Completions of Rings and Modules

Zhan Jiang

July 12, 2017

Contents

1 Completion of Rings

1.1 Definition

Definition 1.1. *Let R be a ring and $I \subseteq R$ an ideal. Then the completion $\widehat{R^I}$ of R with respect to I is* $\mathrm{Lim}(R/I^t)$

This is called I-adic completion of R. If we have $\cap_t I^t = \{0\}$, then R is called I-adically seperated. If $R \to \widehat{R^I}$ is an isomorphism, then R is I-adically complete. Note that if R is I-adically complete, then R is I-adically seperated: Choose $r \in \cap_t I^t$, then the image of r is zero in $\widehat{R^I}$, which implies that $r = 0$.

1.2 Properties

Proposition 1.2. *Let $J = \mathrm{Ker}(\widehat{R^I} \to R/I)$, then J is contained in the Jacobson ideal of $\widehat{R^I}$.*

Proof. We only need to show that for any unit u and any $j \in J$, $u + j$ is a unit. But we have $u + j = u(1 + u^{-1}j)$. So it's enough to show this for $1 + j$.

Let $r_0, r_1, \ldots$ be a Cauchy sequence represents j, consider the sequence $1 - r_0, 1 - r_1 + r_1^2, \ldots, 1 - r_n + r_n^2 - r_n^3 + \cdots + (-1)^{n-1} r_n^{n+1}, \ldots$. Call its n^{th} term v_n. First it is a Cauchy sequence since $v_{n+1} - v_n = (r_{n+1} - r_n)(\ldots) + (-1)^n r_{n+1}^{n+2}$, so if r_n and r_{n+1} differ by an element of I^t, then v_n and v_{n+1} differ by an element in $I^t + I^{n+2}$.

But then $v = (v_0, v_1, ...)$ is an inverse for $1 + j$: n^{th} term of $v(1 + j) - 1$ is a power of r_n, therefore this Cauchy sequence converges to zero. □

Note that the map $\widehat{R}^I \to R/I$ is a surjection, therefore maximal ideals of R/I contracts to maximal ideals of $\widehat{R}^I$. From Proposition 1.2 above, we see that there is a bijection between maximal ideals of $\widehat{R}^I$ and R/I. This observation is extremely useful when we are in the quasilocal case, which we record in the following remark

Remark 1.3. *If R/I is a quasilocal ring, then so is $\widehat{R}^I$. In particular, this holds if R is quasilocal.*

We have following observation:

If $R_1 \to R_2$ maps I_1 into I_2, then a Cauchy sequence in R_1 with respect to I_1 maps to a Cauchy sequence in R_2 with respect to I_2. Therefore we have a ring homomorphism $\widehat{R_1}^{I_1} \to \widehat{R_2}^{I_2}$.

This construction is clearly functorial: If we have a third ring R_3 with $R_2 \to R_3$ mapping I_2 into I_3. Then $\widehat{R_1}^{I_1} \to \widehat{R_3}^{I_3}$ is the composition of $\widehat{R_1}^{I_1} \to \widehat{R_2}^{I_2}$ and $\widehat{R_2}^{I_2} \to \widehat{R_3}^{I_3}$.

If the map $R_1 \to R_2$ is surjection and I_1 maps onto I_2, then the map $\widehat{R_1}^{I_1} \to \widehat{R_2}^{I_2}$ is surjection: we can pick a preimage for each term in the Cauchy sequence.

Example 1.4. *If $S = R[x_1, ..., x_n]$ and $I = (x_1, ..., x_n)S$, then $\widehat{S}^I = R[[x_1, ..., x_n]]$.*

This example enables us to prove following theorem

Theorem 1.5. *If R is Noetherian ring and I is an ideal of R, then $\widehat{R}^I$ is Noetherian*

Proof. Suppose that $I = (f_1, ..., f_n)R$, then we can construct a map

$$\begin{aligned} S = R[x_1, ..., x_n] &\to R \\ x_i &\mapsto f_i \end{aligned}$$

Then $J = (x_1, ..., x_n)S$ maps onto I, therefore $\widehat{S}^J \to \widehat{R}^I$ is surjection. But we have already seen that $\widehat{S}^J = R[[x_1, ..., x_n]]$, which is Noetherian. □

1.3 Completion as metric space

Suppose R is a I-adically seperated, then we can choose a metric on R as following:

$$d(r, s) = \begin{cases} 0, & r = s \\ \epsilon^n & r - s \in I^n \text{ and } r - s \notin I^{n+1} \end{cases}$$

where ϵ is a real number between 0 and 1, i.e. $0 < \epsilon < 1$. To show that $d(-, -)$ is a metric, we just need to show triangle inequality:

Let r_1, r_2, r_3 be three elements of R: if any of two are equal, then the triangle inequality clearly holds. If not, let n_{12}, n_{23}, n_{31} be the lagest power of I containing $r_1 - r_2, r_2 - r_3, r_3 - r_1$ respectively. Since $r_1 - r_2 = -(r_2 - r_3) - (r_3 - r_1)$. Therefore $r_{12} \geq \min\{r_{23}, r_{31}\}$. If r_{12} is the smallest, then either r_{23} or r_{31} must equal r_{12}. Therfore the two largest sides are equal, hence triangle inequality is automatic.

