TIGHT CLOSURE

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Contents

1. Preliminiary

Throught out this note, all rings are assumed commutative, associative with identity.

1.1. **Base change.** Given a ring homomorphism $f : R \to S$, we have a base change functor $\mathcal{B} = S \otimes_R \_$ from R-modules to S-modules. It has following properties:

- Proerpties of the Functor
  - The functor $\mathcal{B}$ is right exact

  - The functor $\mathcal{B}$ commutes with arbitrary direct sums and with arbitrary direct limits
  - Given two homomorphisms $R \to S$ and $S \to T$, the base change functor $\mathcal{B}_{R\to T}$ is the composition of $\mathcal{B}_{R\to S}$ and $\mathcal{B}_{S\to T}$.
- Properties of Modules
  - The functor $\mathcal{B}$ takes R^n to S^n and free modules to free modules
  - The functor $\mathcal{B}$ takes projective R-modules to projective S-modules
  - The functor $\mathcal{B}$ takes flat R-modules to flat S-modules
  - The functor $\mathcal{B}$ takes faithfully flat R-modules to faithfully flat S-modules.
  - The functor $\mathcal{B}$ takes finitely generated R-modules to finitely generated S-modules.
- Properties of Elements
  - The functor $\mathcal{B}$ takes the cokernel of the matrix (r_{ij}) to the cokernel of the matrix $\big(f(r_{ij})\big)$
  - The functor $\mathcal{B}$ takes R/I to S/IS.
- If we denote the restriction-of-scalar functor as $\mathcal{R}$, then there is a natural transformation from the identity functor to $\mathcal{R} \circ \mathcal{B}$, i.e. for any R-module M, there is a natural R-linear map:

$$\begin{aligned} M &\to S \otimes_R M \\ ru &\mapsto f(r) \otimes u \end{aligned}$$

1.2. **Characteristic of a ring.** The characteristic of a ring is the smallest integer such that 1 times it gets 0. If we have a homomorphism $R \to S$, then the characteristic of S must divides the characteristic of R. A quick example is that $\mathbb{Z}/n\mathbb{Z}$ has characteristic n. So the characteristic of a ring doesn't need to be prime.

If the ring is assumed to be without zerodivisors, then its characteristic must be either 0 or a prime number. This situation applies to fields, integral domains and division rings.

1.3. **Frobenius functor.** Suppose R has characteristic p, the Frobenius map

$$\begin{aligned} F : R &\to R \\ r &\mapsto r^p \end{aligned}$$

induces a base change functor $\mathcal{F}$ from R-modules to R-modules. It certainly has all the properties listed above. We may also consider the e-fold iterated composition of this functor with itself, which we denote $\mathcal{F}^e$. This is the same functor induced by the e-fold iterated composition of the Frobenius map F^e.

Let $q = p^e$, then $\mathcal{F}^e$ takes cokernel of (r_{ij}) to the cokernel of (r_{ij}^q). And $\mathcal{F}^e$ takes R/I to $R/I^{[q]}$ where

$$I^{[q]} = \{a^q | a \in I\}R$$

(It's different from I^q, where we take all products of q elements, here we only allow the qth power).

For every module M, the natural map is

$$\begin{aligned} M &\to \mathcal{F}^e(M) \\ ru &\mapsto 1 \otimes (ru) = r^q \otimes u \end{aligned}$$

If we write $1 \otimes u$ as u^q, then the map takes ru to $r^q u^q$, which matches the form of a Forbenius map in the ring case.

Given $N \subseteq M$, the map $\mathcal{F}^e(N) \to \mathcal{F}^e(M)$ is not necessarily injective, we denote the image of this map by $N^{[q]}$ or $N_M^{[q]}$. It's quite easy to see that $N^{[q]}$ is spanned by u^q where $u \in N$ in $\mathcal{F}^e(M)$. When $N = I$ an ideal of $M = R$, the map takes $a \in I$ to $1 \otimes a = a^q \otimes 1$ in R. So it takes I to $I^{[q]}$.

Remark 1.1. The module $N^{[q]}$ is a submodule of $\mathcal{F}^e(M)$, not of M itself.

2. Tight Closure

2.1. **Definition.** Let R be a Noetherian ring of prime characteristic $p > 0$.

Definition 2.1 (Tight Closure for Modules)**.** Given two R-modules $N \subseteq M$, the **tight closure** N^*_M of N in M is the set of elements $u \in M$ such that there is some $c \in R^\circ$ (depending on u)

$$cu^q \in N^{[q]}_M$$

for all large enough q.

