The Hilbert Scheme

$k = \mathbb{C}$

Prelude: <u>Grassmannians</u>

V v.s. of dim $n+1$ $G = Gr(r, \mathbb{P}V) = Gr(r+1, V) =$ $r+1$ dim'l linear subspace

it has a universal bundle: $\Phi = \{(\Lambda, P) \in G \times \mathbb{P}V \mid P \in \Lambda\} \subset G \times \mathbb{P}V \to G$

<u>Univ property</u>: $\forall$ variety S, $\mathcal{Y} \subset S \times \mathbb{P}V$ family of r dim'l linear subspace of $\mathbb{P}V$

$$\exists! \ \alpha : S \to G \quad \text{s.t.} \quad \begin{array}{ccc} \mathcal{Y} & \to & \Phi \\ \downarrow & \times & \downarrow \\ S & \to & G \end{array}$$

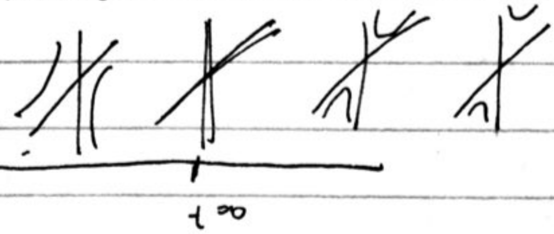
<u>Upshot</u>: $Gr(r, \mathbb{P}V)$ encodes how r-dim'l subspaces can "move" in $\mathbb{P}V$

~~Gen~~ <u>Generalize</u>: Given proj. var. X, want parameter space of closed subschemes Y of X ~~that~~ encodes how Y can move in X

More precisely: Parametrize flat families $\mathcal{Y} \subset S \times X$ of closed subschemes

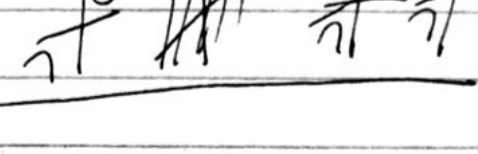
<u>ex</u> 1) $V(xy - t) \subset \mathbb{A}^3$ (flat)

$$\begin{array}{c} (x,y,t) \\ \downarrow \\ t \quad \mathbb{A}^1 \end{array}$$



2) $V(xyt - t) \subset \mathbb{A}^3$

$$\begin{array}{c} (x,y,t) \\ \downarrow \\ t \quad \mathbb{A}^1 \end{array}$$



(Thm) (Grothendieck) let X be a proj variety. then $\exists !$ scheme Hilb_X called the Hilbert scheme of X, together with a universal flat family
$$\begin{tikzcd} Z \subset \mathrm{Hilb}_X \times X \arrow[d] \\ \mathrm{Hilb}_X \end{tikzcd}$$
of closed subschemes of X

for all flat families
$$\begin{tikzcd} Y \subset S \times X \arrow[d] \\ S \end{tikzcd}$$
$\exists !\ S \xrightarrow{\alpha} \mathrm{Hilb}_X$

s.t.
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ S \arrow[r] & \mathrm{Hilb}_X \end{tikzcd}$$

Rmk: (1) Works for all quasi-projective schemes / Noeth base S

(2) Not true for projective replaced by proper (Hironaka)

(3) Uniqueness easy

Stratification of Hilb_X

Def/Thm: Y a proj. var. Then $\exists !$ polynomial $\Phi_Y \in \mathbb{Q}[z]$ such that $\Phi_Y(m) = \dim(\Gamma(\mathcal{O}_Y(m)))$ for ~~n~~ $m \gg 0$

Encodes numerical invariants of Y

1) $r = \dim Y = \deg \Phi_Y$

2) $\deg Y = (\text{leading coefficient of } \Phi_Y) \cdot r!$

Ex 1) $Y = d$ distinct points in $\mathbb{P}^n$

$\Phi_Y(z) = d$

2) $Y \subset \mathbb{P}^n$ hypersurface of deg d

$\Leftrightarrow \Phi_Y(z) = \Phi_d(z) = \binom{n+z}{n} - \binom{n-d+z}{n}$

Fact: $Y \subset S \times X$ flat family of closed subschemes, $f \downarrow S$

then $S \ni s \mapsto \Phi_{f^{-1}(s)}(z)$ is locally constant

closed point

Thus $\mathrm{Hilb}_X = \coprod_{\Phi \in \mathbb{Q}[z]} \mathrm{Hilb}_X^{\Phi}$

1) Noeth
2) proj
3) $X = \mathbb{P}^n$, connected (no idea otherwise)

Ex (1) $X = \mathbb{P}^n$, $\Phi = \Phi_d(z) = \binom{n+z}{n} - \binom{n-d+z}{n}$

$\mathrm{Hilb}_{\mathbb{P}^n}^{\Phi_d} = \mathbb{P}^N \quad Z = \{ (f,p) \in \mathbb{P}^N \times \mathbb{P}^n \mid p \in V(f) \}$

(2) X = sm proj. var $\Phi = d$ Hilb_X^d
(3) X = curve, $\mathrm{Hilb}_X^d = \mathrm{Sym}^d(X) := X^d / S_d$

Pf of existence of Hilb_X^d

idea: Realize Hilb_X^d as subspace of some Grassmannian

Toy ex: fix ~~some~~ $Z \subset X$ length d subscheme $h^0(X, \mathcal{O}_Z) = d$

$\mathcal{L}$: ample line bundle on X

$0 \to I_Z \otimes \mathcal{L} \to \mathcal{O}_X \otimes \mathcal{L} \to \mathcal{O}_Z \otimes \mathcal{L}$

$H^0(\mathcal{L}) \twoheadrightarrow H^0(\mathcal{L}|_Z)$ after passing to higher dimension

$\rightsquigarrow [P_Z] \in Gr(d, H^0(\mathcal{L})^*)$

① Uniform positive lemma:

$\exists$ very ample line bundle $\mathcal{L}$ on X s.t.

$H^0(\mathcal{L}) \to H^0(\mathcal{L}|_Z)$

$\forall Z \subset X$ of length $\leq d+1$

Pf: Choose some v.a. M

$\rightarrow$ WLOG, $X = \mathbb{P}^N$, $M = \mathcal{O}(1)$

$\mathcal{L} = \mathcal{O}(d+2)$ #

$X \xhookrightarrow{\mathcal{L}} \mathbb{P}(H^0(\mathcal{L})^*)$

$\begin{matrix} \cup \\ Z \end{matrix} \longmapsto (Z)$