REGULAR RINGS

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1. Definition

Definition 1.1. A local ring (R, m, k) is regular if its embedded dimension $\dim_k m/m^2$ is equal to its Krull dimension.

By Nakayama's lemma, we know that $\dim_K m/m^2$ is also the minimal number of generators of m, hence a ring is regular iff a minimal set of generators is also a system of parameters.

If R is local and $f_1, ..., f_h$ is part of a system of parameters. Then $R/(f_1, ..., f_h)$ is called **local complete intersection**.

2. Projective Resolution

Theorem 2.1 (Auslander-Buchsbaum-Serre)**.** *Let (R, m, k) be a local ring, then TFAE:*

(1) R is regular
(2) k has finite free resolution over R
(3) Every finitely generated R-module has a finite free resolution

3. Regular rings are UFDs

3.1. Regular rings are domains.

Lemma 3.1. *Let (R, m, k) be a local ring. If $\mathrm{gr}_m(R)$ is a domain, then so is R.*

3.2. **Greothedieck Groups.** Let R be a Noetherian ring and let $\mathcal{M}$ denote the set of modules

$$\{R^n/M | n \in \mathbb{N}, M \subseteq R^n \text{ submodule}\}$$

It's clear that all finitely generated R-modules are in $\mathcal{M}$. Let $G_0(R)$ be the abelian group generated by basis in $\mathcal{M}$ and killing the relations: $[M] = [M'] + [M'']$ if we have short exact sequence:

$$0 \to M' \to M \to M'' \to 0$$

This is called the Grothendieck group of R.

Example 3.2. If $R = k$ is a field, then all R-modules are just k vector spaces, so $G_0(k)$ is generated by $[k]$ and isomorphic to $\mathbb{Z}$.

Theorem 3.3. *If R is a regular local ring, then $G_0(R) = \mathbb{Z}[R] \cong \mathbb{Z}$*

Proof. Since every finitely generated R-module has a finite free resolution, we're done. □

For general Noetherian rings we have following description:

Theorem 3.4. *Let R be a Noetherian ring, then*

(1) $G_0(R)$ is generated by the elements $[R/P]$ where P runs through all prime ideals of R.
(2) If P is prime in R and $x \in R - P$, then $[R/(P + xR)] = 0$
(3) All relations in $G_0(R)$ are generated as following: If $R/Q_1, ..., R/Q_k$ are all the factors in a prime filtration of $[R/(P + xR)]$, then $[R/Q_1] + \cdots + [R/Q_k] = 0$.

Proof. (1) follows from all finitely generated R-modules has prime filtration

(2) follows from the short exact sequence:

$$0 \to R/P \xrightarrow{\cdot x} R/P \to R/(P + xR) \to 0$$

(3) Since any relation from $0 \to M' \to M \to M'' \to 0$ is essentially the relation between two prime filtrations of M, we can only consider different prime filtrations of M. Any two different prime filtrations have a common refinement. So the relations breaks into the prime filtration of R/P.

Since R/P has only one associated prime P, any prime filtration of R/P must start with a submodule isomorphic to R/P, i.e. xR/P for some $x \in R$. But then the relation we get is what we described in the theorem. □

We immediately have following corollary:

Corollary 3.5. $G_0(R) \cong G_0(R_{\text{red}})$

Proof. As R and R_{red} has 1-1 correspondence in prime ideals and the same quotients. □

Proposition 3.6. *If R and S are Noetherian rings, then $G_0(R \times S) \cong G_0(R) \times G_0(S)$.*

Proof. Let $e = (1, 0)$ and $f = (0, 1)$, then any $R \times S$ module M factors as $M \cong eM \times fM$. □

Proposition 3.7. *Let R be an Artin ring, then*

(1) If (R, m, k) is local, then $G_0(R) \cong \mathbb{Z}[k] \cong \mathbb{Z}$ where the length map $M \mapsto \ell_R(M)$ gives the isomorphism
(2) If R has maximal ideals $m_1, ..., m_k$, then $G_0(R)$ is the free abelian group generated by $[R/m_j]$.

