ABSOLUTE CLOSURE

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1. Definition

1.1. Definition.

Definition 1.1. Let R be an integral domain. The absolute integral closure R^+ is the integral closure of R in the algebraic closure of its fraction field.

It's easy to see that the absolute closure is unique up to nonunique isomorphisms, just like the algebraic closure of a field.

For any S integral extension of R, we have $S \hookrightarrow R^+$ as R-algebras. Therefore R^+ is a maximal domain extension of R that is integral over R.

Definition 1.2. A domain R is **absolutely integrally closed** if $R = R^+$.

Useful criterion: A domain R is absolutely integrally closed iff every monic polynomial in one variable $f \in R[x]$ factors into monic linear factors over R.

Using this criterion, it's easy to see that if R is absolutely integrally clsoed, then

- A localization at any multiplicative system of R is absolutely integrally closed
- A domain which is also a homomorphic image of R is absolutely integrall closed

Suppose $R \hookrightarrow S$ is an extension of domians, then we have

$$\begin{array}{ccc} R^+ & \hookrightarrow & S^+ \\ \uparrow & & \uparrow \\ R & \hookrightarrow & S \end{array}$$

because $\operatorname{Frac}(S) \supseteq \operatorname{Frac}(R) \Rightarrow$ the integral closure of S contains the integral closure of R.

Now if we have $R \twoheadrightarrow S$, then $S = R/P$ for some prime ideal P. We want to show that there is a surjection $R^+ \twoheadrightarrow S^+$: Since R^+ is integral over R, we have Q lying over P. So $Q \cap R = P \Rightarrow R/P \hookrightarrow R^+/Q$. But then R^+/Q is absolutely integral and integral over $S = R/P$. This shows that $S^+ \cong R^+/Q$, hence

$$\begin{array}{ccc} R^+ & \twoheadrightarrow & S^+ \\ \uparrow & & \uparrow \\ R & \twoheadrightarrow & R \end{array}$$

Now we are ready to show following proposition

Proposition 1.3. *For any homomorphism $R \to S$ of integral domains there is a commutative diagram*

$$\begin{array}{ccc} R^+ & \longrightarrow & S^+ \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

Proof. Since every domain morphism $R \to S$ factors as $R \twoheadrightarrow T \hookrightarrow S$, now apply our discussion above. □