Spectral Sequences

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Contents

1 General Spectral Sequence

1.1 Definition

Definition 1.1. *Let $\mathcal{A}$ be an abelian category. A* ***spectral sequence*** *in $\mathcal{A}$ is a collection of following datas for every $p, q \in \mathbb{Z}$ and $r \geq 0$.*

(1) An object $E_r^{p,q}$.

(2) A morphism $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$.

(3) If we set $Z_{r+1}(E_r^{p,q}) = \operatorname{Ker}\left(d_r^{p,q}\right)$ and $B_{r+1}(E_r^{p,q}) = \operatorname{Im}\left(d_r^{p-r,q+r-1}\right)$, then an isomorphism $\alpha_r^{p,q} : Z_{r+1}(E_r^{p,q})/B_{r+1}(E_r^{p,q}) \to E_{r+1}^{p,q}$.

Consider following diagram:

$$\begin{array}{ccc} E_r^{p,q} & \longrightarrow & E_r^{p,q}/B_{r+1}(E_r^{p,q}) \\ & & \uparrow \\ & & Z_{r+1}(E_r^{p,q})/B_{r+1}(E_r^{p,q}) \longrightarrow E_{r+1}^{p,q} \end{array}$$

For any $k > 0$, if $B_{r+k}(E_{r+1}^{p,q}) \subseteq Z_{r+k}(E_{r+1}^{p,q})$ are submodules of $E_{r+1}^{p,q}$, we can take the preimage of each in $E_r^{p,q}$, denote $B_{r+k}(E_r^{p,q})$, $Z_{r+k}(E_r^{p,q})$ respectively. Clearly we have a pullback:

$$\begin{array}{ccc} B_{r+k}(E_r^{p,q}) & \hookrightarrow & Z_{r+k}(E_r^{p,q}) \\ \downarrow & & \downarrow \\ B_{r+k}(E_{r+1}^{p,q}) & \hookrightarrow & Z_{r+k}(E_{r+1}^{p,q}) \end{array}$$

which induces an isomorphism on the cokernel of each horizontal map, i.e. $Z_{r+k}(E_r^{p,q})/B_{r+k}(E_r^{p,q}) \to Z_{r+k}(E_{r+1}^{p,q})/B_{r+k}(E_{r+1}^{p,q})$. This shows that we have canonical isomprohisms

$$Z_{r+k}(E_r^{p,q})/B_{r+k}(E_r^{p,q}) \to E_{r+k}^{p,q}$$

The picture is that we have inclusion of submodules

$$0 \subseteq B_r(E_r^{p,q}) \subseteq B_{r+1}(E_r^{p,q}) \subseteq B_{r+2}(E_r^{p,q}) \subseteq \cdots$$
$$\cdots \subseteq Z_{r+2}(E_r^{p,q}) \subseteq Z_{r+1}(E_r^{p,q}) \subseteq Z_r(E_r^{p,q}) \subseteq E_r^{p,q}$$

where the quotient $(E_{r+k}^{p,q})$ becomes smaller and smaller.

By saying that a spectral sequence convergent to $\{E^n\}_{n\in\mathbb{Z}}$, we mean a filtration $F^p(E^n)$ such that $E_\infty^{p,q} \cong F^p(E^{p+q})/F^{p+1}(E^{p+q})$

A spectral sequence is degenerate on page $r > a$ if for every p, q the morphism $d_r^{p,q} = 0$, follows from which we have $E_r^{p,q} \cong E_{r+1}^{p,q} \cong \cdots \cong E_\infty^{p,q}$.

1.2 Filtration

Proposition 1.2. *Let $\mathcal{A}$ be an abelian category and C a complex in $\mathcal{A}$ with a decreasing filtration $\{F^p(C)\}_{p\in\mathbb{Z}}$. Then there is a canonical spectral sequence $(E_r^{p,q}, E^n)$ starting on page zero, with*

$$E_0^{p,q} = (F^pC)^{p+q}/(F^{p+1}C)^{p+q}$$
$$E^n = H^n(C)$$

In other words, $E_r^{p,q} \Rightarrow H^{p+q}(C)$.

