TEST ELEMETS

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Contents

1. Test Elements

1.1. Definition.

Definition 1.1. Let R be a Noetherian ring of prime characteristic $p > 0$.

- An element $c \in R^\circ$ is called a **test element** for R if for every inclusion of finitely generated modules $N \subseteq M$ and every $u \in M$, we have: $u \in N^*_M$ if and only if $cu^q \in N^{[q]}_M$ for every $q = p^e \geq 1$
- An element $c \in R^\circ$ is called a **big test element** for R if the above assertion is true for any R-modules $N \subseteq M$.
- A (big) test element is called **locally stable** if it is a (big) test element in every localization of R.
- A (big) test element is called **completely stable** if it is a (big) test element in the completion of every local ring of R.

1.2. Basic properties.

Proposition 1.2. *Let R be a Noetherian ring of prime characteristic $p > 0$ and let $c \in R$. Then:*

- *c is a big test element for R if and only if $c \in R^\circ$ and $cN^*_M \subseteq N$ for any modules $N \subseteq M$.*
- *c is a test element for R if and only if $c \in R^\circ$ and $cN^*_M \subseteq N$ for any finitely generated modules $N \subseteq M$.*

Proof. The "only if" part comes from the definition by choosing q to be 1. For the "if" part, if $u \in N^*_M$, then $u^q \in (N^{[q]})^*_{\mathcal{F}^e(M)}$ for any q. But then $cu^q \in N^{[q]}$ for any q. □

Next proposition tells us that test elements behave well under faithfully flat morphisms:

Proposition 1.3. *Let R be a Noetherian ring of prime characteristic $p > 0$ and let $c \in R^\circ$. S is faithfully flat over R.*

(1) If c is a (big) test element for S, then it is a (big) test element for R
(2) If c is a completely stable (big) test element for S, then it is a completely stable (big) test element for R.

Proof. (1):Suppose $u \in N^*$, want to show that $cu \in N$. First we notice that

$$c(1 \otimes u) \in c(S \otimes N^*) \subseteq c((S \otimes N)^*) \subseteq S \otimes N$$

So $1 \otimes (cu) \in S \otimes N$. Consider the module $(N + Rcu)/N$, after tensor with S we get $(S \otimes N + S(1 \otimes cu))/S \otimes N$, this is zero since $1 \otimes cu \in S \otimes N$. Therefore $(N + Ru)/N = 0$ as S is faithfully flat, but then $u \in N$.

(2):It follows from (1), for any prime P of R, there is a prime ideal Q lying over P in S. Then $R_P \to S_Q$ is still faithfully flat, so is their completion $\widehat{R_P} \to \widehat{S_Q}$. Since c is a (big) test element for $\widehat{S_Q}$, it is a (big) test element in $\widehat{R_P}$. □

Now we can give following criterion: test element could be checked localizing at every maximal ideal.

Proposition 1.4. *Let R be a Noetherian ring of prime characteristic $p > 0$ and let $c \in R^\circ$. If c is a (big) test element in R_m for every maximal ideal m, then c is a (big) test element for R.*

Proof. If we have a counterexample, i.e. for some $N \subseteq M$, there is an element $u \in N^*$ such that $cu \notin N$. Then choose m to be the maximal ideal in the support of $(N + Ru)/N$, then pass to R_m the counterexample still holds (because $((N + Ru)/N)_m \neq 0$). □

We have following corollary:

Corollary 1.5. *Let R be a Noetherian ring of prime characteristic $p > 0$ and let $c \in R^\circ$. If c is a (big) test element in R_P for every prime ideal P, then c is a locally stable (big) test element for R.*

Proof. Since any localization at maximal ideals of $W^{-1}R$ is a localization of R at some prime ideal. The result follows from Proposition 1.4 □

Now we have a furthur corollary which reveals the connection between "completely stable" and "locally stable".