Remark 2.2. Here R° is the set of elements not contained in any minimal prime of R, i.e. $R^\circ = R - \cup_{P \text{ minimal}} P$

An ideal I of R is also a submodule of R, therefore take N to be I and M to be R, we obatin the notion of tight closure I^* of an ideal I.

2.2. **Basic properties.** Let R be a Noetherian ring of prime characteristic $p > 0$ and let M, N, Q be R-modules.

Proposition 2.3. *N^*_M is an R-module.*

Proof. For any $u \in N^*_M$ and $r \in R$, we need to show that $ru \in N^*_M$. We know that there is some $c \in R^\circ$ such that $cu^q \in N^{[q]}_M$ for $q >> 0$. But then $c(ru)^q \in N^{[q]}_M$ for $q >> 0$. Therefore $ru \in N^*_M$. □

Proposition 2.4. *If $N \subseteq M \subseteq Q$, then*

(1) $N^*_Q \subseteq M^*_Q$
(2) $N^*_M \subseteq N^*_Q$

Proof. The first statement is true because $N^{[q]}_Q \subseteq M^{[q]}_Q$ for all q. The second statement is true because the map $\mathcal{F}^e(M) \to \mathcal{F}^e(Q)$ takes $N^{[q]}_M$ into $N^{[q]}_Q$. □

Proposition 2.5. *If I is an ideal of R, then $I^* N^*_M \subseteq (IN)^*_M$*

Proof. Only need to show that every product au where $a \in I^*$ and $u \in N^*_M$ is in the right hand side. There is some $c_a \in R^\circ$ such that $c_a a^q \in I^{[q]}$ for all $q >> 0$ and some $c_u \in R^\circ$ such that $c_u u^q \in N^{[q]}_M$ for all $q >> 0$. Notice that $I^{[q]} N^{[q]} = (IN)^{[q]}$ for any q(both sides are generated by qth power of products). so we have $c_a c_u (au)^q \in (IN)^{[q]}_M$ for all $q >> 0$. Therefore $au \in (IN)^*_M$. □

Tight closure behaves well under arbitrary direct sum and finite product.

Proposition 2.6. *If $N_\lambda \subseteq M_\lambda$ is any family of inclusions, let $N = \oplus_\lambda N_\lambda$ and $M = \oplus_\lambda M_\lambda$, then $N^*_M = \oplus_\lambda (N_\lambda)^*_{M_\lambda}$.*

Proof. Any element in N^*_M is a sum of finitely many elements in those M_λ, therefore when considering a particular element, we could pass to the finite case. Since $N^{[q]}_M = \oplus_\lambda (N_\lambda)^{[q]}_{M_\lambda}$ as direct sum commutes with tensor product, the result is now clear. □

Proposition 2.7. *If $R = R_1 \times \cdots \times R_n$ and $N_i \subseteq M_i$ are R_i-modules ($1 \leq i \leq n$). Let $M = M_1 \times \cdots \times M_n$ and $N = N_1 \times \cdots \times N_n$, then $N^*_M = (N_1)^*_{M_1} \times \cdots \times (N_n)^*_{M_n}$.*

Proof. Notice that $R^\circ = R^\circ_1 \times \cdots \times R^\circ_n$. The result is clear. □

2.3. **Behaviour under R-linear map.** Next we want to discuss the behaviour of tight closure under R-linear maps.

Proposition 2.8. *If $N \subseteq M$ and $U \subseteq V$ are R-modues and $f : M \to V$ is an R-linear map such that $f(N) \subseteq U$, then $f(N^*_M) \subseteq U^*_V$.*

Proof. Let $u \in N^*_M$, then there is an element $c \in R^\circ$ such that $cu^q \in N^{[q]}_M$ for all $q >> 0$. But $f(N^{[q]}_M) \subseteq U^{[q]}_V$, so $c(f(u)^q) \in U^{[q]}_V$ for all $q >> 0$, therefore $f(u) \in U^*_V$. □

Proposition 2.9. *An element $u \in N_M^*$ if and only if the image $\bar{u} \in 0^*_{M/N}$. Hence if G maps into M and let H be the preimage of N. Let v be a preimage of u, then $u \in N_M^*$ if and only if $v \in H_G^*$.*

