1. Preliminary

1.1. Mapping cones

If we have two complexes $f_\bullet : (B_\bullet, \delta_\bullet) \to (A_\bullet, d_\bullet)$, we can form a mapping cone $C_\bullet$ to be $C_n = A_n \oplus B_{n-1}$ and the differential map is given by

$$a_n \oplus b_n \mapsto (d_n(a_n) + (-1)^n f(b_{n-1})) \oplus \delta_{n-1}(b_{n-1})$$

It's not hard to verify that this map is a differential map. Note that under this construction, we have that $A_\bullet \subseteq C_\bullet$ is a subcomplex while the quotient is $B_{\bullet-1}$, i.e., we have an exact sequence

$$0 \to A_\bullet \to C_\bullet \to B_{\bullet-1} \to 0$$

which induces a long exact sequence

$$\cdots \to H_n(A_\bullet) \to H_n(C_\bullet) \to H_{n-1}(B_\bullet) \to H_{n-1}(A_\bullet) \to \cdots$$

Immediately we have following consequences:

**Proposition 1.1.** If $f_\bullet : B_\bullet \to A_\bullet$ is a map of acyclic complexes, then

- $H_n(C_\bullet) = 0$ for $n \geq 2$
- $H_1(C_\bullet) = \text{Ker}(H_0(B) \to H_0(A))$, which is the kernel of the induced map $B_0/\delta_1(B_1) \to A_0/d_1(A_1)$.
- $H_0(C_\bullet) = \text{Coker}(H_0(B) \to H_0(A))$ which is $A_0/(d_1(A_1) + f_0(B_0))$.

Note that we also have a quotient complex of $f_\bullet : B_\bullet \to A_\bullet$, namely $Q_\bullet$. We have following proposition

**Proposition 1.2.** $H_i(C_\bullet) \cong H_i(Q_\bullet)$. 

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THE KOSZUL COMPLEX

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In particular, we can apply this to a Koszul complex and get

The Koszul complex of modules.

2.4. complex is Koszul complex.

The connecting map in the long exact sequence is given by multiplication by \( x \). So we get SES's:

\[ \begin{align*}
0 & \rightarrow R u_1 \rightarrow R \rightarrow 0 \\
& \quad \quad u_1 \mapsto x_1
\end{align*} \]

(2) Suppose \( K_\bullet(x_1, \ldots, x_{n-1}; R) \) is defined, then multiplication by \( x_n \) gives a map of complexes:

\[ K_\bullet(x_1, \ldots, x_{n-1}; R) \rightarrow K_\bullet(x_1, \ldots, x_{n-1}; R) \]

and \( K_\bullet(x; R) \) is defined to be the mapping cone of above map.

2.2. Via tensor product. The Koszul complex of one element \( x_i \) is defined to be \( K_\bullet(x_i) = 0 \rightarrow K_1 \rightarrow x_i; K_0 \rightarrow 0 \) where \( K_1 = K_0 = R \) and the Koszul complex for a sequence \( x_1, \ldots, x_n \) is \( K_\bullet(x_1, \ldots, x_n) = K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n) \).

2.3. Direct description. Let \( \sigma \subseteq [n] \) be a subset of the \( n \) elements set \( [n] = \{1, 2, \ldots, n\} \). Assume that \( \sigma = \{i_1, ..., i_k\} \) where \( i_1 \leq \cdots \leq i_k \). Let \( |\sigma| \) denote the cardinality of \( \sigma \).

Let \( u_\sigma \) be indeterminants on \( R \) for each \( \sigma \). Then \( K_j = \oplus_{|\sigma|=j} R u_\sigma \) is a free \( R \)-module with \( \binom{n}{j} \) generators. Define maps

\[ K_j \rightarrow K_{j-1} \]

\[ u_\sigma \mapsto \sum_{t=1}^{j} (-1)^{t-1} x_{i_t} u_{\sigma\setminus\{i_t\}} \]

These maps are differentials and \( K_\bullet \) is a complex. We identify \( u_{\emptyset} \), the generator of \( K_0 \), with \( 1 \in R \). This complex is Koszul complex.

2.4. The Koszul complex of modules. \( K_\bullet(x; M) = K_\bullet(x; R) \otimes_R M \).

3. The Koszul Homology

3.1. Basic properties. \( K_\bullet(x; R) \) is a free complex of length \( n \) in which the degree \( j \) term is isomorphic to the free \( R \)-module on \( \binom{n}{j} \) generators.

It’s symmetric in the sequence: if we move around elements in the sequence, we still get the same Koszul complex.

Observation: If \( C_\bullet \) is a complex, then there is a SES of complexes:

\[ \begin{align*}
0 & \rightarrow C_\bullet \rightarrow C_\bullet \otimes K_\bullet(x) \rightarrow C_\bullet(-1) \rightarrow 0
\end{align*} \]

given by \( 0 \rightarrow C_n \rightarrow C_n \otimes C_{n-1} \rightarrow C_{n-1} \rightarrow 0 \), etc.

The connecting map in the long exact sequence is given by multiplication by \( x \).

