F-RATIONAL RINGS

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1. Definition

1.1. Definition.

Definition 1.1. A local ring (R, m, K) is F-rational if it's a holomorphic image of a Cohen-Macaulay ring and every ideal generated by a system of parameters is tightly closed.

Theorem 1.2. *If R is F-rational, then*

- *R is Cohen-Macaulay.*
- *R is normal.*
- *Every ideal generated by part of a system of parameters is tightly closed.*

Proof. Let $x_1, ..., x_k$ be part of a system of parameters and let $I = (x_1, ..., x_k)$. Let $x_1, ..., x_n$ be a system of parameters for R and let $I_t = (x_1, ..., x_k, x_{k+1}^t, ..., x_n^t)$. Then for all t, $I \subseteq J_t$ and J_t is tightly closed. So $I^* \subseteq J_t \Rightarrow I^* \subseteq \cap_t J_t = I$, as required.

We see that (0) and principal ideals generated by nonzerodivisors are tightly closed, so R is a normal domain.

By colon-capturing, we know R is C-M. □

Theorem 1.3. *Let (R, m, K) be a local ring and C-M. If one ideal generated by a system of parameters is tightly closed, then R is F-rational, i.e. every ideal generated by part of any system of parameters is tightly closed.*

Proof. Let $I_t = (x_1^t, ..., x_n^t)$. We shall show that all I_t are tightly closed. Choose any nonzero element $u \in I_t^*/I_t$, since I_t^*/I_t has finite length. There is a nonzero multiple of u has annihilator m, i.e. it's in $\mathrm{soc}(I_t^*/I_t) \subseteq \mathrm{soc}(R/I_t)$.

Notice that we have an isomorphism $\mathrm{soc}(R/I) \to \mathrm{soc}(R/I_t)$ given by multiplication by $x_1^{t-1} \cdots x_n^{t-1}$. So $v = x_1^{t-1} \cdots x_n^{t-1} w$ for some $w \in \mathrm{soc}(R/I)$.

Now since $v = x_1^{t-1} \cdots x_n^{t-1} w \in I_t^*$, so we have

$$c(x_1^{t-1} \cdots x_n^{t-1} w)^q \in I_t^{[q]}$$

for some $c \in R^\circ$ and all sufficiently large q. which is

$$(x_1^q \cdots x_n^q)^{t-1}(cw^q) \in ((x_1^q)^t, ..., (x_n^q)^t)R$$

Since $x_1^q, ..., x_n^q$ also form a regular sequence, we see that

$$cw^q \in I_t^{[q]}$$

for all sufficiently large q. Therefore $w \in I^* = I$. But we have $w = 0$ in $\mathrm{soc}(R/I)$, a contradiction!

Now for any other system of parameters $y_1, ..., y_n$, there is some t such that $(x_1^t, ..., x_n^t) \subseteq (y_1, ..., y_n)$. Hence we have an injection $R/(y_1, ..., y_n)R \to R/(x_1^t, ..., x_n^t)R$. Since (0) is tightly closed in the latter one, it is necesarily closed in the former one. □

Next we show that we don't need the C-M condition:

Theorem 1.4. *Let (R, m, K) be a reduced local ring that is a holomorphic image of a C-M ring and let $x_1, ..., x_n)$ be a sequence of elements of m such that*

- *$I_k = (x_1, ..., x_k)$ has height k modulo every minimal prime of R for every k.*
- *R has a test element.*
- *$(x_1, ..., x_n)R$ is tightly closed.*

Then I_k is tightly closed and $x_1, ..., x_n$ is a regular sequence in R.

Proof. We show this by reverse induction on k. $k = n$ is true by assumption. Now assume that I_{k+1}^* is tightly closed, let $u \in I_k^*$ be given. Since $u \in I_k^* \subseteq I_{k+1}^* = I_{k+1} = I_k + x_{k+1}R$, we can write

$$u = v + x_{k+1}r$$

where $v \in I_k$ and $r \in R$. Then both u and v are in I_k^* so we have $r \in (I_k^* : x_{k+1})$. Since R has a test element, by the second part of colon capturing we have

$$r \in (I_k^* : x_{k+1}) \subseteq I_k^*$$

so $u \in I_k + x_{k+1}I_k^*$ and u is arbitrary, we have $I_k^* \subseteq I_k + x_{k+1}I_k^*$. Now passsing to modulo I_k we have $I_k^* \subseteq x_{k+1}I_k^* \subseteq I_k^*$, which implies that it's zero by Nakayama's lemma. Hence $I_k = I_k^*$. □

Now following result follows immediately

Theorem 1.5. *Let (R, m, K) be reduced, equidimensional local ring that is a homomorphic image of a C-M ring. Suppose that R has a test element and one ideal generated by some system of parameters is tightly closed, then R is F-rational.*

Proposition 1.6. *A localization of an F-rational ring at any prime is F-rational.*

Proof. TO BE ADDED. □

F-rational rings behaviour extremely well in the case of Gorenstein rings:

Theorem 1.7. *Let (R, m, K) be a reduced Gorenstein local ring. Let $x_1, ..., x_n$ be a system of parameters for R and let $u \in R$ be a generator of* $\operatorname{soc}(R/(x_1, ...x_n)R)$. *TFAE:*

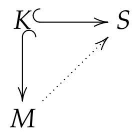
(1) R is weakly F-regular
(2) R is F-rational
(3) $(x_1, ..., x_n)R$ is tightly closed
(4) $u \notin ((x_1, ..., x_n)R)^*$

Proof. Clearly (1)⇒(2), while (2)⇔(3)⇔(4) is easy: First of all, (2)⇒(3) is obvious. Since R is C-M, we also have that (3)⇒(2). If (3) holds, then 0 is tightly closed therefore $u \notin 0^*$ in $\operatorname{soc}(R/(x_1, ..., x_n)R)$. So $u \notin ((x_1, ..., x_n)R)^*$. If (4) holds and $(x_1, ..., x_n)R$ is not tightly closed, then the image of its tight closure in $\operatorname{soc}(R/(x_1, ..., x_n)R)$ is nonzero submodule, therefore it must contain u, a contradiction!

Now we want to show (2)&(3)&(4)⇒(1): Assume R is F-rational, let $N \subseteq M$ be two finitely generated R-modules, we want to show that N is tightly closed.

Suppose not, choose $v \in N^* - N$. We may replace N by a maximal submodule N' not containing v. Then $v \in N'^* - N'$. Kill N' we can assume WLOG that M is finite length and 0 is not tightly closed, i.e. there is some nonzero element $v \in 0^*$.

Since M is finite length, some power of m will kill M. In other words, every x_i^t kills M. So M is actually an $S = R/(x_1^t, ..., x_n^t)R$-module. Now S is a Gorenstein Artin local ring. Since both M and S [See Useful Lemma in F-regularity] has a one-dimensional socle. We have following diagram:



The dotted map comes from the injectivity of S over S. We claim that this is also an injection: If not, v is in the kernel. But the scoles are isomorphic. So we have an injection $M \to S$. But 0 is tightly closed in S. So 0 is tightly closed in M, a contradiction! □