Proof. By the right exactness of tensor products, $\mathcal{F}^e(M/N) \cong \mathcal{F}^e(M)/N^{[q]}$. Therefore an elment $cu^q \in N^{[q]}$ iff $c\bar{u}^q = 0$ in $\mathcal{F}^e(M/N)$. □

Remark 2.10. By the proposition above (PROP 2.9), we could always map a free module G onto M and then take the preimage H of N, therefore the problem of N_M^* becomes a problem of H_M^*. If G is of rank n, we could choose a free basis and identify G with R^n. If we write $u = r_1 \oplus \cdots \oplus r_n \in H$, then $H_G^{[q]}$ is generated by $u^q = r_1^q \oplus \cdots \oplus r_n^q$. So we could explicitly define the tight closure like the ideal case. However we have to prove that this definition is independent of those choices we made.

2.4. **Behaviour under base change.** Suppose R and S are two Noetherian rings with prime characteristic $p > 0$. Let $f : R \to S$ be a ring homomorphism. We want to know what happens under the base change functor $S \otimes_R \_$. We have following proposition:

Proposition 2.11. *Suppose $f : R \to S$ maps R° into S°, then the image of $S \otimes_R N_M^*$ is contained in the tight closure of $S \otimes_R N$ in $S \otimes_R M$.*

Before giving the proof, notice that the Frobenius map commutes with any ring homomorphism, i.e. $f \circ F_R^e = F_S^e \circ f$, which is saying $f(r^q) = (f(r))^q$. Therefore we have following observation:

Observation 2.12. For every R-module M, we have a natural isomorphism $S \otimes_R \mathcal{F}^e(M) \cong \mathcal{F}^e(S \otimes_R M)$.

Proof. We have to show that for any $u \in N_M^*$, we have $1 \otimes u \in (S \otimes_R N)^*$. First of all we know that there is an element $c \in R^\circ$ such that $cu^q \in N_M^{[q]}$ for all $q >> 0$. So $f(c) \otimes u^q \in S \otimes_R N_M^{[q]} \subseteq S \otimes_R \mathcal{F}^e(M) \cong \mathcal{F}^e(S \otimes_R M)$ and clearly $f(c) \otimes u^q = f(c)(1 \otimes u)^q$. So $1 \otimes u \in (S \otimes_R N)^*$. □

Remark 2.13. The condition "R° maps into S°" holds when

(1) $R \subseteq S$ are domains
(2) $R \to S$ is flat
(3) $S = R/P$ where P is a minimal ideal of R

Here both (1) and (3) are quite clear. For (2), note that for any minimal prime Q of S, the contraction $Q^c = P$ is a prime ideal of R and we have a map $R_P \to S_Q$. This is a local flat map of local rings, therefore faithfully flat. Then it's injection.(See [faithfully flat notes]) Since Q is minimal, QS_Q is nilpotent, which implies that PR_P is nilpotent. So P is minimal in R. Therefore R° maps into S°.

2.5. **The main theorem.** Tight closure can be checked modulo every minimal prime of R.

Theorem 2.14. *Consider following settings:*

- *Let R be a Noetherian ring of prime characteristic $p > 0$ and $N \subseteq M$ are R-modules.*
- *Let $P_1, \cdots, P_n$ be minimal primes of R and let $D_i = R/P_i$.*
- *Let $M_i = M \otimes D_i = M/P_iM$ and let N_i be the image of $D_i \otimes N$ in M_i.*
- *Suppose $u \in M$ and let u_i be the image of u in M_i*

Then $u \in N_M^*$ over R if and only if for all $1 \leq i \leq n$, $u_i \in (N_i)^*_{M_i}$ over D_i.

Proof. The "only if" part follows from proposition 2.11 applying to the case $S = R/P_i$. It remains to prove the "if" part:

For every i there is some $c_i \in R - P_i$ such that for all $q >> 0$, we have $\bar{c}_i u_i^q \in N_i^{[q]}$. Here $N_i^{[q]}$ is the image of $\mathcal{F}^e(N_i)$ in $\mathcal{F}^e(M_i) = \mathcal{F}^e(M \otimes R/P_i)$. Since Frobenius functor commutes with tensor, we have

$$\mathcal{F}^e(M \otimes R/P_i) = R/P_i \otimes \mathcal{F}^e(M) = \mathcal{F}^e(M)/P_i\mathcal{F}^e(M)$$

So the image could be identified as the image of $N_M^{[q]}$. Thus

$$c_i u^q \in N^{[q]} + P_i \mathcal{F}^e(M)$$

Choose d_i in all P_j but not in P_i, and let J be the intersection of all P_i. Then J is nilpotent and

$$d_i c_i u^q \in N^{[q]} + J\mathcal{F}^e(M)$$

since every $d_i P_i \subseteq J$.

