F-finite Rings

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1 Definition

1.1 Definition

Definition 1.1. *A Noetherian ring R of prime characteristic $p > 0$ is **F-finite** if the Frobenius map $F : R \to R$ makes R into a module-finite R-algebra.*

Note that this is equivalent to say that R is module-finite over $R^p = \{r^p : r \in R\}$.

If furthur more we have R reduced, this is also equivalent to $R^{1/p}$ is module-finite over R.

1.2 Properties of F-fintie rings

Proposition 1.2. *Let R be a Noetherian ring of prime characteristic $p > 0$. If R is F-finite, then so is every ring essentially of finite type over R*

Proof. Since R is module-finite over R^p, assume that $u_1, ..., u_n$ spans R over R^p.

We need to show three cases:

(1) If $S = R[x_1, ..., x_n]$, then S is F-finite. By induction it sufficies to show that $R[x]$ is finite. But clearly $u_i x^j (1 \leq i \leq n, 1 \leq j \leq p-1$ spans $R[x]$ over $(R[x])^p$.

(2) If $S = R/I$ for some ideal I. Then the image of $u_1, ..., u_n$ will span R/I over $(R/I)^p$.

(3) If $S = W^{-1}R$, then the image of $u_1, ..., u_n$ will span $W^{-1}R$ over $(W^{-1}R)^p$. Because $(W^{-1}R)^p = W^{-1}(R^p)$ as inverting r^p is the same as inverting r. ☐

By the same idea in the proof, we can prove following

Proposition 1.3. *Let R be a Noetherian ring of prime characteristic $p > 0$. If R is F-finite, so is the formal power series ring $R[[x_1, ..., x_n]]$*

Proof. Again by induction we only have to show $R[[x]]$ is F-finite. Let $u_1, ..., u_n$ span R over R^p, then $u_i x^j$ spans $R[[x]]$ over $R^p[[x^p]]$. □

Also note that Proposition 1.2 has following corollary:

Corollary 1.4. *If K is a field finitely generated as a field over a perfect field, then every algebra essentially of finite type over K is F-finite.*

Proof. Note that Frobenius map is an automorphism of a perfect field. So a perfect field is always F-finite. K is an algebra essentially of finite type over this perfect field. So K is F-finite. The result follows by Proposition 1.2. □

Proposition 1.5. *Let R be a Noetherian ring of prime characteristic $p > 0$, then R is F-fintie iff R_{red} is F-finite.*

Proof. The "only if" part is clear by Proposition 1.2 above. We only need to show the "if" part. Let J be the nilradical of R. Then R/J is module-finite over $(R/J)^p$ by $u_1, ..., u_n$. Suppose J is generated by $v_1, ..., v_m$, then we have

$$\begin{aligned} R &= R^p u_1 + \cdots + R^p u_n + J \\ &= R^p u_1 + \cdots + R^p u_n + R v_1 + \cdots + R v_m \end{aligned}$$

If we replace R in the second expansion by the first expansion, we get

$$R = \sum_{i=1}^{n} R^p u_i + \sum_{i,j} R^p u_i v_j + \sum_{j=1}^{m} J v_j$$

The last term is J^2, so R/J^2 is module-finite over $(R/J^2)^p$. By induction we have R/J^{2^k} is module-finite over $(R/J^{2^k})^p$. Since J is the nilradical, there is some power 2^k kills J. Then R is F-finite. □

Proposition 1.6. *Let R be a Noetherian ring of prime characteristic $p > 0$. TFAE:*

(1) $F : R \to R$ is module-finite (R is F-finite)

(2) $F^e : R \to R$ is module-finite for all $e \geq 1$

(3) $F^e : R \to R$ is module-finite for some $e = e_0$

Proof. Clearly (1)⇒(2)⇒(3). We only need to show (3)⇒(1): If F^e is module-finite, so is its reduced ring R_{red}. Then we have $R_{\text{red}} \subseteq R_{\text{red}}^{1/p} \subseteq R_{\text{red}}^{1/q}$. Thus $R_{\text{red}}^{1/p}$ is module-finite since it's a subring of a module-finite extension. Then R_{red} is module-finite therefore R is module-finite by Proposition 1.5. □

Proposition 1.7. *Let R be a Noetherian ring of prime characteristic $p > 0$. If (R, m, K) is a complete local ring, then R is F-finite iff K is F-finite.*

Proof. If R is F-finite, then $K = R/m$ is F-finite. Assume that K is F-finite, then by the structure theorem of complete local ring [see Complete-local], R is a holomorphic image of a formal power series ring $K[[x_1, ..., x_n]]$. Then by Proposition 1.3 and Proposition 1.2, R is F-finite. □

2 Knuz’s Theorem

2.1 The theorem

Theorem 2.1 (E. Knuz)**.** *Every F-finite ring is excellent.*