F-REGULARITY

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Contents

1. Motivation

We want to show that every ideal in a regular ring is tightly closed. This depneds on knowing that the frobenius map $F : R \to R$ is flat. This could be derived from following theorem

Theorem 1.1. *Let (R, m, K) be a regular Noetherian local ring and M an R-module. Then M is a big C-M module over R iff M is faithfully flat over R.*

First since the Frobenius map is flat: $R_P \to R_P$ is flat hence R_P is a R_P C-M module, hence R is a R C-M module and therefore a faithfully flat module.

We need following result on colon operations:

Proposition 1.2. *Let $R \to S$ be flat and let I and J be ideals of R such that J is finitely generated. Then $(IS :_R JS) = (I :_R J)S$.*

Proof. Let $J = (f_1, ..., f_n)R$, then we have an exact sequence

$$0 \to (I :_R J) \to R \to (R/I)^{\oplus n}$$

where the rightmost map is $r \mapsto (rf_1, ..., rf_n)$. This stays exact after we apply $- \otimes_R S$, hence

$$0 \to (I :_R J)S \to S \to (S/IS)^{\oplus n}$$

But the kernel is $(IS :_R JS)$. So we're done. □

We see immediately that $(I^{[q]} :_R J^{[q]}) = (I :_R J)^{[q]}$ where J is finitely generated since the Frobenius map is flat.

Theorem 1.3. *All ideals in a regular Noetherian ring is tightly closed.*

Proof. Let I be an ideal of R such that $I^* \neq I$, let $u \in I^* - I$. Chhose a prime in the support of $(I + Ru)/I$, then we have $u \in I^* R_P - IR_P$. So we can reduce to the local case: (R, m, K) is regular local and $u \in I^* - I$. We know that there is some nonzero $c \in R^\circ$ such that

$$c \in (I^{[q]} : u^q) = (I : u)^{[q]} \subseteq m^{[q]}$$

for $q \geq q_0$. Then $c \in \cap_n m^n = 0$, which is a contradiction! □

2. Weakly F-regular & F-regular rings

2.1. **Definition.**

Definition 2.1. Let R be a Noetherian ring of prime characteristic $p > 0$. R is *weakly F-regular* if every ideal is tightly closed. R is *F-regular* if all of its localizations are weakly F-regular.

2.2. **Useful Lemma.**

Lemma 2.2. *Let R be any Noetherian ring and M be a finitely generated module. Let $u \in M$. Suppose that $N \subseteq M$ is maximal with respect to the condition that $u \notin N$. Then*

(1) M/N has finite length
(2) $\mathrm{Ass}(M/N)$ contains a unique maximal ideal m
(3) u spans the scole $\mathrm{Ann}_{M/N}\, m$ of M/N

Proof. The maximality of N implies that the image of u is in every nonzero submodule of M/N. We change notation:

- Replace M by M/N
- Replace N by 0
- Replace u by its image in M/N

So u is a nonzero element in M such that every nonzero sudmodule contains u. We have to show that $\mathrm{Ass}(M) = \{m\}$ for some maximal ideal m, we show this in two steps:

First we show that $\mathrm{Ass}(M)$ contains only one prime. Suppose it has two associated primes P_1 and P_2, then we have injections $R/P_1 \to M$ and $R/P_2 \to M$, which shows that there are submodules N_1 and N_2 isomorphic to R/P_1 and R/P_2 respectively. Since $u \in N_1 \cap N_2$, the intersection is nonzero. But this is impossible unless $P_1 = P_2$.

Next we show that the unique associated prime (denote P) is maximal. Consider the injection $R/P \to M$, u is contained in R/P therefore in any ideal of R/P. If P is not maximal, then there is some other prime ideal Q in R/P and $u \in Q$. Since u is in every nonzero submodule, u is contained in any power of Q. In particular, this is true when we pass to the local ring $(R/P)_Q$, but then $0 \neq u \in \cap_n (Q(R/P)_Q)^n = 0$, which is a contradiction!