$\widehat{R^I}$ is literally the completion of R as a metric space with repsect to this metric.

2 Completion of Modules

2.1 Definition

Definition 2.1. *The completion* $\widehat{M}^I$ *of* M *with respect to an ideal* $I \subseteq R$ *is* $\mathrm{Lim}\, M/I^t M$.

Similarily, M is called I-adically seperated if $\cap_t I^t M = 0$. M is called I-adically complete if $M \to \widehat{M}^I$ is isomoprhism.

Completion $\widehat{*}^I$ is a covariant functor from R-modules to $\widehat{R}^I$-modules, which we denote $\mathcal{C}$.

Then $\mathcal{C}$ preserves epimorphism: If $M \to N$ is surjection, then so is $\widehat{M}^I \to \widehat{N}^I$. Any element in $\widehat{N}^I$ is represented by a partial sum $u_1 + u_2 + \cdots$ such that $u_k \in I^k N$. Since $I^k M$ surjects onto $I^k N$ we have a $v_k \in I^k M$ maps to u_k. Then $v_1 + v_2 + \cdots$ maps onto the given elements.

There is natural transformation from the base change functor $\mathcal{B} = \widehat{R}^I \otimes_R$ to the completion functor $\mathcal{C}$. This natrual transformation is an isomorphism if we restrict to the category of finitey generated modules.

The I-adically completion functor is exact if restricted to the finitely generated functors. We shall prove all these in the next section

2.2 Artin-Rees Lemma

Lemma 2.2 (Artin-Rees)**.** *Let* $N \subseteq M$ *be Noetherian modules over the Noetherian ring* R *and let* I *be an ideal of* R. *Then there is a positive integer* c *such that for all* $n \geq c$,

$$I^n M \cap N = I^{n-c}(I^c M \cap N)$$

Proof. We consider the module $R[t] \otimes M$. Within this module:

$$\mathcal{M} = M + IMt + I^2 M t^2 + \cdots + I^k M t^k + \cdots$$

is a finitely generated module over $R[It]$. It's generated by generators for M over R. Therefore it's Noetherian over $R[It]$. Consider the submodule

$$\mathcal{N} = N + (IM \cap N)t + (I^2 M \cap N)t^3 + \cdots + (I^k M \cap N)t^k + \cdots$$

it's finitely generated over $R[It]$, so we can choose a set of generators with degree at most c.

Now look at $ut^n \in I^n M \cap N$ where $n \geq c$, it's a $R[It]$ linear combination of generators in $(I^j M \cap N)t^j$ where $j \leq c$. Clearly we only need those homogeneous terms and we can take out t^n. Therefore we end up with

$$u = \sum_j a_{n-j} t^{n-j} v_j t^j$$

where $a_{n-j} \in I^{n-j}$ and $v_j \in (I^j M \cap N)$, but since $I^{c-j}(I^j M \cap N) \subseteq I^c M \cap N$. So we have $I^{n-j}(I^j M \cap N) \subseteq I^{n-c}(I^c M \cap N)$. □

Next we prove those assertions

Proposition 2.3. *Let* R *be a Noetherian ring and* $I \subseteq R$ *an ideal of* R, *then* $\mathcal{C}$ *is an exact functor on the category of finitely generated* R*-modules.*

Proof. Let $0 \to N \to M \to Q \to 0$ be an exact sequence, we have to prove exactness at N, M, Q. The surjection is already proved. So $M \to Q \to 0$ remains exact.

The exactness at $\widehat{M}$: If u is in the kernel of $\widehat{M} \to \widehat{Q}$, then assume that u is the limit of $(u_1, u_2, ...)$ where $u_i - u_{i+1} \in I^i M$. Since the image is zero in $Q = M/N$, we may assume WLOG that $u_i \in I^i(M/N)$, which is $u_i \in I^i M + N$. Let $u_i = w_i + v_i$ where $w_i \in I^i M$ and $v_i \in N$. Then v_i is a Cauchy sequence in N whose image is u in M: $v_i - v_{i+1} \in I^i M \cap N$ for all i. Therefore it's a Cauchy sequence by Artin-Rees lemma.

