$

-  $y^2-x^2-x^3=0$

$$C_0(y^2-x^2-x^3) = 1$$

Geometrically $X \xrightarrow{\pi} \mathbb{A}^n$

Log resolution of f :
- X smooth
- $\widetilde{(f=0)}$ smooth
- $\mathrm{Jac}(\pi) + \pi^{-*}(f)$ simple normal crossing

$$\mathrm{Jac}(\pi) = K_{X/\mathbb{A}^n} = K_X - \pi^* K_{\mathbb{A}^n}$$

$$K_{X/\mathbb{A}^n} = \sum k_i E_i$$

$$\pi^*(f=0) = \sum a_i E_i$$

$\frac{1}{|f|^{2c}}$ is integrable at $0 \iff \frac{|\mathrm{Jac}_{\mathbb{C}}(\pi)|^2}{|\pi\circ f|^{2c}}$ is integrable near q, $\forall q \in \pi^{-1}(0)$

choose $q \in \pi^{-1}(0)$, $q \in E_{i_j}$ for $1 \le j \le s$

choose local coordinates at $(w_1, \dots, w_n)$

$$\{w_j = 0\} \leftrightarrow E_{i_j} \text{ for } 1 \le j \le s$$

$$\mathrm{Jac}_{\mathbb{C}}(\pi) = u \cdot w_1^{k_{i_1}} \cdots w_s^{k_{i_s}}$$

$$\pi\circ f = v \cdot w_1^{a_{i_1}} \cdots w_s^{a_{i_s}}$$

$\frac{|u w_1^{k_{i_1}} \cdots w_s^{k_{i_s}}|^2}{|v w_1^{a_{i_1}} \cdots w_s^{a_{i_s}}|^{2c}}$ integrable near q

$$\iff 2a_{i_j}c - 2k_{i_j} - 1 < 1 \iff c < \frac{k_{i_j}+1}{a_{i_j}}$$

$$\mathrm{lct}_0(f) = \min\left\{\frac{k_i+1}{a_i}\right\}$$

$a_i = \mathrm{ord}_{E_i}(f)$ correspond to discrete valuation $(\mathcal{O}_{X,E_i})$

T_n = { LCT in dim n }

= { $c_0(f)$ | $f \in \mathbb{C}[z_1, \cdots, z_n]$ }

we know $T_1 = \{1, 1/2, 1/3, \cdots\}$

$5/6 \in T_2$

no one can say anything about higher dim T_n

ACC conjecture

- T_n satisfies a.c.c
- {the limit points on T_{n+1}} = T_n

Pf $\supset$ is easy, $\forall f \in \mathbb{C}[z_1, \cdots, z_n]$

wts $c_0(f)$ is a limit point of T_{n+1}

$$c_0(f + z_{n+1}^m) \xrightarrow[m \to +\infty]{} c_0(f)$$