Thus $\mathrm{Ass}(M) = \{m\}$ and M has finite length (a power of m will kill M as M has a finite filtration with factors R/m). The scole $\mathrm{Ann}_M\, m$ is a submodule of M, hence, contains u. If u doesn't span $\mathrm{Ann}_M\, m$, then any nonzero element not in Ru will span a submodule doesn't contain u. So $\mathrm{Ann}_M\, m$ must be one-dimensional as a R/m-vector space. □

2.3. **Criterion for weak F-regularity.** We want to give equivalent characterizations of weak F-regularity, we begin with a proposition

Proposition 2.3. *Let R be a Noetherian ring and let W be a multiplicative system. Then every element of $(W^{-1}R)^{\circ}$ has the form $\frac{c}{w}$ where $c \in R^{\circ}$ and $w \in W$*

Before giving the proof, we need the prime avoidance lemma for cosets, which we stated below

Lemma 2.4. *Let R be any commutative ring and r an element of R. Let I be an ideal of R and $P_1, ..., P_k$ prime ideals of R. Suppose that the coset $r + I$ is contained in $\cup_{i=1}^{k} P_i$, then there exists some j such that $rR + I \subseteq P_j$.*

Proof of the Proposition 2.3. Suppose that $\frac{c}{w} \in (W^{-1}R)^{\circ}$ where $c \in R$ and $w \in W$. Let $P_1, ..., P_k$ be the minimal primes of R that do not meet W, then $P_i W^{-1}R$ are all minimal prime ideals in $W^{-1}R$. The intersection $P_1 \cap \cdots \cap P_k$ is nilpotent in $W^{-1}R$ therefore we could choose N such that $(P_1 \cap \cdots \cap P_k)^N = 0$ in $W^{-1}R$. Let $I = (P_1 \cap \cdots \cap P_k)^N$.

If $c + I$ is contained in the union of all minimal primes of R, then by the Lemma 2.4 above we have $cR + I \subseteq P$ for some minimal prime ideal P. Then $I \subseteq P$, we have that $P_1 \cap \cdots \cap P_k \subseteq P$, and it follows that $P_j = P$ for some j. But then $c \in P_j \to \frac{c}{1} \in P_j W^{-1}R$, a contradiction, since $\frac{c}{1} \in (W^{-1}R)^{\circ}$. Thus we could choose $a \in I$ such that $c + a$ is in R° and we have $\frac{c}{w} = \frac{c+a}{w}$. □

Lemma 2.5. *Let R be a Noetherian ring of prime characteristic $p > 0$. Let I be an ideal of R primary to a maximal ideal m of R. Then I is tightly closed in R if and only if IR_m is tightly closed.*

Proof. First we note that R_m is flat over R, so by [PROP 2.11 in Tight-closure] we see that $I^* R_m \subseteq (IR_m)^*$. Therefore, if $a \in I^* - I$, then $a \in I^* R_m - IR_m \subseteq (IR_m)^* - IR_m$. So if IR_m is tightly closed, then I is tightly closed.

For the converse direction, if $\frac{r}{w} \in (IR_m)^* - IR_m$ where $r \in R$ and $w \in W$, then we can clear the denominator and assume that $\frac{r}{1} \in (IR_m)^* - IR_m$. Let $c \in (R_m)^{\circ}$ be the element such that $c(\frac{r}{1})^q \in (IR_m)^{[q]}$ for all $q >> 0$. By the Propostion 2.3 above, we know that $c = \frac{c'}{w'}$ where $c' \in R^{\circ}$ and $w' \in W$. Again we can clear denominator and replace c by c'. Then $\frac{c' r^q}{1} \in (IR_m)^{[q]} = I^{[q]} R_m$ for all $q >> 0$. Notice that $I^{[q]}$ is still m-primary and we have following commutative diagram:

$$\begin{array}{ccc} R & \longrightarrow & R_m \\ \downarrow & & \downarrow \\ R/I^{[q]} & \xrightarrow{\simeq} & R_m/I^{[q]}R_m \end{array}$$

The bottom isomorphism is because $R/I^{[q]}$ is already a local ring with maximal ideal m. Since the image of $c' r^q$ from the upper corner is zero, it follows that $c' r^q \in I^{[q]}$ for all $q >> 0$. So $r \in I^* = I \to \frac{r}{1} \in IR_m$, a contradiction! □

We have following theorem

Theorem 2.6. *Let R be a Noetherian ring of prime characteristic $p > 0$, then TFAE:*

(a) R is weakly F-regular
(b) R_m is weakly F-regular for every maximal ideal m
(c) Every ideal primary to a maximal ideal is tightly closed.

