Injective Modules and Injective Hulls

Zhan Jiang

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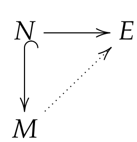
1 Injective Modules

1.1 Definition

Definition 1.1. *An R-module E is* ***injective*** *if equivalently,*

- $\mathrm{Hom}_R(\_, E)$ *is an exact functor.*
- *For any injection* $N \to M$*, the map* $\mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(N, E)$ *is surjection.*

In other words, every R-linear map from a submodule N of M to E can be extended to a map of all of M to E.



Proposition 1.2. *An R-module E is injective iff for every ideal I of R and R-linear map* $\phi : I \to E$ *extends to a map* $R \to E$.

Proof. "Only if" part is clear. We only need to prove "if" part: Let $N \subseteq M$ and $f : N \to E$ be given. we want to extend f to all of M.

First of all let Λ be the set of R-linear maps from submodules of M to E. We define a partial order on Λ : $g \geq h$ if the domain of h is contained in the domain of g and h is a restriction of g. The set $\Lambda_{\geq f}$ consisting of all R-linear maps larger than or equal to f has a maximal element: we can take the union of a chain and define the R-linear map in the obvious way. Then by Zorn's lemma, there is a maximal element (f', N').

If $N' \neq M$, we can pick an element $x \in M - N'$ and define $I = \{r \in R : rx \in N'\}$. Define a map

$$\begin{aligned} \phi : I &\to E \\ r &\mapsto f'(rx) \end{aligned}$$

Then we can extend this to a map $\psi : R \to E$. Therefore there is a map $f' \oplus \psi : N' \oplus Rx \to E$. We have a surjection $N' \oplus Rx \to N' + Rx$ and we claim that $f' \oplus \psi$ kills the kernel: If $u \oplus rx \mapsto 0$, then $u + rx = 0 \Rightarrow r \in I$. Then $f' \oplus \psi(u, rx) = f'(u) + \psi(r) = f'(u) + f'(rx) = f'(u) + f'(-u) = 0$. So this induces a map $N' \oplus Rx \to E$ which extends f', a contradiction! □

1.2 Divisible Module

Definition 1.3. *A module M is **divisible** if, equivalently:*

- *$rM = M$ for all $r \in R - \{0\}$.*
- *For all $u \in M$ and $r \in R - \{0\}$, there exists $u' \in M$ such that $u = ru'$.*

Corollary 1.4. *We have following*

- *Over a domain R, every injective module is divisible.*
- *Over a PID R, a module is injective iff it is divisible.*

Proof. TO BE ADDED □

1.3 Enough injectives

2 Injective Hulls

2.1 Essential extensions

Definition 2.1. *Let R be a ring, a homomorphism $f : N \to M$ is called an **essential extension** if it's an injection and every nonzero submodule of M has a nonzero intersection with $f(N)$.*