1.3 Double Complex

Let $C^{i,j}$ be a double complex and let d_I and d_{II} be the vertical and horizontal differential respectively,

$$\begin{array}{ccccccc} & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots \to & & C^{i+1,j-1} & \to & C^{i+1,j} & \to & C^{i+1,j+1} & & \cdots \to \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots \to & & C^{i,j-1} & \to & C^{i,j} & \to & C^{i,j+1} & & \cdots \to \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots \to & & C^{i-1,j-1} & \to & C^{i-1,j} & \to & C^{i-1,j+1} & & \cdots \to \\ & & \uparrow & & \uparrow & & \uparrow & & \end{array}$$

The we can take cohomology with respect to d_I and get $H_I^{i,j}(C)$, which is a double complex with zero vertical maps. We can continue to take cohomology of $H_I^{i,j}(C)$ with respect to the induced map of d_{II}. Then we end up with $H_{II}(H_I(C))$.

We can also take the total complex $T(C)^n = \oplus_{i+j=n} C^{i,j}$ and take the cohomology of that.

Consider a filtration of the total complex $T(C)$ by $F_I^p(T(C))^n = \oplus_{r \geq p} C^{r,n-r}$. Then the zeroth page $E_0^{i,j}$ is canonically $C^{i,j}$ and we have $E_2^{i,j} = H_I^{i,j}(H_{II}(C))$. And we know this converges to $H^n(T(C))$.

Similarly, we have $E_2^{i,j} = H_{II}^{j,i}(H_I(C))$ converges to $H^n(T(C))$ because if we switch i, j, the total complex doesn't change. Therefore we have following proposition

Proposition 1.3. *Let $C^{\bullet,\bullet}$ be a double complex, then we have two spectral sequences converge to $H^n(T(C^{\bullet,\bullet}))$:*

$$E_2^{p,q} = H_{II}^{p,q}(H_I(C^{\bullet,\bullet})) \Rightarrow H^n(T(C^{\bullet,\bullet}))$$
$$E_2^{q,p} = H_I^{q,p}(H_{II}(C^{\bullet,\bullet})) \Rightarrow H^n(T(C^{\bullet,\bullet}))$$

where $n = p + q$.

2 Grothendieck Spectral Sequence

2.1 Cartan-Eilenberg resolution

Let $\mathcal{A}$ be an abelian category with enough injectives, and C a complex in $\mathcal{A}$. An injective resolution of C is a commutative diagram

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \to & I^{1,n-1} & \longrightarrow & I^{1,n} & \longrightarrow & I^{1,n+1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \to & I^{0,n-1} & \longrightarrow & I^{0,n} & \longrightarrow & I^{0,n+1} & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \to & C^{n-1} & \longrightarrow & C^{n} & \longrightarrow & C^{n+1} & \to & \cdots
\end{array}$$

For each n we have complexes:

$$0 \to Z^n(C) \to Z^n(I^{0,\bullet}) \to Z^n(I^{1,\bullet}) \to \cdots$$
$$0 \to B^n(C) \to B^n(I^{0,\bullet}) \to B^n(I^{1,\bullet}) \to \cdots$$
$$0 \to H^n(C) \to H^n(I^{0,\bullet}) \to H^n(I^{1,\bullet}) \to \cdots$$

The resolution is fully injective if all complexes above are injective resolutions.

If $\mathcal{A}$ has enough injectives, then every complex C in $\mathcal{A}$ has a fully injective resolution. This is obtained by choosing injective resolutions of $H^n(C)$ and $B^n(C)$, then extend to injective resolutions of $Z^n(C)$ and C^n by following exact sequences:

$$0 \to Z^n(C) \to C^n \to B^{n+1}(C) \to 0$$
$$0 \to B^n(C) \to Z^n(C) \to H^n(C) \to 0$$

This is called Horseshoe lemma. The resolution obtained from this way is called a Cartan-Eilenberg resolution.

2.2 Grothendieck spectral sequence

Theorem 2.1. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be additive functors between abelian categories where $\mathcal{A}, \mathcal{B}$ have enough injectives and $\mathcal{C}$ is complete. Suppose F sends injectives to G-acyclics. Then for any object $A \in \mathcal{A}$ there is a spectral sequence starting on page zero such that*

$$E_2^{p,q} = R^pG(R^qF(A)) \Rightarrow R^{p+q}(GF)(A)$$

Proof. Let A be an object in $\mathcal{A}$, let $A \to J_0 \to J_1 \to \cdots$ be an injective resolution of A, and let $I^{p,q}$ be a Cartan-Eilenberg resolution of $F(J^\bullet)$ in $\mathcal{B}$. Consider the complex $G(I^{\bullet,\bullet})$. We know that there are two spectral sequences with E_2 pages $H_{II}(H_I(G(I^{\bullet,\bullet})))$ and $H_I(H_{II}(G(I^{\bullet,\bullet})))$ respectively, converge to E_∞.