Proof. Clearly (a) implies (c). To see that (c) implies (a), suppose we have a counterexample: there is some element $r \in I^* - I$, then choose J to be maximal in R with respect to the property that J contains I but not r. Then by LEM 2.2, we see that R/J has a unique associated prime m which is also maximal. So J is m-primary. But then $r \notin J$ and $r \in I^* \subseteq J^*$, a contradiction!

Now we know that (b) $\Leftrightarrow$ Every m-primary ideal in R_m is tightly closed. This is equivalent to (c) by LEM 2.5. ☐

3. Strongly F-regular rings

3.1. Definition.

Definition 3.1. Let R be a Noetherian ring of prime characteristic $p > 0$. Suppose that R is F-finite and reduced. Then R is *strongly F-regular* if for every $c \in R^\circ$, there exists q_c such that the map

$$R \to R^{1/q_c}$$
$$1 \mapsto c^{1/q_c}$$

splits over R.

The power q_c usually depends on c. For example, one may need larger q for c^p than for c.

Remark 3.2. This fact is quite useful in the following proof: If $f : R \to S$ is a ring homomorphism and M is an S-module. For any R-linear map $R \to M$, if it splits, then R is a direct summand of S.

Suppose the R-linear map is given by $1 \mapsto u$, and the splitting map is $\theta : M \to R$. Then the splitting map $S \to R$ could be defined as $\phi(s) \coloneqq \theta(su)$.

Note that in the remark, $R \to M$ split means that Ru isomorphic to R and Ru is a direct summand of M.

3.2. Properties. Apply Remark 3.2 to $S = R^{1/q_c}$ we immediately have following:

Proposition 3.3. *A strongly F-regular ring R is F-split.*

Proof. See [F-split ring] ☐

Next we note following:

Proposition 3.4. *Suppose that R is reduced Noetherian of prime characteristic $p > 0$ and $c \in R^\circ$. If the map $R \to R^{1/q_c}$ sending $1 \mapsto c^{1/q_c}$ splits, then for all $q \geq q_c$, the map $R \to R^{1/q}$ sending $1 \mapsto c^{1/q}$ splits.*

Proof. It sufficies to show that we have a splitting map for $R \to R^{1/pq_c}$, then the result follows by induction.

Now assume that $\theta : R^{1/q_c} \to R$ sending c^{1/q_c} to 1 is the splitting map. We can pass to a map $\theta' : R^{1/pq_c} \to R^{1/p}$ by taking pth root on both sides:

$$\begin{array}{ccc} R^{1/pq_c} & \xrightarrow{\theta'} & R^{1/p} \\ \uparrow & & \uparrow \\ R^{1/q_c} & \xrightarrow{\theta} & R \end{array}$$

So $\theta' : R^{1/pq_c} \to R^{1/p}$ sending c^{1/pq_c} to 1 in $R^{1/p}$ and is $R^{1/p}$, hence, R-linear. Now by Proposition 3.3 above, there is a splitting map $\phi : R^{1/p} \to R$. Now $\phi \circ \theta'$ will give the desired splitting map. ☐

Following theorem reveals why we have such a strange definition for strong F-regularity.

Theorem 3.5. *Let R be a strongly F-regular ring. Then for every inclusion of $N \subseteq M$ of modules, N is tightly closed in M.*

Proof. First we may map a free module G onto M and let H be the preimage of N. Then it sufficies to show that H is tightly closed in G. Let $u \in H^*$, then there is some $c \in R^\circ$ such that for all $q >> 0$, $cu^q \in H^{[q]}$.

Since R is strongly F-regular, let q_c be the number such that $R \to R^{1/q_c}$ sending $1 \mapsto c^{1/q_c}$ splits.