Proof. (1) comes from the description of $G_0(R)$ when R is Noetherian. (2) comes from (1) and previous proposition. □

Proposition 3.8. *Let R and S be Noetherian rings*

(1) If $R \to S$ is a flat homomorphism, there is a group homomorphism $G_0(R) \to G_0(S)$ sending $[M]_R \mapsto [S \otimes_R M]_S$. Thus G_0 is a covariant functor from the category of rings and flat homomorphisms to abelian groups.
(2) If $S = W^{-1}R$ is a localization, the map described in (1) is surjective
(3) If P is a minimal prime of R, there is a homomorphism $G_0(R) \to \mathbb{Z}$ given by $[M] \mapsto \ell_{R_P}(M_P)$. If R is a domain, then then $P = 0$ and this is the torsion-free rank map.
(4) If R is a domain, then the map sending $1 \mapsto [R]$ is split by the torsion-free rank map. Thus $G_0(R) = \mathbb{Z}[R] + \bar{G}_0(R)$ where $\bar{G}_0(R)$ is the reduced Grothendieck group.
(5) If S is module-finite over R, then there is a group homomorphism $G_0(S) \to G_0(R)$ sending $[M]_S$ to $[_RM]_R$. Hence G_0 is a contravariant functor from the category of rings and module-finite homomorphisms to abelian groups.

Proof. (1) is obvious as $S \otimes_R -$ preserves exactness. (5) is obvious as restriction of scalars is always exact. We need S module-finite over R for finitely generated S-modules to be finitely R-generated.

(2) is an exercise while the rest is obvious. □

Proposition 3.9. *If S is a Noetherian R-algebra of finite* Tor*-dimension $\leq d$, then there is a map $G_0(R) \to G_0(S)$ that sends $[M]_R$ to*

$$\theta(M) = \sum_{i=0}^{d} (-1)^i [\operatorname{Tor}_i^R(S, M)]_S$$

Proof. Use the long exact sequence of Tor. □

Corollary 3.10. *If x is an NZD in the Noetherian ring R, then there is a map $G_0(R) \to G_0(R/xR)$ that sends $[M]_R \mapsto [M/xM]_{R/xR} - [\operatorname{Ann}_M x]_{R/xR}$.*

Proof. Apply proposition above and note that $d = 1$. □

Corollary 3.11. *Let R be Noetherian and let S be either $R[x]$ or $R[\![x]\!]$, then S is a flat over R so we have an induced map $G_0(R) \to G_0(S)$. This map is injective.*

Proof. Using corollary above we have a map $\theta : G_0(S) \to G_0(R)$, it's not hard to verify that the composition $G_0(R) \to G_0(S) \to G_0(R)$ is the identity map. □

3.3. **Regular local rings are UFDs.** Let $\operatorname{Cl}(R)$ be the divisor class group of R, we list the properties here

Theorem 3.12. *Let R be a Noetherian normal domain. Let I, J be two primes of pure height 1.*

(1) fI is also of pure height 1
(2) $\operatorname{div}(fI) = \operatorname{div}(I) + \operatorname{div}(f)$
(3) $\operatorname{div}(I) = \operatorname{div}(J)$ iff $I = J$
(4) The image of $\operatorname{div}(I)$ and $\operatorname{div}(J)$ are the same in $\operatorname{Cl}(R)$ iff there is f, g such that $fI = gJ$. This holds iff I and J are isomorphic as R-modules
(5) I is principal iff $\operatorname{div}(I) = 0$ in the divisor class group
(6) R is UFD iff $\operatorname{Cl}(R) = 0$.

We are aiming to prove following:

Theorem 3.13 (M.P.Murthy). *Let R be a normal domain and let H be the subgroup of $\bar{G}_0(R)$ spanned by the classes $[R/P]$ for P a prime of height 2 or more. Then*

$$\operatorname{Cl}(R) \cong \bar{G}_0(R)/H$$

with the map sending $[P] \mapsto [R/P]$ for all height one primes P.

Proof. We know that $G_0(R)$ is the free group on the classes of the R/P where P prime, modulo relations obtained from prime cyclic filtrations of $R/(P + xR)$. If we kill $[R]$ and all $[R/Q]$ with $\mathrm{ht}(Q) \geq 2$, all relations are killed except those come from R/xR. We'll show that this relation is the same as $\mathrm{div}(x)$ in $\mathrm{Cl}(R)$.