Corollary 1.6. *Let R be a Noetherian ring of prime characteristic $p > 0$ and let $c \in R^\circ$. If c is a completely stable (big) test element for R, then it is a locally stable (big) test element for R*

Proof. This is immediate from Proposition 1.3 and Corollary 1.5 and the fact that $R_P \to \widehat{R_P}$ is faithfully flat. □

2. Test Ideals

2.1. **Definition.**

Definition 2.1. Let R be a Noetherian ring of prime characteristic $p > 0$ and reduced. We define the **test ideal** $\tau(R)$ to be the set of elements $c \in R$ such that $cN^*_M \subseteq N$ for all finitely generated R-modules $N \subseteq M$.

Similarly we define $\tau_b(R)$ to be the set of elements $c \in R$ such that $cN^*_M \subseteq N$ for all R-modules $N \subseteq M$.

Alternatively, we could write

$$\tau(R) = \cap_{N \subseteq M \text{ f.g. modules}} N :_R N^*M = \cap_{N \subseteq M \text{ f.g. modules}} \operatorname{Ann}_R(N^*_M/N)$$

and

$$\tau_b(R) = \cap_{N \subseteq M} N :_R N^*M = \cap_{N \subseteq M} \operatorname{Ann}_R(N^*_M/N)$$

We immediate have following propositions:

Proposition 2.2. *Let R be a Noetherian ring of prime characteristic $p > 0$ and reduced.*

(1) $\tau_b(R) \subseteq \tau(R)$
(2) $\tau_b(R) \cap R^\circ$ *is the set of big test elements for R*
(3) $\tau(R) \cap R^\circ$ *is the set of test elements for R*

Proof. All are clear from the definition and properties of (big) test elements. ☐

Next we want to observe following

Proposition 2.3. *Let R be a Noetherian ring of prime characteristic $p > 0$ and reduced. If R has at least one (big) test element, then $\tau(R)$ ($\tau_b(R)$) is generated by all (big) test elements.*

This is immediate from following lemma

Lemma 2.4. *Let R be any ring and $P_1, ..., P_k$ finitely many prime ideals of R. Let $W = R - \cup_{i=1}^{k} P_i$. If an ideal $I \cap W \neq \emptyset$, then I is generated by $I \cap W$.*

Proof. Let J be the ideal generated by $I \cap W$, then $I \subseteq J \cup P_1 \cup \cdots \cup P_k$. By Prime avoidance we know that either $I \subseteq J$ or $I \subseteq P_i$ for some i. But if $I \subseteq P_i$, then $I \cap W = \emptyset$. Therefore $I \subseteq J \subseteq I$. ☐

3. Existence of Test Elements in F-finite and reduced case

Lemma 3.1. *Let R be a reduced F-finite ring and suppose that there exists an R-linear map $\theta : R^{1/p} \to R$ such that $\theta(1) = c \in R^{\circ}$. Then for every $q = p^e$, there exists an R-linear map $\eta_q : R^{1/q} \to R$ such that $\eta_q(1) = c^2$.*

Proof. If $q = 1$, take η_q to be c^2 times the identity map.

If $q = p$, take η_q to be $c\theta$.

Now suppose that η_q has been constructed, let $\eta_{pq}(u) = \theta\left(c^{(p-2)/p}\eta_q^{1/p}(u)\right)$. Then

$$\eta_{pq}(1) = \theta\left(c^{(p-2)/p}\eta_q^{1/p}(1)\right) = \theta\left(c^{(p-2)/p}c^{2/p}\right) = \theta(c) = c^2$$

☐

Theorem 3.2. *Let R be F-finite and reduced. If $c \in R^{\circ}$ and R_c is strongly F-regular, then c has a power that is a big test element*

If there exists an R-linear map $\theta : R^{1/p} \to R$ such that $\theta(1) = c$, then c^3 is a big test element.

Proof. Since R_c is strongly F-regular, it is F-split. We claim that there is a map $\theta : R^{1/p} \to R$ sending $1 \mapsto c^N$ for some N: First we can choose a split map $\beta : R_c^{1/p} \to R_c$ sending $1 \mapsto 1$. Since $\mathrm{Hom}_{R_c}(R_c^{1/p}, R_c) = (\mathrm{Hom}_R(R^{1/p}, R))_c$, we have $\beta = \frac{\alpha}{c^N}$ for some N. Now α is what we want.