Let $c = \sum_i c_i d_i$, then c is not contained in any P_i. And we have

$$cu^q \in N^{[q]} + J\mathcal{F}^e(M)$$

for all $q >> 0$, say $q \geq q_0$. Choose q_1 such that $J^{[q_1]} = 0$. Then $c^{q_1} u^{qq_1} \in N^{[qq_1]}$ for all $q \geq q_0$. But then $c^{q_1} u^q \in N^{[q]}$ for all $q \geq q_0 q_1$. ☐

Remark 2.15. If $M = R$ and $N = I$ in the theorem above (THM 2.14), then an element $r \in I^*$ iff $\bar{u} \in (IR/P_i)^*$ for every i. So we could pass to the reduced ring $R_{\text{red}} = R/J$

Let J be the intersection of minimal primes $P_1, ..., P_n$ of R, then $(R/J)/(P_i/J) \cong R/P_i$ so we have the following corollary:

Corollary 2.16. *Let R be a Noetherian ring of prime characteristic $p > 0$ and J be the nilpotent ideal of R. Let $N \subseteq M$ be R-modules and let $u \in M$. Then $u \in N_M^*$ if and only if the image of u in M/JM is in the tight closure of the image of N/JN over R/J.*

2.6. Iteration of tight closure.

We have following theorem

Theorem 2.17. *Let R be a Noetherian ring of prime characteristic $p > 0$ and let $N \subseteq M$ be R-modules. Consider the following condition:*

There exists an element $c \in R^\circ$ and $q_0 = p^{e_0}$ such that for all $u \in N^*$, $cu^q \in N^{[q]}$ for all $q \geq q_0$.

If the above condition holds, then $(N_M^*)_M^* = N_M^*$.

Proof. If the condition holds, then for any $u \in (N^*)^*$, there is some $c \in R^\circ$ such that $cu^q \in (N^*)^{[q]}$ for all $q \geq q_0$. Therefore cu^q is in the R-span of w^q for $w \in N^*$. Since there is another c' such that $c'w^q \in N^{[q]}$ for all $w \in N^*$ and all $q \geq q_1$, multiply cu^q by c' we get $c'cu^q \in N^{[q]}$ for all $q \geq \max\{q_0, q_1\}$. Thus $u \in N^*$. ☐

It's quite easy to see that when N^*/N is finitely generated, the boxed condition is automatic. Furthurmore, if M is Noetherian, then both N^* and N are finitely generated, therefore boxed condition holds. So we have following corollary

Corollary 2.18. *Let R be a Noetherian ring of prime characteristic $p > 0$, and let $N \subseteq M$ be finitely generated modules. Then $(N_M^*)_M^* = N_M^*$.*

2.7. One more proposition.

Proposition 2.19. *Let R be a Noetherian ring of prime characteristic $p > 0$. Let $N \subseteq M$ be R-modules. If $u \in N_M^*$, then for any $q_0 = p^{e_0}$, we have $u^{q_0} \in (N^{[q_0]})^*_{\mathcal{F}^{e_0}(M)}$.*

Proof. This is immediate from the fact that $(N^{[q_0]})^{[q]} \subseteq \mathcal{F}^e(\mathcal{F}^{e_0}(M))$. ☐

3. Briançon-Skoda Theorem

3.1. Preliminary.

Proposition 3.1. *Let R be a Noetherian ring of prime characteristic $p > 0$. The tight closure of 0 is the ideal J of all nilpotent elements of R*

Proof. If $u \in 0^*$, then there is some $c \in R^\circ$ such that $cu^q = 0$ for all $q >> 0$. Since c is not in any minimal prime, we have that u^q is in J therefore u is in J. So $0^* \subseteq J$.

