Vanishing Theorem.

Thm (Serre) X some proj var / k
$\mathcal{L}$ ample
$\mathcal{F}$: coherent sheaf $\exists\ m_0 = m(\mathcal{F})$
s.t. $\forall\ m \geq m_0$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^m) = 0$

Cohomology of $\mathbb{P}^n$: $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z} \cdot \mathcal{O}(1)$ ← ample

$$X = \mathbb{P}^n \qquad H^0(X, \mathcal{O}(m)) = \begin{cases} 0 & m < 0 \\ \mathbb{C}\text{-span of degree } m \text{ monomials } x_0, \dots, x_n & \\ \quad \mathrm{Sym}^m(H^0(\mathcal{O}(1))) & \end{cases}$$

$$H^i(X, \mathcal{O}(m)) = 0$$
$$H^n(X, \mathcal{O}(m)) = (H^0(X, -n-1-m))^*$$

Recall: X sm proj
Ω_X cotangent sheaf
$\Omega^p_X := \overset{p}{\wedge} \Omega^1_X$
$(K_X)\ \omega_X = \wedge^n \Omega^1_X$ } canonical bundle
$\forall$ locally free F
$$H^i(X, F) = (H^{n-i}(X, F^* \otimes \omega_X))^*$$
$$(\Omega^p_X)^* \otimes \omega_X \simeq (\Omega^{n-p}_X)$$

Thm E vector bundle on $\mathbb{P}^1$, $\mathrm{rk}(E) = r$
$E \simeq \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ $\quad a_1 \geq \cdots \geq a_r$

Pf. ~~$H^0(\mathbb{P}^1, E(-a))$~~
$h^0(\mathbb{P}^1, E(-a))$ $\quad$ if $a \ll 0$, $h^1 = 0$ (Serre)
$h^0(\mathbb{P}^1, E(-a)) = \deg E - (1-a) r$ $\quad$ (Riemann-Roch)
$h^0(\mathbb{P}^1, E(-a)) = h^1(\mathbb{P}^1, E^*(a-2)) = 0$ for $a \gg 0$ (Serre)
$\exists$ maximal a_1 such that $H^0(\mathbb{P}^1, E(-a_1)) \neq 0$

$$0 \to \mathcal{O}(a_1) \to E \to E' \to 0$$

$E' = \bigoplus_{i=2}^{r} \mathcal{O}(a_i)$ If $a_2 > a_1$, twist by $\mathcal{O}(-a_1-1)$

$$0 \to (E')^*(a_1) \to E^*(a_1) \to \mathcal{O} \to 0$$

$$H^1((E')^*(a_1)) = 0$$

<u>Ex 2</u> Ample line bundles on globally F-split vars.

X/k char $k = p > 0$

$F: X \to X$

$F^\#: \mathcal{O}_X \to F_* \mathcal{O}_X$

X is globally F-split if $\exists\, \varphi$ s.t.

$$\begin{tikzcd} F_*\mathcal{O} \arrow[r, "\varphi"] & \mathcal{O} \\ \mathcal{O} \arrow[u, "F^\#"] \arrow[ur, "id"'] & \end{tikzcd}$$

<u>Eg.</u> $\mathbb{P}^n$, some toric var, some abelian var.

$\mathcal{L}$ ample on X, F-split

$$H^i(X, \mathcal{L}) = 0 \text{ for } i > 0$$

<u>Pf</u>:

$$\begin{tikzcd} F^e_*\mathcal{O} \arrow[r] & \mathcal{O} \\ \mathcal{O} \arrow[u] \arrow[ur] & \end{tikzcd} \Rightarrow \begin{tikzcd} F^e_*(\mathcal{O})\otimes\mathcal{L} \arrow[r] & \mathcal{L} \\ \mathcal{L} \arrow[u] \arrow[ur] & \end{tikzcd}$$

$$(F^e\mathcal{O}) \otimes \mathcal{L} \cong F^e_*(F^{e*}\mathcal{L} \otimes \mathcal{O})$$

$$\cong F^e_*(\mathcal{L}^{p^e})$$

$$\begin{tikzcd} 0 = H^i(X, \mathcal{L}^{p^e}) \arrow[r] & H^i(\mathcal{L}) \\ H^i(\mathcal{L}) \arrow[u] \arrow[ur, "id"'] & \end{tikzcd}$$

by Serre vanishing,

$H^i(X, \mathcal{L}^{p^e}) = 0$

$\Rightarrow H^i(\mathcal{L}) = 0$ #

<u>Thm</u> (Kodaira - Akizuki - Nakano)

$X/\mathbb{C}$ smooth projective $\quad H^q(X, \Omega_X^p \otimes \mathcal{L}) = 0$ for $p+q > n$

$\mathcal{L}$ ample

false in char $= p$ in general.

<u>Thm</u> (Weak Lefschetz)

X smooth proj $/\mathbb{C}$ dim n

$\subset \mathbb{P}^N_{\mathbb{C}}$, H hypersurface such that $Y = H \cap X$ is smooth

$H^k(X, \mathbb{C}) \xrightarrow{\alpha_k} H^k(Y, \mathbb{C})$

α_k is $\begin{cases} \text{bijective for } k \leq n-2 \\ \text{injective for } k \leq n-1 \end{cases}$

(Hodge Decomposition)

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q} \qquad H^{p,q} = H^q(X, \Omega_X^p)$$

equivalent to show $H^q(X, \Omega_X^p) \to H^q(Y, \Omega_Y^p)$

is $\begin{cases} \text{bij} & \text{for } p+q \leq n-2 \\ \text{inj} & \text{for } p+q \leq n-1 \end{cases}$

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \qquad (1)$$

$$0 \to \mathcal{O}_Y(-Y) \to \Omega_X|_Y \to \Omega_Y \to 0 \qquad (2)$$

$$\Rightarrow \quad 0 \to \Omega_X^p(-Y) \to \Omega_X^p \to \Omega_X^p|_Y \to 0$$

by taking Λ^p, we end up with

$$0 \to \Omega_Y^{p-1}(-Y) \to \Omega_X^p|_Y \to \Omega_Y^p \to 0$$

$$H^q(X, \Omega_X^p(-Y)) = H^{n-q}(X, (\Omega_X^p)^* \otimes K_X \otimes \mathcal{O}(Y))^*$$

$$= H^{n-q}(X, \Omega_X^{n-p} \otimes \mathcal{O}(Y))$$

$$= 0 \quad \text{if } (n-p)+(n-q) > n \Leftrightarrow p+q < n$$