Now we can choose q to be larger than q_c and fix it. $cu^q \in H^{[q]}$ tells us that

$$cu^q = r_1 h_1^q + \cdots + r_n h_n^q$$

Work in $G \otimes R^{1/q}$ so then we could take qth root on both side and get

$$c^{1/q} u = r_1^{1/q} h_1 + \cdots + r_n^{1/q} h_n$$

But then we can apply the split map $\theta : R^{1/q} \to R$ to both sides and get

$$u = \theta(r_1^{1/q}) h_1 + \cdots \theta(r_n^{1/q}) h_n$$

The right hand side is clearly in H. □

Theorem 3.6. *Let R be an F-finite reduced ring. TFAE:*

(1) R is strongly F-regular.
(2) R_m is strongly F-regular for every maximal ideal m.
(3) $W^{-1}R$ is strongly F-regular for every multiplicative system W.

Proof. (1)→(3): By Proposition 2.3, we know that elements in $(W^{-1}R)^\circ$ has the form $\frac{c}{w}$ where $c \in R^\circ$ and $w \in W$. Given such an element $\frac{c}{w}$, choose q such that $R \to R^{1/q}$ splits. Assume the split map is $\theta : R^{1/q} \to R$ such that $\theta(c^{1/q}) = 1$. Define $\eta : R^{1/q} \to R$ by $\eta(u) = \theta(w^{1/q} u)$. Then η induces a map $W^{-1}R^{1/q} \to W^{-1}R$. Notice that $W^{-1}R^{1/q} = (W^{-1}R)^{1/q}$ and we have $\eta((\frac{c}{w})^{1/q}) = \theta(c^{1/q}) = 1$.

(3)→(2): This is obvious.

(2)→(1): Fix $c \in R^\circ$, for any maximal ideal m of R, the image of c is in R_m°. So there exist q_m such that $R_m \to R_m^{1/q_m}$ sending 1 to c^{1/q_m} splits. There is a Zariski open neighbourhood containing m such that at every prime P in this open neighbourhood the map $R_P \to R_P^{1/q_m}$ splits. These open neighbourhood covers Spec(R). Since Spec(R) is quasicompact, there is a finite cover, thus there finitely many q_m's. Let q_c be the maximal one. Then $R \to R^{1/q_c}$ splits at every prime ideal, hence it splits. □

Corollary 3.7. *A strongly F-regular ring is F-regular.*

Corollary 3.8. *R is strongly F-regular iff it is a finite product of strongly F-regular domains.*

Proposition 3.9. *If S is strongly F-regular and R is a direct summand of S, then R is strongly F-regular.*

Proof. If R and S are domains then let $c \in R^\circ$ be given. Since S is strongly F-regular, we may choose q and S-linear map $\theta : S^{1/q} \to S$ such that $\theta(c^{1/q}) = 1$. Let $\alpha : S \to R$ be R-linear map such that $\alpha(1) = 1$. Then $\alpha \circ \theta : S^{1/q} \to R$ is R-linear and sends $c^{1/q}$ to 1. We may restrict this map to $R^{1/q}$. □

Proposition 3.10. *If $R \to S$ is faithfully flat and S is strongly F-regular, then R is strongly F-regular.*

Proof. Let $c \in R^\circ$, then $c \in S^\circ$. So there exists q and an S-linear map $S^{1/q} \to S$ such that $c^{1/q} \mapsto 1$. There is an obvious map $S \otimes_R R^{1/q} \to S^{1/q}$. This yields a map $S \otimes_R R^{1/q} \to S$. Thus $R \to R^{1/q}$ splits after a faithfully flat base change, which implies that itself splits. □

Theorem 3.11. *An F-finite regular ring is strongly F-regular.*

Proof. We may assume that (R, m, K) is local so it's a domain. Since $\mathcal{F}$ is flat, we have that $R^{1/q}$ is flat over R, hence, free over R since it's also module-finite over R.

Let $c \neq 0$ be given. Choose q so large that $c \notin m^{[q]}$. Then $c^{1/q} \notin mR^{1/q}$, so $c^{1/q}$ is part of a minimal basis for the R-free module $R^{1/q}$. Now we can choose a split map. □

The following result makes the property of being a strongly F-regular ring much easier to test.

Theorem 3.12. *Let R be a reduced F-finite ring of prime characteristic $p > 0$ and let $c \in R^\circ$ be such that R_c is strongly F-regular. Then R is strongly F-regular iff*

There exists q_c such that the map $R \to R^{1/q_c}$ sending $1 \mapsto c^{1/q_c}$ splits.