Look at $H_I(G(I^{\bullet,\bullet}))$, it is $R^pG(F(J^\bullet))$. But $J^\bullet$ is injective so $F(J^\bullet)$ is G-acyclic. Therefore all are zero but $p = 0$, hence the only survive term is $GF(J^\bullet)$. Once take H_{II}, we have $E_2^{p,q} = R^q(GF)(A)$. Therefore $E_\infty^n = R^n(GF)(A)$.

Now look at $H_{II}(G(I^{\bullet,\bullet}))$, notice that we have a fully injective resolution, so

$$\begin{aligned}
&0 \to Z^p(FC) \to Z^{p,0} \to Z^{p,1} \to \cdots \\
&0 \to B^p(FC) \to B^{p,0} \to B^{p,1} \to \cdots \\
&0 \to H^p(FC) \to H_{II}^{p,0}(I^{\bullet,\bullet}) \to H_{II}^{p,1}(I^{\bullet,\bullet}) \to \cdots
\end{aligned}$$

and exact sequences

$$\begin{aligned}
&0 \to Z^{p,q} \to I^{p,q} \to B^{p+1,q} \to 0 \\
&0 \to B^{p,q} \to Z^{p,q} \to H^{p,q} \to 0
\end{aligned}$$

which all split because everything is sight splits. So it remains split after applying G, therefore we have $G(H_{II}(I^{\bullet,\bullet})) = H_{II}(G(I^{\bullet,\bullet}))$. But $H_{II}(I^{\bullet,\bullet})$ is an injective resoltion of $H(F(J^\bullet)) = R^qF(A)$. So after taking cohomology H_I we have $R^pG(R^qF(A))$. Now we're done. □