Let $xR = P_1^{(k_1)} \cap \cdots \cap P_n^{(k_n)}$ be its primary decomposition, we can ignore those primes of height 2. Since $\mathrm{div}(x) = k_1[P_1] + \cdots$, we only need to show that the occurrences of P in the prime filtration of R/xR is exactly k. We can check this after localizing at P: any factor of the from R/P is unaffected while other factors are killed. But then we have $xR_P = P^{(k)}R_P = P^kR_P$, note that (R_P, PR_P) is a DVR, the claim follows. □

Corollary 3.14. *A regular local ring is a UFD*

Corollary 3.15. *If R is a Dedekind domain, then $\bar{G}_0(R) \cong \mathrm{Cl}(R)$ and $G_0(R) \cong \mathbb{Z}[R] \oplus \mathrm{Cl}(R)$.*

3.4. **More discussions.**

Theorem 3.16. *$G_0(R) \cong G_0(R[x])$ under the map sending $[M] \mapsto [M[x]]$, where $M[x] := M \otimes_R R[x]$.*

Proof. We've already seen that the map is injective and has a left inverse:

$$[N]_{R[x]} \mapsto [N/xN]_R - [\mathrm{Ann}_N\, x]_R$$

But we'll take a different approach to show surjectivity. Any primes in $R[x]$ lies over some prime $P \subseteq R$, There are two types of primes lying over P: Look at the fiber $(R-P)^{-1}(R/P)[x] = \kappa_P[x]$, the primes are either (0) or generated by some monic irreducible polynomial. If we clear the denominator and lift the nonzero coefficients to $R-P$, then we get a polynomial f with leading coefficient in $R-P$ whose image is irreducible in $\kappa_P[x]$. Note that Q is recovered from P and f as the set of all elements of $R[x]$ multiplied into $P + fR[x]$ by an element of $R - P$, i.e. $Q = (PR[x] + fR[x]) :_{R[x]} (R - P)$.

Since $R[x]/PR[x] = (R/P) \otimes_R R[x]$ is evidently in the image, we only need to show that primes Q of the form described above is in the image. Look at the exact sequence:

$$0 \to (R/P)[x] \xrightarrow{f} (R/P)[x] \to M \to 0$$

where $M = R[x]/(PR[x] + fR[x])$ and

$$0 \to N \to M \to R[x]/Q \to 0$$

Since $(R-P)^{-1}M = (R-P)^{-1}R[x]/Q$, we have that N is a finitely generated module that is a torsion module over R/P. Hence we can choose some $a \in R - P$ such that $aN = 0$.

From the first exact sequence we know $[M] = 0$, hence in the second exact sequence we have $[N] = [R[x]/Q]$. Since $P + aR$ kills N, the prime cyclic filtration of N only involves primes lying over some prime strictly containing P. Now $[N]$ is in the image by Noetherian induction, so is $[R[x]/Q]$. □

Theorem 3.17. *Let R be a Noetherian ring and W a multiplicative system of R, then the kernel of the map $G_0(R) \to G_0(W^{-1}R)$ is spanned by the set of classes $\{[R/P]| : P \cap W \neq \emptyset\}$.*

Proof. The specified classes are clearly in the kernel of the map, we just need to show that they span the whole kernel. It sufficies to show that all the spanning relations on the classes $[W^{-1}R/QW^{-1}R]$ hold in $G_0(R)/\{[R/P]|P \cap W \neq \emptyset\}$.

Consider a prime cyclic filtration of $W^{-1}R/(Q, x)W^{-1}R$ where $x \in R$. We may take the inverse image of this filtration to get a filtration of R/Q. Each factor N_i contains an element u_i such that u_i generates $W^{-1}N_i \cong W^{-1}R/Q_i$. So we have short exact sequences:

$$0 \to Ru_i \to N_i \to C_i \to 0$$
$$0 \to D_i \to Ru_i \to R/Q_i \to 0$$

where C_i and D_i vanish after localization at W. So they have prime cyclic filtrations with factors meeting W. Now the relation $\sum_{i>1}[W^{-1}S/W^{-1}Q_i] = 0$ comes from the relations on $[R/Q_i]$ and $[C_i], [D_i]'s$:

$$[N_i] = [Ru_i] + [C_i] = [R/Q_i] + [C_i] + [D_i]$$
$$0 = [C_1] + [D_1] + \sum_{i>1}[R/Q_i] + [C_i] + [D_i]$$

So we're done. □

Corollary 3.18. *For any $x \in R$ we have an exact sequence:*

$$G_0(R/xR) \to G_0(R) \to G_0(R_x) \to 0$$

Now we define the Grothendieck group $K_0(R)$ of projective modules over a Noetherian ring R by forming the same Grothendieck group but with generators of finitely generated projective R-modules.