Proof. The boxed condition for R is obviously necessary. Now assume it, we have to show that R is strongly F-regular. First we see that R is F-split: because the map R to $R^{1/p}$-module R^{1/q_c} splits, so $R \to R^{1/p}$ splits.

For any $d \in R^\circ$, since R_c is strongly F-regular, we can choose q_d and an R_c-linear map $\beta : R_c^{1/q_d} \to R_c$ such that $\beta(d^{1/q_d}) = 1$. Since $\mathrm{Hom}_{R_c}(R_c^{1/q_d}, R_c)$ is the localization of $\mathrm{Hom}_R(R^{1/q_d}, R)$ at c. We have that $\beta = \frac{\alpha}{c^q}$ for some $q = p^e$ where α is a map $R^{1/q_d} \to R$ such that $\alpha(d^{1/q_d}) = c^q$.

By taking qq_cth root we obtain a map:

$$\alpha^{1/qq_c} : R^{1/qq_cq_d} \to R^{1/qq_c}$$
$$d^{1/qq_cq_d} \mapsto c^{1/q_c}$$

Since R is F-split: $R \to R^{1/q}$ splits. So there is a map $\gamma : R^{1/q} \to R$. Therefore we have a map

$$\gamma^{1/q_c} : R^{1/qq_c} \to R^{1/q_c}$$
$$1 \mapsto 1$$
$$c^{1/q_c} \mapsto c^{1/q_c}$$

Finally we have the split map

$$\theta : R^{1/q_c} \to R$$
$$c^{1/q_c} \mapsto 1$$

The composition of all three maps gives what we want. □

3.3. **Cohen-Macaulayness.**

Theorem 3.13. *If an F-finite local ring $(R, \mathfrak{m})$ is strongly F-regular, then R is Cohen-Macaulay.*

Proof. We prove by induction on the dimension of R. First note that everything is preserved by completing. Hence we may assume that $R = S/I$ where S is a complete regular local domain. We prove it by induction on the dimension of R. The case $d = \dim(R) = 0$ is easy.

Suppose that it holds for any $d < \dim(R)$. Let $\dim(S) = n$. Consider $\mathrm{H}^i_{\mathfrak{m}}(R)$. By local duality, we have $\mathrm{H}^i_{\mathfrak{m}}(R)^\vee \cong \mathrm{Ext}^{n-i}_S(R, S)$. Since $\mathrm{Ext}^{n-i}_S(R, S)$ is noetherian and its formation commutes with localization. For any prime $P \subseteq R$ (we also write $P \subseteq S$), we have $\mathrm{Ext}^{n-i}_S(R, S)_P \cong \mathrm{Ext}^{n-i}_{S_P}(R_P, S_P)$. Since S is a local domain, $\dim S_P + \dim S/P = n$ and $S/P \cong R/P$. So $n - i = \dim S_P - (i - \dim R/P)$. By local duality over S_P, we have $\mathrm{Ext}^{n-i}_{S_P}(R_P, S_P) \cong \mathrm{H}^{i-\dim(R/P)}_{PS_P}(R_P)^\vee$. But by induction assumption, we conclude that $\mathrm{H}^{i-\dim(R/P)}_{PS_P}(R_P) = 0$ whenever $i - \dim(R/P) < \mathrm{ht}\, P \Leftrightarrow i < \dim(R)$. Therefore $\mathrm{H}^i_{\mathfrak{m}}(R)$ is of finite length for any $i < \dim(R)$.

We can choose $c \in \mathfrak{m}$ such that c kills all $\mathrm{H}^i_{\mathfrak{m}}(R)$ where $i < \dim(R)$. Now consider the composition map $R \subseteq \mathcal{F}^e_*(R) \overset{\mathcal{F}^e_* c}{\to} \mathcal{F}^e_*(R) \to R$. This is identity on R, which will becom identity on $\mathrm{H}^i_{\mathfrak{m}}(R)$. But the middle map will be zero because $\mathcal{F}^e_* c$ kills $\mathrm{H}^i_{\mathfrak{m}}(\mathcal{F}^e_* R)$ under the isomorphism $R \cong \mathcal{F}^e_* R$. □

3.4. **Geometrically regular maps.**

We want to show that if $R \to S$ is geometrically regular and R is strongly F-regular, then S is strongly F-regular.

