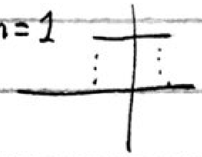
09 / 22 / 2015

Oscillatory Integrals:

Q: $B(1) \subset \mathbb{R}^n$ Indicator function χ_1

$$\hat{\chi}_1(\xi) \sim \cancel{(1+|\xi|)^{?}} \; |\xi|^{-?} \text{ as } \xi \to \infty$$

$n=1$  $\hat{f}(x) = \frac{\sin x}{x} \sim \frac{1}{x}$

for $n=1$, decays like $1/x$

$$\hat{\chi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$$

$$= \int_{B(1)} e^{-ix\cdot\xi}\,dx \qquad \xi = (\xi_1, \cdots, \xi_n)$$

$$\hat{\chi}(\xi_n) = \int_{B(1)} e^{-ix_n\xi_n}\,dx$$

$$= \int_{-1}^{1} e^{-ix_n\xi_n}\, \text{Vol}(\text{Slice at } x_n)\,dx_n$$

$$\sim \int_{-1}^{1} e^{-ix_n\xi_n} \left(\sqrt{1-x_n^2}\right)^{n-1} dx_n$$

$$x_n = \cos\theta$$

$$\sim \int_0^{\pi} e^{-i\xi_n\cos\theta} (\sin\theta)^n\,d\theta \Bigg\} \Rightarrow \underbrace{\int_0^{\infty} e^{-i\varphi(x)\lambda} f(x)\,dx}_{= I(\lambda)}$$

Aim: what happens as $\xi_n \to +\infty$

Principle: The only x which matter are where $\varphi'(x) = 0$ and $\varphi''(x) \neq 0$

Assume only one critical point x_0

$\varphi'(x_0) = 0$ $\qquad f(x) \sim \cancel{\quad}\,(x-x_0)^{\alpha}$ near x_0

$\varphi''(x_0) \neq 0$ $\qquad$ smooth compact support

then $I(\lambda) \sim \lambda^{-\frac{1}{2}-\frac{\alpha}{2}}$

Application: $x=0$ $\quad \sin x \sim x \quad \alpha=1$

$$\text{so } I(\lambda) \sim \lambda^{-1}$$

$$\Rightarrow n=1 \quad \hat{\chi}(\xi) \sim |\xi|^{-1}$$

$$n=k \quad \hat{\chi}(\xi) \sim |\xi|^{-\frac{k+1}{2}}$$

<u>Intuitive</u> assume $x_0=0$, $\varphi'(0)=0$, $\varphi''(0)\neq 0$

$\varphi(x)=c+x^2$

$f(x)=x^\alpha \cdot \psi(x)$ ← smooth, compact support

$\psi(0)=1$

$$\int_0^\infty e^{-i\lambda\varphi(x)} f(x)\,dx$$

$$\sim \int_0^\infty e^{-i\lambda(c+x^2)} x^\alpha \cdot \psi(x)\,dx$$

$$= e^{-i\lambda c}\int_0^\infty e^{-i\lambda x^2} x^\alpha \cdot \psi(x)\,dx$$

$$\sim \int_0^\infty e^{-i\lambda x^2} x^\alpha \cdot \psi(x)\,dx$$

$$\sim \int_0^\delta \cdots + \int_\delta^\infty \cdots$$

$$\alpha=0 \quad \left|\int_0^\infty\right| \leq \left|\int_0^\delta\right| + \left|\int_\delta^\infty\right|$$

$$\leq C\cdot\delta + |\square|$$

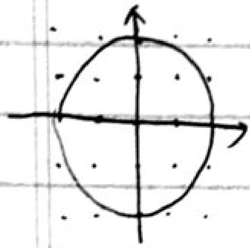
$$\left|\int_\delta^\infty -\frac{\psi(x)}{2i\lambda x}\frac{d}{dx}\left(e^{-i\lambda x^2}\right)dx\right|$$

$$\leq \left|\int e^{-i\lambda x^2}\left(\frac{\psi'(x)}{\lambda x}-\frac{\psi(x)}{\lambda x^2}\right)dx\right| + \left|e^{-i\lambda\delta^2}\frac{\psi(\delta)}{\lambda\delta}\right|$$

$$\leq \frac{C}{\lambda\delta} + \frac{1}{\lambda}\int_\delta^\infty\left(\left|\frac{1}{x}\right| + \left|\frac{1}{x^2}\right|\right)dx$$

