Bochner Techniques

<u>Proto-Thm</u> geometric space, curvature condition
$\Rightarrow$ cohomology vanishes

Pf: understand a PDE. #

Ex (Kodaira Vanishing Thm)

X compact complex manifold (Kähler)

$L \to X$ positive line bundle

$\Rightarrow H^q(X, \Omega^p \otimes L) = 0$ when $p+q > n = \cancel{\deg}\ \dim_{\mathbb{C}} X$

Pf: $H^q(X, \Omega^p \otimes L) = \mathrm{Ker}(\Delta_{\bar\partial})$

$\bar\partial \curvearrowright \{L\text{-valued } (p,q)\text{-forms}\}$

Kodaira-Nakano Identities

$\Rightarrow \Delta_{\bar\partial} = [\text{non-neg op}] + [\text{curv form}]$

$0 = (\Delta_{\bar\partial} w, w) = \text{—}$!! #

Move to the Riemann Setting.

<u>Goal</u> (Bochner Vanishing Thm)

(M, g) compact orientable Riemann manifold $\mathrm{Ric}(g) > 0$

$\Rightarrow H^1_{dR}(M) = 0$ [Recall: $H^p_{dR}(M) = \{\text{closed } p\text{-forms}\} / \{\text{exact } p\text{-forms}\}$]

Related result [Myers theorem]

Same condition $\Rightarrow \pi_1(M)$ finite

<u>Pf</u>. "variation of energy" [do Carmo §9]

<u>Ref</u>: A book by Rosenberg "Laplacian on a Riemann manifold"

<u>Step 1</u>: Find our PDE

- put an L^2-inner product on p-forms

$$(\alpha, \beta) = \int_M g(\alpha, \beta)\, d\mathrm{vol}$$

$\rightsquigarrow$ the exterior derivative has an adjoint d^*

$\rightsquigarrow$ define the Laplacian $\Delta^p = dd^* + d^*d$

• <u>Hodge Thm</u>: each de Rham class has a unique harmonic representative

[Idea: find ξ in a deRham class that's orthogonal to the exact forms

i.e. $0 = (\xi, dv) = (d^*\xi, v)\ \forall v \Rightarrow d^*\xi = 0 \Rightarrow \Delta\xi = 0$]

new ~~proot~~ problem: $H^1_{dR}(M) = \mathrm{Ker}(\Delta^2)$

$\Rightarrow$ show there are no harmonic 1-forms.

$$\nabla : \Gamma(TM) \to T^*M \otimes \Gamma(TM)$$
$$\nabla \tfrac{\partial}{\partial x^j} = dx^j \otimes \Gamma^i_{jk} \tfrac{\partial}{\partial x^i}$$

Step 2 " ~~St~~ Completing the square"

• (M, g) has a Levi-Civita connection ∇

$\rightsquigarrow$ extend this to an operator on all tensors $\nabla : T^p_q \to T^{p+1}_q$

[here, p = contravariant (dx^j's)

q = covariant ($\frac{\partial}{\partial x^i}$'s)

$\rightsquigarrow$ when acting on forms, it has

Thm (Weitzenbock formula)

$$\Delta^p = \nabla^* \nabla + R^p$$

where R^p is the curvature operator

• Why is this not unexpected?

<u>Exercise</u>: $\Delta(a_j dx^j) = g^{ik} \frac{\partial^2 a_j}{\partial x^i \partial x^k} dx^j$ + [lower order terms]

$\nabla^*\nabla(a_j dx^j) =$ ~~~~~~~~

• What is this curvature operator?

work locally: pick an orthonormal frame $\{e_i\}$ of vector fields

$\rightsquigarrow$ get dual coframe $\{\eta_i\}$

These come with operators on terms

• a_i = wedging with η_i

• a_i^* = interior product with e_i

These satisfies the ~~Ch~~ Clifford relations $a_j a_k^* + a_k^* a_j = \delta_{kj}$

Define the curvature operator $R = -\sum_{i,j} a_i^* a_j R(e_i, e_j)$

[here, $R(e_i, e_j) = \nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i} - \nabla_{[e_i, e_j]}$]

<u>Step 3</u> Pf of the theorem

- When working with one forms, $R = Ric$ (← Exercise)
- Assume that we have nonzero $\omega \in Ker(\Delta^1)$

$$\Delta^1 \omega = \nabla^* \nabla \omega + R\omega$$

$$\text{so } (\Delta^1 \omega, \omega) = (\nabla^* \nabla \omega, \omega) + (R\omega, \omega)$$

$$\Rightarrow 0 = \underbrace{\|\nabla \omega\|^2}_{\geq 0} + \underbrace{(Ric \cdot \omega, \omega)}_{> 0} \quad !!$$

<u>Easy Cor</u>: If $R^p > 0$, then $H^p_{dR}(M) = 0$ ($R^p = R$ acts on dim p)

- <u>With a bit more work Co.</u> (1) If $Ric \geq 0$, $\dim H^1_{dR}(M) \leq \dim M$

(2) If $R^p \geq 0$, then $\dim H^p_{dR}(M) \leq \binom{\dim M}{p}$

[with equality hold iff (M, g) flat torus]

These methods can be taken to an extreme

<u>Thm</u> [A] (M, g) as before + $Vol(M, g) = 1$

Pick $R_0 > 0$, $K < 0$, $\exists\, \varepsilon = \varepsilon(R_0, K, \dim M) > 0$ st. if $Ric \geq R_0$ except on $A \subseteq M$, where $diam(A) \leq \varepsilon$, ~~where~~ $Ric \geq K$

than $H^1_{dR}(M) = 0$

<u>Thm</u> (B] (Bonaus) As in thm A, if $\pi_1(M)$ has a solvable finite-index subgp, then $\pi_1(M)$ finite

(Elworthy & Rosenberg "Compact mfd with a little neg curv")