We need following result:

Theorem 3.14 (Radu-André)**.** *Let R and S be F-finite rings such that $R \to S$ is geometrically regular. Then for all e, $R^{(e)} \otimes_R S \to S^{(e)}$ is faithfully flat.*

Proof. TO BE ADDED □

If R and S are reduced F-finite rings, then $R^{1/q} \otimes_R S \to S^{1/q}$ is faithfully flat. Note that we also have following proposition

Proposition 3.15. *Let $R \to S$ be faithfully flat map of Noetherian rings such that S is module-finite over R. Then R is a direct summand of S as an R-module*

Proof. Since B is finitely generated A-module, the split issue is local in R. Assume WLOG that (R, m, K) is local, then S is free over R and 1 is not in mS. We can extend 1 to be a set of free basis for S by Nakayama's lemma. Then the split map follows. □

Corollary 3.16. *Let $R \to S$ be geometrically regular of F-finite rings. Then for all q, $R^{1/q} \otimes S \to S^{1/q}$ makes $R^{1/q} \otimes S$ a direct summand of $S^{1/q}$.*

Now we are ready to prove following:

Theorem 3.17. *If $R \to S$ is geometrically regular map of F-finite rings and R is strongly F-regular, then so is S.*

Proof. Since F-finite rings are excellent, we can choose $c \in R^\circ$ such that R_c is regular, then S_c is regular. By Theorem 3.12 we only need to show that $S \to S^{1/q}$ sending $1 \mapsto c^{1/q}$ splits over S. By Corollary 3.16 we already have a split map $S^{1/q} \to R^{1/q} \otimes S$ sending $c^{1/q} \mapsto c^{1/q} \otimes 1$. Now composte it with the map $R^{1/q} \otimes S \to R \otimes S = S$ which sending $c^{1/q} \otimes 1 \mapsto 1 \otimes 1$ and we are done. □

4. Gorenstein case

4.1. Preliminary.

Proposition 4.1. *Let R be a Noetherian ring of prime characteristic $p > 0$, then TFAE:*

(1) If $N \subseteq M$ are arbitrary modules, than N is tightly closed in M
(2) For every maximal ideal m of R, 0 is tightly closed (over R) in $E_R(R/m)$
(3) For eveery maximal ideal m of R, if u generates the $\operatorname{soc}(E_R(R/m))$, then u is not in the tight closure of 0 in $E_R(R/m)$.

Proof. Evidently we have (1)→(2)→(3). We also note that (3)→(2) is obvious: since $R/m \to E_R(R/m)$ is essential, if 0^* is nonzero, it will contain u.

Now assume (2) and (3), we want to prove (1): Let $u \in N_M^* - N$, we may replace N by a module maximal with respect to containing N and not containing u. By passing to M/N we see that $u \in 0^* - 0$ and u generates the socle of the finite length module M. M is an essential extension of $Ru \cong K$, therefore $K \to E \to M$ and u also generates the socle in E. Now $u \in 0_M^* \to u \in 0_E^*$. □

4.2. **Gorenstein local ring.** Suppose that $(R, \mathfrak{m})$ is a F-finite Gorenstein local ring, then R has a canonical module ω_R. Apply the functor $\operatorname{Hom}_R(-, \omega_R)$ to the natural inclusion $R \subseteq R^{1/p^e}$ yilds a map $\operatorname{Hom}_R(R^{1/p^e}, \omega_R) \to \omega_R$.

We have $\operatorname{Hom}_R(R^{1/p^e}, \omega_R) \cong \omega_{R^{1/p^e}}$ and $\omega_R \cong R, \omega_{R^{1/p^e}} \cong R^{1/p^e}$, Hence we get almost splitting map $R^{1/p^e} \to R$, call it Φ.

Lemma 4.2. *The R-linear map $\Phi : R^{1/p^e} \to R$ generates $\operatorname{Hom}_R(R^{1/p^e}, R)$ as an R^{1/p^e}-module.*

Proof. If we dual everything back, Φ will corresponds to the natural inclusion map taking 1 to 1. Hence Φ has to be the generator up to a unit. (DETALIS TO ADD LATER). □