$\vdots$

Gauss Circle problem



$B(R)$ $\quad N(R) = \#$ of lattice points in $B(R)$

Q: What is $N(R)$?

$N(R) \approx \pi R^2$

Q: $\left|N(R) - \pi R^2\right| \sim R^{?}$

Gauss: $\left|N(R) - \pi R^2\right| \leq 2\sqrt{2}\pi R$

Hardy-Littlewood $\left|N(R) - \pi R^2\right| \leq C R^{2/3}$

known: $R^{1/2}$ not possible

Conj: $R^{1/2+\varepsilon}$ works for all $\varepsilon > 0$

Idea: χ_R $\quad N(R) = \sum_{x \in \mathbb{Z}^2} \chi_R(x)$

$$N(\varepsilon, R) = \sum_{x \in \mathbb{Z}^2} \chi_{\varepsilon,R}(x)$$

(where $\chi_{\varepsilon,R}(x) = \chi_R * \varphi_\varepsilon$

φ bump function. $\int \varphi = 1$, $\varphi \geq 0$ $\operatorname{supp} \varphi \subseteq B(\frac{1}{2})$

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^2}\varphi\left(\frac{x}{\varepsilon}\right))$$

$$N(\varepsilon, R+\varepsilon) \geq N(R) \geq N(\varepsilon, R-\varepsilon)$$

$$N(\varepsilon, R) = \sum_{x \in \mathbb{Z}^n} \chi_{\varepsilon,R}(x)$$

Poisson Summation formula

$$= \sum_{\xi \in \mathbb{Z}^n} \hat{\chi}_{\varepsilon,R}(\xi)$$

$$= \hat{\chi}_{\varepsilon,R}(0) + \sum_{\substack{\xi \in \mathbb{Z}^2 \\ \xi \neq 0}} \hat{\chi}_{\varepsilon,R}(\xi)$$

$$= \pi R^2 +$$

$$\chi_{\varepsilon,R} = \chi_R * \varphi_\varepsilon$$

$$\hat{\chi}_{\varepsilon,R}(\xi) = (\hat{\chi}_R(\xi)) \cdot (\hat{\varphi}_\varepsilon)(\varepsilon, \xi)$$

$$\hat{\chi}_{\varepsilon,R}(\xi) = \hat{\chi}_R(\xi) \cdot \hat{\varphi}(\varepsilon \cdot \xi)$$

$$\cancel{\frac{1}{R^2}\hat{\chi}_1\left(\frac{\xi}{R}\right)}$$

$$(\hat{\chi}_{\varepsilon,R})(\xi) = R^2 \hat{\chi}_1(\xi R)\, \hat{\varphi}(\varepsilon \cdot \xi)$$

$$\text{Error} \leq \sum_{\substack{\xi \in \mathbb{Z}^2 \\ \xi \neq 0}} R^2 \hat{\chi}_1(R\xi)\, \hat{\varphi}(\varepsilon\xi)$$

$$\sim \int_{|\xi| \geq 1} R^2 \hat{\chi}_1(R\xi)\, \hat{\varphi}(\varepsilon\xi)\, d\xi$$

$$\hat{\chi}_1(\xi) \sim |\xi|^{-\frac{1}{2}-\frac{2}{2}} = |\xi|^{-\frac{3}{2}}$$

$$\sim \int_{\varepsilon \leq |\xi| \leq R} \text{——} + \int_{|\xi| \geq R} \text{——}$$