The injectivity is proved by Artin-Rees lemma: if a Cauchy sequence converges to zero in M, then it has to converge to zero in N. □

Proposition 2.4. *For finitely generated module M, the natural transformation $\widehat{R} \otimes_R M \to \widehat{M}$ is an isomoprhism.*

Proof. Take a finite presentation $R^m \to R^n \to M \to 0$ of M, and apply both functors and the natural transformation to them:

$$\begin{array}{ccccccc} \widehat{R}^I \otimes_R R^n & \longrightarrow & \widehat{R} \otimes_R R^m & \longrightarrow & \widehat{R} \otimes_R M & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ \widehat{R^n}^I & \longrightarrow & \widehat{R^m}^I & \longrightarrow & \widehat{M}^I & \longrightarrow & 0 \end{array}$$

The first two vertical arrows are isomorphism by the fact that both functors commutes with direct sum and they are identity on R. By five lemma, the third vertical arrow must be an isomorphism. □

Finally, we want to show that $\mathcal{C}$ is actually an exact functor, this comes from the criterion for flatness:

Lemma 2.5. *Let M be an R-module, if for any finitely generated modules $N \subseteq Q$ we have an injection $M \otimes N \to M \otimes Q$, then M is flat over R.*

Then the exactness follows from above lemma.

3 Completion of Local Rings

For a local ring (R, m, K), the map $R \to \widehat{R}^m$ is faithfully flat: Consider the exact sequence

$$0 \to m \to R \to K \to 0$$

Since tensor with $\widehat{R}$ is an exact functor and get identified with completion on finitely generated modules, we have an exact sequence:

$$0 \to m \otimes \widehat{R} \to \widehat{R} \to K \to 0$$

In particular, $m\widehat{R}$ is not zero. Therefore $\widehat{R}$ is faithfully flat over R. This also proves following proposition.

Proposition 3.1. *The maximal ideal of $\widehat{R}$ is the expansion of m.*

Next we note following: For every ideal I of R, there are three operations:

(1) The expansion $I\widehat{R}$

(2) The tensor product $I \otimes \widehat{R}$

(3) The completion $\widehat{I}$ as a module over R

The first two are idetified because $\widehat{R}$ is flat. The latter two get identified if I is finitely generated. Therefore in the case of local ring, we know that (1) and (3) are identified.

Now it's resonable to write the maximal ideal of $\widehat{R}$ as $\widehat{m}$ as it will not cause confusion in any sense. It follows quite easily that $\widehat{m}^n = m^n\widehat{R}$, which is true for any expansion of an ideal.

We also have following lemma due to the flatness of completion:

Lemma 3.2. *For any ideal I of R, we have $\widehat{R/I} = \widehat{R}/\widehat{I}$*

Proof. Completion preserves the exactness of $0 \to I \to R \to R/I \to 0$. □

We already know that the completion of K is still K itself. But this easy fact enables us to prove following:

Proposition 3.3. *If M is an R-module of finite length, then $\widehat{M} = M$.*

Proof. Choose a filtraion for M and each factor is a finite copy of K, therefore invariant under completion. Now the conclusion follows. □

Proposition 3.3 enables us to show following proposition

Proposition 3.4. *Expansion and contraction gives a bijection between m-primary ideals in R and $\widehat{m}$-primary ideals in $\widehat{R}$.*

Proof. Note that m-primary ideals always contains some power of m. Those ideals containing m^n corresponds bijectively to ideals in R/m^n. But $R/m^n \cong \widehat{R/m^n} \cong \widehat{R}/\widehat{m^n} = \widehat{R}/\widehat{m}^n$. And the result follows. □

3.1 Regularity

We already know that $\dim(\widehat{R}) \geq \dim(R)$, but we can prove more:

Proposition 3.5. *Every system of parameters in R is a system of parameters in $\widehat{R}$. Therefore* $\dim(\widehat{R}) = \dim(R)$.

Proof. Let $x_1, ..., x_n$ be a system of parameters for R, then $m^N \subseteq (x_1, ..., x_n)R$ and so that $\widehat{m}^N \subseteq (x_1, ..., x_n)\widehat{R}$. Therefore $x_1, ..., x_n$ remains a system of parameters. And we have $\dim(\widehat{R}) \leq n$. □

Now we can prove following theorem about regularity:

Theorem 3.6. *A local ring (R, m, K) is regular iff its completion $\widehat{R}$ is regular*

Proof. We just one more equality:the embedded dimension of R equals the embedded dimension of $\widehat{R}$. This comes from the fact that m/m^2 is a finite-length module over R. □