On the other hand, any element u in J satisfies $u^{q_0} = 0$ for some q_0, therefore for any $q \geq q_0$, we have $1 \times u^q = 0$. So $J \subseteq 0^*$. □

Proposition 3.2. *Let R be a Noetherian ring of prime characteristic $p > 0$. For every ideal I of R, we have $I^* \subseteq \overline{I} \subseteq \sqrt{I}$.*

Remark 3.3. Here $\overline{I}$ is the integral closure of I [See Integral-closure-of-an-ideal]

Proof. Suppose $u \in I^*$,to show $u \in \overline{I}$, it suffices to verify this modulo every minimal prime: So we pass to R/P hence we may assume that R is a domain, then we have some $c \neq 0$ such that $cu^q \in I^{[q]}$ for large enough q. This suffices to say that $u \in \overline{I}$.

If $u \in \overline{I}$, then u satisfies a monic polynomial

$$u^n + f_1 u^{n-1} + \cdots + f_n = 0$$

where $f_j \in I^j$. Thus $u^n \in I \Rightarrow u \in \sqrt{I}$. □

Then we have an obvious corollary:

Corollary 3.4. *Let R be a Noetherian ring of prime characteristic $p > 0$. Then any radical ideal, prime ideal or integrally closed ideal is tightly closed.*

3.2. **The theorem.** Now we are ready to prove this theorem

Theorem 3.5 (Briançon-Skoda)**.** *Let R be a Noetherian ring of prime characteristic $p > 0$. Let I be an ideal generated by n elements, then $\overline{I^n} \subseteq I^*$*

Proof. We can work modulo each minimal prime in turn so we assume that R is a domain. If $u \in \overline{I^n}$ then there exists $c \neq 0$ such that for all $k \geq 0$, $cu^k \in (I^n)^k = I^{nk}$. If we choose k to be $q = p^e$, then

$$u \in I^{nq} \subseteq I^{[q]}$$

and we're done. □

Corollary 3.6. *Let R be a Noetherian ring of prime characteristic $p > 0$. Let I be a principal ideal, then $\overline{I} = I^*$.*

3.3. **Application.**

Proposition 3.7. *Let R be a Noetherian domain of prime characteristic $p > 0$. If the ideal* (0) *and those pincipal ideals generated by NZDs are tightly closed, then R is normal.*

Proof. Suppose $\frac{f}{g} \in R$ is algebraic over R, then we have an equation

$$(f/g)^s + r_1(f/g)^{s-1} + \cdots + r_s = 0$$

with $r_j \in R$. Multiplying by g^s we obtain

$$f^s + r_1 f^{s-1} g + \cdots + r_s g^s = 0$$

So f is in the integral closure of gR, which is $(gR)^* = gR$. Then $f = gr \Rightarrow \frac{f}{g} = r$, so R is normal. □

Theorem 3.8 (Symbolic Power Theorem)**.** *Let P be a prime ideal of height h in a regular ring R of prime characteristic p. Then for every integer $n \geq 1$, we have $P^{(hn)} \subseteq P^n$*

To prove the theore, we need two preliminary results:

Lemma 3.9. *Let P be a prime ideal of height h in a regular ring R of prime characteristic p. Then*

(1) $P^{[q]}$ is primary to P

(2) $P^{(qh)} \subseteq P^{[q]}$

Proof. (1): Clearly $\sqrt{P^{[q]}} = P$, we only need to show that every $f \in R - P$ is an NZD on $R/P^{[q]}$. Since

$$0 \to R/P \to^{f} R/P$$

is exact, it stays exact after applying the Forbenius functor $\mathcal{F}$, hence

$$0 \to \mathcal{F}^e(R/P) \to^{f^q} \mathcal{F}^e(R/P)$$

so f^q is a NZD on $R/P^{[q]}$, hence is f.

(2): Suppose $u \in P^{(qh)}$. Make a base change to R_P, then the image of u is in $P^{qh}R_P$. Now PR_P is generated by h elements, so $P^{[q]}R_P \supseteq P^{(qh)}R_P$ hence $u \in P^{[q]}R_P$. Since $R - P$ elements are NZD on $R/P^{[q]}$, we see that $u \in P^{[q]}$. □

4. Colon-capturing

4.1. **Height of ideals.** First we need some results for Cohen-Macaulay rings

Proposition 4.1. *Let R be a Noetherian ring and let $x_1, ..., x_d$ generate a proper ideal I of height d. Then there exists elements $y_1, ..., y_d \in R$ such that for every i*

- $y_i \in x_i + (x_{i+1}, ..., x_d)R$
- $y_1, ..., y_i$ *generates an ideal of height i*

and moreover, $(y_1, ..., y_d)R = I$ and $y_d = x_d$.

If R is CM, then $y_1, ..., y_d$ is a regular sequence.