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\begin{array}{ccccccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
E_0^{5,0} & \longrightarrow & E_0^{5,1} & \longrightarrow & E_0^{5,2} & \longrightarrow & E_0^{5,3} & \longrightarrow & E_0^{5,4} & \longrightarrow & E_0^{5,5} & \longrightarrow & \cdots \\
E_0^{4,0} & \longrightarrow & E_0^{4,1} & \longrightarrow & E_0^{4,2} & \longrightarrow & E_0^{4,3} & \longrightarrow & E_0^{4,4} & \longrightarrow & E_0^{4,5} & \longrightarrow & \cdots \\
E_0^{3,0} & \longrightarrow & E_0^{3,1} & \longrightarrow & E_0^{3,2} & \longrightarrow & E_0^{3,3} & \longrightarrow & E_0^{3,4} & \longrightarrow & E_0^{3,5} & \longrightarrow & \cdots \\
E_0^{2,0} & \longrightarrow & E_0^{2,1} & \longrightarrow & E_0^{2,2} & \longrightarrow & E_0^{2,3} & \longrightarrow & E_0^{2,4} & \longrightarrow & E_0^{2,5} & \longrightarrow & \cdots \\
E_0^{1,0} & \longrightarrow & E_0^{1,1} & \longrightarrow & E_0^{1,2} & \longrightarrow & E_0^{1,3} & \longrightarrow & E_0^{1,4} & \longrightarrow & E_0^{1,5} & \longrightarrow & \cdots \\
E_0^{0,0} & \longrightarrow & E_0^{0,1} & \longrightarrow & E_0^{0,2} & \longrightarrow & E_0^{0,3} & \longrightarrow & E_0^{0,4} & \longrightarrow & E_0^{0,5} & \longrightarrow & \cdots
\end{array}
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\begin{array}{ccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
E_1^{5,0} & E_1^{5,1} & E_1^{5,2} & E_1^{5,3} & E_1^{5,4} & E_1^{5,5} & \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
E_1^{4,0} & E_1^{4,1} & E_1^{4,2} & E_1^{4,3} & E_1^{4,4} & E_1^{4,5} & \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
E_1^{3,0} & E_1^{3,1} & E_1^{3,2} & E_1^{3,3} & E_1^{3,4} & E_1^{3,5} & \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
E_1^{2,0} & E_1^{2,1} & E_1^{2,2} & E_1^{2,3} & E_1^{2,4} & E_1^{2,5} & \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
E_1^{1,0} & E_1^{1,1} & E_1^{1,2} & E_1^{1,3} & E_1^{1,4} & E_1^{1,5} & \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\
E_1^{0,0} & E_1^{0,1} & E_1^{0,2} & E_1^{0,3} & E_1^{0,4} & E_1^{0,5} & \cdots
\end{array}
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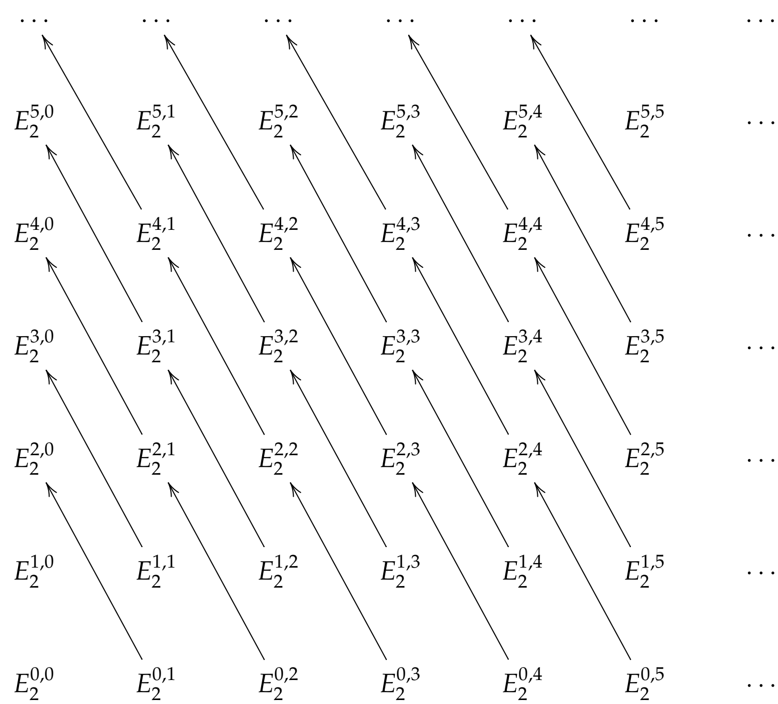
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$E_2^{3,0}$ $E_2^{3,1}$ $E_2^{3,2}$ $E_2^{3,3}$ $E_2^{3,4}$ $E_2^{3,5}$ . . .
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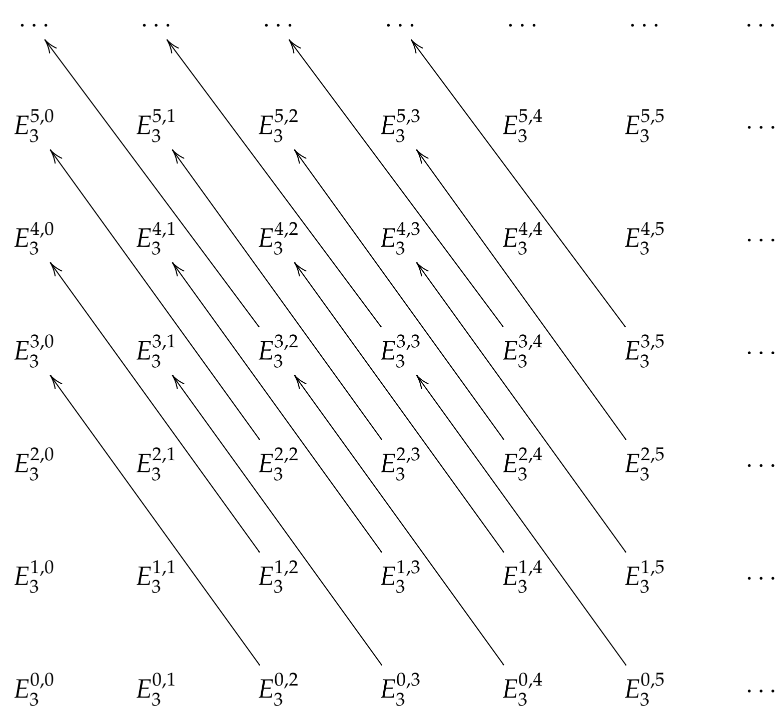
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