There is obviously a canonical map $K_0(R) \to G_0(R)$, and we have following theorem:

Theorem 3.19. *If R is regular, the map $K_0(R) \to G_0(R)$ is an isomorphism.*

Proof. The map from $G_0(R) \to K_0(R)$ is given by sending $[M]$ to the finite projective resolution of M. We have to show that this is independent of the choice of the resoltion. Of course they both equal to $[M]$ in $G_0(R)$, but we don't have $[M]$ in $K_0(R)$. The way to amend this is to use the mapping cone of two projective resolutions. The rest is a direct check. □

Note that K_0 is a functor on maps of Noetherian rings as short exact sequences of projectives remain exact after base change (they split). Restriction of scalars from S to R will not induce a map on K_0 unless S is module-fintie and projective over R.

Observe that $K_0(R)$ has a commutative ring structure induced by tensor, with $[R]$ as multiplicative identity.

Proposition 3.20. *Let P and Q be finitely generated projective module over a Noetherian ring R, then $[P] = [Q]$ in $K_0(R)$ iff there is a free module G such that $P \oplus G \cong Q \oplus G$.*

Proof. If $[P] = [Q]$, then $[P] - [Q]$ is in the span of the relations, i.e.

$$[P] - [Q] = \sum_{i=1}^{h}[P_i \oplus Q_i] - [P_i] - [Q_i] + \sum_{j=1}^{k}[P'_j] + [Q'_j] - [P'_j \oplus Q'_j]$$

which could be rewritten as

$$[P] + \sum_{i=1}^{h}[P_i] + [Q_i] + \sum_{j=1}^{k}[P'_j \oplus Q'_j] = [Q] + \sum_{i=1}^{h}[P_i \oplus Q_i] + \sum_{j=1}^{k}[P'_j] + [Q'_j]$$

This equation tells us that the number of occurence of any given projective module on both side are the same, hence we can change every + sign to $\oplus$ sign and get isomorphic modules:

$$P \oplus \left(\bigoplus_{i=1}^{h} P_i \oplus Q_i \oplus \bigoplus_{j=1}^{k} P'_j \oplus Q'_j\right) \cong Q \oplus \left(\bigoplus_{i=1}^{h} P_i \oplus Q_i \oplus \bigoplus_{j=1}^{k} P'_j \oplus Q'_j\right)$$

which is

$$P \oplus N \cong Q \oplus N$$

where N is projective. So there is some N' such that $N \oplus N' \cong G$, a free module. So

$$P \oplus N \oplus N' \cong Q \oplus N \oplus N' \Rightarrow P \oplus G \cong Q \oplus G$$

□

Corollary 3.21. *Let R be Noetherian, then $K_0(R)$ is generated by $[R]$ iff every projective module P has a finitely generated free component, i.e. iff for every finitely generated projective module M there is some h, k such that $P \oplus R^h \cong R^k$.*

We already know that

$$K_0(R = K[x_1, ..., x_n]) \cong G_0(K[x_1, ..., x_n]) \cong G_0(K) \cong \mathbb{Z}$$

is generated by the class of R. By corollary above, every finitely generated projective module over R has a finitely generated free complement. If we want to prove that they are in fact free, we need to show that the kernel of any map $R^n \twoheadrightarrow R$ is free. Such map is always represented by a $1 \times n$ matrix $(r_1, ..., r_n)$. The surjectivity of the map corresponds to the condition that r_j's generates the unit ideal. The kernel being free is equivalent to the possibility of extending $(r_1, ..., r_n)$ to be a free basis of R^n. This is called "unimodular column" problem. This question is first raised by Serre in the mid 1950s and was open until 1976, when it was settled in the affirmative, independently, by D. Quillen and A. Suslin. A bit later, Vaserstein gave another proof which is very short, albeit very tricky. So it's true that every finitely generated projective modules over R are free.