Proof. First we see that $x_1 + (x_2, ..., x_d)R$ is not contained in the union of all minimal primes of R by the coset form of prime avoidace lemma: otherwise $\text{ht}(I) = 0$. So we can choose $x_1 + \delta_1$ not in any minimal prime. Hence $y_1 = x_1 + \delta_1$ is a NZD in R. So $\text{ht}(y_1) = 1$ and $(y_1, x_2, ..., x_d)R = I$. Now we apply induction to R/y_1R. □

Proposition 4.2. *Let R be a Noetherian ring and let P be a minimal prime of R. Let $x_1, ..., x_d$ be elements of R such that $(x_1, ..., x_i)(R/P)$ has height i. Then there exists $\delta_i \in P$ such that let $y_i = x_i + \delta_i$, then $(y_1, ..., y_i)R$ has height i.*

Proof. We construct δ_i recursively: Let $\delta_1, ..., \delta_t$ be chosen. If $t < d$, we cannot have $x_{t+1} + P$ contained in the union of minimal primes of the ideal $(y_1, ..., y_t)R$. Otherwise by prime avoidance we have $x_{t+1} + P \subseteq Q$. Then on one hand $\text{ht}(Q) \leq t$. On the other hand, after modulo P, we get $(x_1, ..., x_{t+1})R/P \subseteq QR/P \Rightarrow \text{ht}(QR/P) \geq t + 1$, a contradiction! Then we can choose $\delta_{t+1} \in x_{t+1} + P$ not in any minimal prime of $(y_1, ..., y_t)R$. □

4.2. **The theorem.** First we have a lemma:

Lemma 4.3. *Let P be a minimal ideal of height n in a Cohen-Macaulay ring S. Let $x_1, ..., x_{k+1}$ be elements of $R = S/P$ such that $\text{ht}(x_1, ..., x_k)R = k$ in R while $\text{ht}(x_1, ..., x_{k+1})R = k + 1$. Then we can choose elements $y_1, ..., y_n \in P$ and $z_1, ..., z_{k+1} \in S$ such that:*

(1) $y_1, ..., y_n, z_1, ..., z_{k+1}$ *is a regular sequence in S*
(2) The image of $z_1, ..., z_k$ in R generates the ideal $(x_1, ..., x_k)R$.
(3) The image of z_{k+1} in R is x_{k+1}.

Proof. By PROP 4.1 we may assume WLOG that $x_1, ..., x_i$ generate an ideal of height i in R, $1 \leq i \leq k$. We also know this for $i = k + 1$.

Choose z_i arbitrarily such that z_i maps to x_i for $1 \leq i \leq k + 1$. Choose a regular sequence $y_1, ..., y_h$ of length h in P. Then P is minimal over $(y_1, ..., y_h)S$. By applying PROP 4.2 to the image of z_i in $S/(y_1, ..., y_h)S$ with minimal prime $P/(y_1, ..., y_h)$, we may alter the z_i by adding elements of P so that the height of the image of the ideal generated by the images of $z_1, ..., z_i$ in $S/(y_1, ..., y_h)S$ is i for $1 \leq i \leq k + 1$.

Since $S/(y_1, ..., y_h)S$ is again Cohen-Macaulay, it follows from PROP 4.1 that the images of $z_1, ..., z_{k+1}$ modulo $(y_1, ..., y_h)S$ form a regular sequence, which shows that $y_1, ..., y_h, z_1, ..., z_{k+1}$ form a regular sequence. □

We can now prove this colon-capturing property:

Theorem 4.4. *Let R be a reduced Noetherian ring of prime characteristic p. Assume that R is a homomorphic image of a C-M ring. Let $x_1, ..., x_{k+1}$ be elements of R and let $I_i = (x_1, ..., x_i)R$. Suppose that the image of I_k has height k modulo every minimal prime of R and the image of I_{k+1} has height $k+1$ modulo every minimal prime of R. Then:*

(1) $I_k :_R x_{k+1} \subseteq I_k^*$
(2) If R has a test element, then $I_k^* :_R x_{k+1} \subseteq I_k^*$

Proof. To prove the first statement, it sufficies to prove it modulo every minimal prime of R, hence we may assume that R is a domain and $R = S/P$ where S is C-M. By LEM 4.3 above we can choose a regular sequence $y_1, ..., y_h, z_1, ..., z_{k+1}$ such that $y_1, ..., y_h \in P$ where $h = \mathrm{ht}(P)$. We may also replace these x_i by the image of z_i.

