THE KOSZUL COMPLEX

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CONTENTS

1. PRELIMINARY

1.1. **Mapping cones.** If we have two complexes $f_\bullet : (B_\bullet, \delta_\bullet) \to (A_\bullet, d_\bullet)$, we can form a mapping cone $C_\bullet$ to be $C_n = A_n \oplus B_{n-1}$ and the differential map is given by

$$a_n \oplus b_n \mapsto (d_n(a_n) + (-1)^{n-1} f(b_{n-1})) \oplus \delta_{n-1}(b_{n-1})$$

It's not hard to verify that this map is a differential map. Note that under this construction, we have that $A_\bullet \subseteq C_\bullet$ is a subcomplex while the quotient is $B_{\bullet-1}$, i.e. we have an exact sequence

$$0 \to A_\bullet \to C_\bullet \to B_{\bullet-1} \to 0$$

which induces a long exact sequence

$$\cdots \to H_n(A_\bullet) \to H_n(C_\bullet) \to H_{n-1}(B_\bullet) \to H_{n-1}(A_\bullet) \to \cdots$$

Immediately we have following consequences:

Proposition 1.1. *If $f_\bullet : B_\bullet \to A_\bullet$ is a map of acyclic complexes, then*

- $H_n(C_\bullet) = 0$ *for* $n \geq 2$
- $H_1(C_\bullet) = \mathrm{Ker}(H_0(B) \to H_0(A))$*, which is the kernel of the induced map* $B_0/\delta_1(B_1) \to A_0/d_1(A_1)$.
- $H_0(C_\bullet) = \mathrm{Coker}(H_0(B) \to H_0(A))$ *which is* $A_0/(d_1(A_1) + f_0(B_0))$.

Note that we also have a quotient complex of $f_\bullet : B_\bullet \to A_\bullet$, namely $Q_\bullet$. We have following proposition

Proposition 1.2. $H_i(C_\bullet) \cong H_i(Q_\bullet)$.

Proof. TO BE ADDED □

2. Construction

2.1. **Via mapping cones.** Let $\underline{x} = x_1, ..., x_n$ be a sequence of elements in R, then the Koszul complex $\mathcal{K}_\bullet(\underline{x}; R)$ is defined recursively:

(1) If $n = 1$, then $\mathcal{K}_\bullet(\underline{x}; R)$ is

$$0 \to Ru_1 \to R \to 0$$
$$u_1 \mapsto x_1$$

(2) Suppose $\mathcal{K}_\bullet(x_1, ..., x_{n-1}; R)$ is defined, then multiplication by x_n gives a map of complexes:

$$\mathcal{K}_\bullet(x_1, ..., x_{n-1}; R) \to \mathcal{K}_\bullet(x_1, ..., x_{n-1}; R)$$

and $\mathcal{K}_\bullet(\underline{x}; R)$ is defined to be the mapping cone of above map.

2.2. **Via tensor product.** The Koszul complex of one element x_i is defined to be $K_\bullet(x_i) = 0 \to K_1 \to^{\times x_i} K_0 \to 0$ where $K_1 = K_0 = R$ and the Koszul complex for a sequence $x_1, ..., x_n$ is $K_\bullet(x_1, ..., x_n) = K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n)$.

2.3. **Direct description.** Let $\sigma \subseteq [n]$ be a subset of the n elements set $[n] = \{1, 2, ..., n\}$. Assume that $\sigma = \{i_1, ..., i_k\}$ where $i_1 \leq \cdots \leq i_k$. Let $|\sigma|$ denote the cardinality of σ.

Let u_σ be indeterminants on R for each σ. Then $\mathcal{K}_j = \oplus_{|\sigma|=j} Ru_\sigma$ is a free R-module with $\binom{n}{j}$ generators. Define maps

$$\mathcal{K}_j \to \mathcal{K}_{j-1}$$
$$u_\sigma \mapsto \sum_{t=1}^{j} (-1)^{t-1} x_{i_t} u_{\sigma - \{i_t\}}$$

These maps are differentials and $\mathcal{K}_\bullet$ is a complex. We identify $u_\emptyset$, the generator of $\mathcal{K}_0$, with $1 \in R$. This complex is Koszul complex.

2.4. **The Koszul complex of modules.** $\mathcal{K}_\bullet(\underline{x}; M) = \mathcal{K}_\bullet(\underline{x}; R) \otimes_R M$.

3. The Koszul Homology

3.1. **Basic properties.** $\mathcal{K}_\bullet(\underline{x}; R)$ is a free complex of length n in which the degree j term is isomorphic to the free R-module on $\binom{n}{j}$ generators.

It's symmetric in the sequence: if we move around elements in the sequence, we still get the same Koszul complex.

Observation: If $C_\bullet$ is a complex, then there is a SES of complexes:

$$0 \to C_\bullet \to C_\bullet \otimes K_\bullet(x) \to C_\bullet(-1) \to 0$$

given by $0 \to C_n \to C_n \oplus C_{n-1} \to C_{n-1} \to 0$, etc.