So we get SES’s:

\[ \begin{align*}
0 & \rightarrow H_n(C_\bullet)/x H_n(C_\bullet) \rightarrow H_n(C_\bullet \otimes K_\bullet(x)) \rightarrow \text{Ann}_{H_{n-1}(C_\bullet)}(x) \rightarrow 0
\end{align*} \]

In particular, we can apply this to a Koszul complex and get

\[ \begin{align*}
0 & \rightarrow \frac{H_n(x; M)}{x H_n(x; M)} \rightarrow H_n(x; M) \rightarrow \text{Ann}_{H_{n-1}(x; M)}(x) \rightarrow 0
\end{align*} \]

Let \( H_i(x; M) \) be the \( i \)th homology of \( K_\bullet(x; M) \). We note following propositions:
Proposition 3.1. Let $R$ be a ring and $\underline{x} = x_1, \ldots, x_n \in R$. Let $I = (\underline{x})R$ and let $M$ be an $R$-module:

1. $H_i(\underline{x}; M) = 0$ if $i < 0$ or $i > n$.
2. $H_0(\underline{x}; M) \cong M/IM$.
3. $H_n(\underline{x}; M) = \text{Ann}_M I$.
4. $\text{Ann}_R M$ kills every $H_i(\underline{x}; M)$.
5. If $M$ is Noetherian, then so is $H_i(\underline{x}; M)$.
6. For every $i$, $H_i(\underline{x}; -)$ is a covariant functor from $R$-modules to $R$-modules.
7. If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence, then we have long exact sequences:

$$\cdots \to H_i(\underline{x}; M_1) \to H_i(\underline{x}; M_2) \to H_i(\underline{x}; M_3) \to H_{i-1}(\underline{x}; M_1) \to \cdots$$

8. If $\underline{x}$ is a possibly improper regular sequence on $M$, then $H_i(\underline{x}; M) = 0$, $\forall i \geq 1$.

Proof. (1) is trivial from definition

(2): The first map is identified as $M^n \to M$ sending $(v_1, \ldots, v_n) \mapsto v_1 x_1 + \cdots + v_n x_n$ then the result follows

(3): The last map is identified as $M \to M^n$ sending $v \mapsto (x_1 v, -x_2 v, \cdots, (-1)^{n-1} x_n v)$ then the result follows

(4)&(5): Since every term in the Koszul complex is itself a direct sum of copies of $M$

(6): This is standard homological algebra argument

(7): Since each term in the Koszul complex is free, any exact sequence induces an exact sequence of complexes, hence induces a long exact sequence.

(8): by induction on the length of the regular sequence. □

Theorem 3.2. If $(R, m)$ is local and $\underline{x} \in m$ and $M$ is finitely generated, then $H_i(\underline{x}; M) = 0$ for all $i > 0$ implies that $\underline{x}$ is a regular sequence on $M$.

Proof. TO BE ADDED □

3.2. Independence of Koszul homology of the base ring. Suppose we have a map $R \to S$ and an $S$-module $M$. By restriction of scalars $M$ is also an $R$-module. Let $\underline{x}$ be a sequence in $R$ and let $\underline{y}$ be its image in $S$. Note that the action of $x_i$ and $y_i$ are the same for every $i$. This shows that $\mathcal{K}_*(\underline{x}; M)$ and $\mathcal{K}_*(\underline{y}; M)$ are the same. Therefore $H_i(\underline{x}; M) \cong H_i(\underline{y}; M)$ for every $i$.

Note that even if we treat $M$ as an $R$-module at the very beginning, we can still recover the $S$-module structure of $H_i(\underline{x}; M)$: since $M \to^S M$ is $R$-linear and $H_i(\underline{x}; -)$ is a covariant functor, we can recover the action of $s$ on $H_i(\underline{x}; M)$.

3.3. Koszul homology and Tor. Let $A = \mathbb{Z}[X_1, \ldots, X_n]$, $K[X_1, \ldots, X_n]$ or $R[X_1, \ldots, X_n]$ where $X_i$ acts on $M$ exactly as $x_i$, then

Proposition 3.3. $H_i(\underline{x}; M) \cong \text{Tor}_i^R(A/(X_1, \ldots, X_n)A, M)$

Corollary 3.4. Let $\underline{x}$ be a sequence in $R$ and let $I = (\underline{x})R$, then $I$ kills $H_i(\underline{x}; M)$ for all $i$.

3.4. Cohomological Koszul complex. $\mathcal{K}^*(\underline{x}; M) \cong \text{Hom}_R(\mathcal{K}_*(\underline{x}; M))$ The cohomological Koszul complex of $R$ (or $M$) is isomorphic with the homological Koszul complex numbered “backward”.

Theorem 3.5. Let $\underline{x}$ be a possibly improper sequence in $R$ and let $M$ be an $R$-module, then

$$\text{Ext}_R^i(R/(\underline{x})R, M) \cong H^i(\underline{x}; M) \cong H_{n-i}(\underline{x}; M) \cong \text{Tor}_{n-i}^R(R/(\underline{x})R, M)$$