Let $J = (y_1, ..., y_h)S$, then P is nilpotent over $J \Rightarrow$ there is some $c \in S - P$ such that $cP^{[q_0]} \subseteq J$.

Now suppose we have a relation

$$rx_{k+1} = r_1x_1 + \cdots + r_kx_k$$

in R. Then we can lift $r, r_1, ..., r_k$ to elements $s, s_1, ..., s_k \in S$ such that

$$sz_{k+1} = s_1z_1 + \cdots + s_kz_k + v$$

for some $v \in P$. Raise both side to q^{th} power and multiply by c to get

$$cs^qz_{k+1}^q = cs_1^qz_1^q + \cdots + cs_k^qz_k^q + cv^q$$

Notice that we have $cv^q \in (y_1, ..., y_h)$. Therefore

$$cs^qz_{k+1}^q \in (z_1^q, cdots, z_k^q, y_1, ..., y_h)S$$

But $y_1, ..., y_h, z_1^q, ..., z_{k+1}^q$ form a regular sequence in S, so

$$cs^q \in (z_1^q, cdots, z_k^q, y_1, ..., y_h)S$$

Let $\bar{c}$ be the image of c in R, then $\bar{c} \in R^\circ$. Modulo P we have

$$\bar{c}r^q \in (x_1, ..., x_k)^{[q]}$$

for all $q \geq q_0$. Hence $r \in (x_1, ..., x_k)^*$ in R. This completes the proof of the first part.

For the second part: Suppose R has a test element $d \in R^\circ$, that $r \in R$ and that $rx_{k+1} \in I_k^*$. Then there exists $c \in R^\circ$ such that $c(rx_{k+1})^q \in (I_k^*)^{[q]}$ for all $q >> 0$. Note that $(I_K^*)^{[q]} \subseteq (I_k^{[q]})^*$. So we have

$$c(rx_{k+1})^q \in (I_k^{[q]})^* \Rightarrow dcr^qx_{k+1}^q \in I_k^{[q]}$$

Now apply part 1 we get

$$dcr^q \in (I_k^{[q]})^* \Rightarrow d^2cr^q \in I_k^{[q]} \Rightarrow r \in I_k^*$$

□

Corollary 4.5. *Let R be a holomorphic image of a C-M ring and assume that R is weakly F-regular, then R is C-M.*

Proof. Consider R_m where m is a maximal ideal of R, it's still weakly F-regular, hence it's normal. So we may assume that (R, m) is local domain. Now choose a system of parameters and apply the colon capturing theorem. □

5. Relations to Other Closures

5.1. **Plus Closure.** We note following lemma:

Lemma 5.1. *Let D be a domain and let M be a finitely generated torsion-free module over D. Then*

- *M can be embedded in R^n where n is the torsion-free rank of M*
- *For any nonzero element $u \in M$, there is an R-linear map $\theta : M \to R$ such that $\theta(u) \neq 0$.*

Proof. We can choose n elements $v_1, ..., v_n$ of M that are linearly independent over Frac(R) and let $u_1, ..., u_n$ be a set of generators of M. Then each u_i is a Frac(D)-linear combination of v_i's. So we can clear the denominator and assume that $c_i u_i \in D^n$. Let $c = c_1 \cdots c_n$, then $cu_i \in D^n$. So $cM \subseteq D^n$, but $M \cong cM$.

If $u \neq 0$, then the image of u in D^n is not zero, i.e. some coordinate is nonzero. Let θ be the composition of $M \to D^n$ with the projection. □

Theorem 5.2. *Let R be a Noetherian ring and $R \subseteq S$ is an integral extension. Let $I \subseteq R$ be an ideal of R, then $IS \cap R \subseteq I^*$*

Proof. Let $r \in IS \cap R$, again we can work modulo every minimal prime P of R in turn. Let Q be the prime of S lying over P, then $R/P \hookrightarrow S/Q$ and the image of r in R/P is in IS/Q. We hence assume that R and S are domains.

Let $f_1, ..., f_h$ generate I so we can write

$$r = s_1 f_1 + \cdots + s_h f_h$$

where $s_i \in S$. Now we can replace S by $R[f_1, ..., f_h]$ and assume that S is module-finite over R. By the lemma above (LEM 5.1) we know that S is a solid algebra, now the result is easy to prove. □