The connecting map in the long exact sequence is given by multiplication by x.

So we get SES's:

$$0 \to H_n(C_\bullet)/xH_n(C_\bullet) \to H_n(C_\bullet \otimes K_\bullet(x)) \to \mathrm{Ann}_{H_{n-1}(C_\bullet)}(x) \to 0$$

In particular, we can apply this to a Koszul complex and get

$$0 \to \frac{H_n(\underline{x}; M)}{xH_n(\underline{x}; M)} \to H_n(\underline{x}; M) \to \mathrm{Ann}_{H_{n-1}}(\underline{x}; M) \to 0$$

Let $H_i(\underline{x}; M)$ be the i^{th} homology of $\mathcal{K}_\bullet(\underline{x}; M)$. We note following propositions:

Proposition 3.1. *Let R be a ring and $\underline{x} = x_1, ..., x_n \in R$. Let $I = (\underline{x})R$ and let M be an R-module:*

(1) $H_i(\underline{x}; M) = 0$ *if* $i < 0$ *or* $i > n$.
(2) $H_0(\underline{x}; M) \cong M/IM$.
(3) $H_n(\underline{x}; M) = \mathrm{Ann}_M I$.
(4) $\mathrm{Ann}_R M$ *kills every* $H_i(\underline{x}; M)$.
(5) If M is Noetherian, then so is $H_i(\underline{x}; M)$.
(6) For every i, $H_i(\underline{x}; \_)$ is a covariant functor from R-modules to R-modules.
(7) If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence, then we have long exact sequences:

$$\cdots \to H_i(\underline{x}; M_1) \to H_i(\underline{x}; M_2) \to H_i(\underline{x}; M_3) \to H_{i-1}(\underline{x}; M_1) \to \cdots$$

(8) If $\underline{x}$ is a possibly improper regular sequence on M, then $H_i(\underline{x}; M) = 0, \forall i \geq 1$

Proof. (1) is trivial from definition

(2): The first map is identified as $M^n \to M$ sending $(v_1, ..., v_n) \mapsto v_1x_1 + \cdots + v_nx_n$ then the result follows

(3): The last map is identified as $M \to M^n$ sending $v \mapsto (x_1v, -x_2v, \cdots, (-1)^{n-1}x_nv)$ then the result follows

(4)&(5): Since every term in the Koszul complex is itself a direct sum of copies of M

(6): This is standard homological algebra arguement

(7): Since each term in the Koszul complex is free, any exact sequence induces an exact sequence of complexes, hence induces a long exact sequence.

(8): by induction on the length of the regular sequence. □

Theorem 3.2. *If (R, m) is local and $\underline{x} \in m$ and M is finitely generated, then $H_i(\underline{x}; M) = 0$ for all $i > 0$ implies that $\underline{x}$ is a regular sequence on M.*

Proof. TO BE ADDED □

3.2. Independence of Koszul homology of the base ring.

Suppose we have a map $R \to S$ and an S-module M. By restriction of scalars M is also an R-module. Let $\underline{x}$ be a sequence in R and let $\underline{y}$ be its image in S. Note that the action of x_i and y_i are the same for every i. This shows that $\mathcal{K}_\bullet(\underline{x}; M)$ and $\mathcal{K}_\bullet(\underline{y}; M)$ are the same. Therefore $H_i(\underline{x}; M) \cong H_i(\underline{y}; M)$ for every i.

Note that even if we treat M as an R-module at the very beginning, we can still recover the S-module structure of $H_i(\underline{x}; M)$: since $M \to^s M$ is R-linear and $H_i(\underline{x}; \_)$ is a covariant functor, so we can recover the action of s on $H_i(\underline{x}; M)$.

3.3. Koszul homology and Tor.

Let $A = \mathbb{Z}[X_1, ..., X_n]$, $K[X_1, ..., X_n]$ or $R[X_1, ..., X_n]$ where X_i acts on M exactly as x_i, then

Proposition 3.3. $H_i(\underline{x}; M) \cong \mathrm{Tor}_i^A(A/(X_1, ..., X_n)A, M)$

Corollary 3.4. *Let $\underline{x}$ be a sequence in R and let $I = (\underline{x})R$, then I kills $H_i(\underline{x}; M)$ for all i.*

3.4. Cohomological Koszul complex.

$\mathcal{K}^\bullet(\underline{x}; M) \cong \mathrm{Hom}_R(\mathcal{K}_\bullet(\underline{x}, M))$ The cohomological Koszul complex of R (or M) is isomorphic with the homological Koszul complex numbered "backward".

Theorem 3.5. *Let $\underline{x}$ be a possibly improper sequence in R and let M be an R-module, then*

$$\mathrm{Ext}^i_R(R/(\underline{x})R, M) \cong H^i(\underline{x}; M) \cong H_{n-i}(\underline{x}; M) \cong \mathrm{Tor}^R_{n-i}(R/(\underline{x})R, M)$$