By the second part, c^{3N} will be a big test element. So it sufficies to prove the second part and it sufficies to show that if $u \in N^*$, then $c^3 u \in N$.

The idea is as following: First we map a free module G onto M, let H be the inverse image of N and let v be a preimage of u. Then we show that $v \in H^* \Rightarrow c^3 v \in H$. Since $v \in H^*$, there is some $d \in R^{\circ}$ such that $dv^q \in H^{[q]}$ for all sufficiently large q, which, if we write out, is

$$dv^q = \sum_{i=1}^{n} r_i h_i^q \tag{3.1}$$

Now tensor G with $R^{1/q}$ and we can take q^{th} root of (3.1) and get

$$d^{1/q}v = \sum_{i=1}^{n} r_i^{1/q} h_i \tag{3.2}$$

Once we obtain a map $R^{1/q} \to R$ such that sending $d^{1/q}$ to c^3, apply that map to (3.2) and we are done.

Since R_c is strongly F-regular, we have q_d such that $R \to R^{1/q_d}$ sending $1 \mapsto d^{1/q_d}$ splits. Choose q_d large enough such that $dv^{q_d} \in H^{[q_d]}$. Let β be the split map, again we have $\beta = \frac{\alpha}{c^q}$. Hence $\alpha = c^q\beta$ and $\alpha(d^{1/q_d}) = c^q$.

Take q^{th} root of this we have $\alpha^{1/q} : R^{1/qq_d} \to R^{1/q}$ such that $d^{1/qq_d} \mapsto c$. Now by Lemma 3.1 we know that there exists a map $\theta : R^{1/q} \to R$ sending $1 \mapsto c^2$. So we have

$$\theta \circ \alpha^{1/q} : R^{1/qq_d} \to R^{1/q} \to R$$
$$d^{1/qq_d} \mapsto c \mapsto c^3$$

Now we are done. □

Note that F-finite rings are execellent and execellent rings has an element $c \in R^\circ$ such that R_c is regular, hence, strongly F-regular. See[E.Knuz *Characterizations of regular local rings of characteristic p*, Amer. J. Math. 91(1969) 772-784]. So we have proved

Corollary 3.3. *If R is reduced and F-finite, then R has a big test element.*

Actually we can prove a stronger statement, provided following theorem, see[F-regularity]

Theorem 3.4. *If $R \to S$ is geometrically regular map of F-finite rings and R is strongly F-regular, then so is S.*

Then we can prove following:

Theorem 3.5. *Let R be a reduced F-finite ring and let $c \in R^\circ$ such that R_c is strongly F-regular. Assume further that there is an R-linear map $R^{1/p} \to R$ that sends 1 to c. If S is F-finite and $R \to S$ is geometrically regular map, then c^3 is a big test element for S*

In particular, c^3 is a completely stable big test element.

Proof. Since the map is flat, we know that $c \in S^\circ$. We only need to show that S_c is strongly F-regular and the map $S \to S^{1/p}$ sending $1 \mapsto c$ splits.

Now since R_c is strongly F-regular and $R_c \to S_c$ is still geometrically regular, we know that S_c is strongly F-regular.

We have an R-linear split map $R^{1/p} \to R$. Tensor with S we get $R^{1/p} \otimes S \to S$. Compose it with the split map $S^{1/p} \to R^{1/p} \otimes S$ we get what we want.

Note $R \to R_P$ is geometrically regular. F-finite rings are excellent and the map from an excellent local ring to is completion is geometrically regular. □

4. Existence of Test Elements in Excellent Case

We aim to prove following result

Theorem 4.1. *Let R be a Noetherian ring. Suppose that R is a reduced algebra essentially of finite type over excellent semilocal ring B. Then there are elements $c \in R^\circ$ such that R_c is regular and any such c has a power that is a completely stable big test element.*

MORE DETAIL LATER.