COHEN-MACAULAY RINGS

ZHAN JIANG

CONTENTS

1. PRELIMINARY

1.1. Regular sequences.

Definition 1.1. If $x_1, ..., x_n \in R$ and M is an R-module, the sequence $x_1, ..., x_n$ is called **possibly improper regular sequence** on M if x_1 is an NZD on M and for all i, x_{i+1} is an NZD on $M/(x_1, ..., x_i)M$. A possibly improper regular sequence is a **regular sequence** if $(x_1, ..., x_n)M \neq M$. If $(x_1, ..., x_n)M = M$, then the regular sequence is **improper**.

Note that if $M = 0$, then any sequence is an improper regular sequence. And hence any sequence starting with an unit is an improper regular sequence on M.

Remark 1.2. Suppose (R, m, K) is local and M is finitely generated, then any sequence $x_1, ..., x_n \in m$ that is a possibly improper regular sequence is automatically a regular sequence by Nakayama's lemma.

If S is a flat R-algebra, then $x_1, ..., x_n$ continues to be a possibly improper regular sequence on $S \otimes M$: By induction on n we only need to show the case $n = 1$. The map $M \hookrightarrow^x M$ stays as injection after applying $- \otimes_R S$.

Note that in general, a regular sequence is not permutable: In the polynomial ring $K[x, y, z]$, $x - 1, xy, xz$ is a regular sequence but $xy, xz, x-1$ is not. But in the local case or homogeneous case we do have permutability. We shall make this precise later

Lemma 1.3. *Let R be a ring and M an R-module. Let $x_1, ..., x_n$ be a possibly improper regular sequence on M. If $u_1, ..., u_n \in M$ such that $\sum_{i=1}^n x_i u_i = 0$. Then $u_i \in (x_1, ..., x_n)M$ for every i.*

Proof. Prove by induction on n. The case $n = 1$ is obvious.

We have $u_n \in (x_1, ..., x_{n-1})M$, which is $u_n = x_1 v_1 + \cdots + x_{n-1} v_{n-1}$. So $x_1(u_1 + x_n v_1) + ... + x_{n-1}(u_{n-1} + x_n v_{n-1}) = 0$. By induction hypothesis, $u_i + x_n v_i \in (x_1, ..., x_{n-1})M$. Thus $u_i \in (x_1, ..., x_n)M$. □

Proposition 1.4. *Let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$ be a finite filtration of M. If $x_1, ..., x_n$ is a possibly improper regular sequence on every factor M_{k+1}/M_k where $0 \leq k \leq m-1$, then it's a possibly improper regular sequence on M. If moreover it's a regular sequence on M/M_{m-1}, then it is a regular sequence on M*

Proof. If we prove the first part, the last statement follows. Let $I = (x_1, ..., x_n)R$, then $IM = M \Rightarrow IM/M_{m-1} = M/M_{m-1}$.

We prove by induction on m. The case $m = 1$ is obvious. Assume $m = 2$: we have

$$0 \to M_1 \to M \to N \to 0$$

$x_1, ..., x_n$ is already an improper regular sequence on M_1 and N. Since $\text{Ass}(M) \subseteq \text{Ass}(M_1) \cup \text{Ass}(N)$. So if x_1 is an NZD on M_1 and N, then x_1 is an NZD on M. So we have an exact sequence:

$$0 \to x_1 M_1 \to x_1 M \to x_1 N \to 0$$

Therefore we have an exact sequence of quotients:

$$0 \to M_1/x_1 M_1 \to M/x_1 M \to N/x_1 N \to 0$$

Then we can inductively prove that $x_1, ..., x_n$ is a possibly improper regular sequence on M by 9-lemma.

Now we can carry through the induction on m. Suppose we know the result for filtrations of length $m-1$. Then $x_1, ..., x_n$ is a possibly improper regular sequence on M_{m-1} by induction hypothesis. On the other hand, it's also possibly improper regular sequence on M/M_{m-1}. So the result follows from the case $m = 2$. □

Theorem 1.5. *Let $x_1, ..., x_n \in R$ and let M be an R-module. $x_1, ..., x_n$ is a (possibly improper) regular sequence on M iff $x_1^{m_1}, ..., x_n^{m_n}$ is a (possibly improper) regular sequence on M where n_i are integers.*

Proof. Let $I = (x_1, ..., x_n)R$ and $J = (x_1^{m_1}, ..., x_n^{m_n})R$. If $IM = M$, then $I^k M = M \Rightarrow M = I^k M \subseteq JM \subseteq M$. If $JM = M$, then $M = JM \subseteq IM \subseteq M$. So the properness is taken care of.

Suppose that $x_1, ..., x_n$ is a p.i.r.s(possibly improper regular sequence) on M, by induction on n, it will suffices to show that $x_1^{m_1}, x_2, ..., x_n$: we may pass to $x_2, ..., x_n$ and $M/x_1^{m_1}M$ and apply induction hypothesis. First $x_1^{m_1}$ is an NZD as x_1 is. Since $M/x_1^{m_1}M$ has a finite filtration by submodules $x_1^j M/x_1^{m_1}M$. Each factor is isomorphic to $M/x_1 M$ on which $x_2, ..., x_n$ is p.i.r.s. So by Proposition 1.4 we see that they continue to be a p.i.r.s. on $M/x_1^{m_1}M$. Now we can do induction.

For the other implication, it will suffice to show that if $x_1^{m_1}, .., x_j^{m_j}, x_{j+1}, ..., x_n$ is a p.i.r.s. then $x_1^{m_1}, ..., x_{j-1}^{m_{j-1}}, x_j, x_{j+1}, ..., x_n$ is a p.i.r.s. After this we can prove by induction. After killing $x_1^{m_1}, ..., x_{j-1}^{m_{j-1}}$, we only need to do the case $x_1^{m_1}, x_2, ..., x_n$ implies $x_1, x_2, ..., x_n$.

First $x_1^{m_1}$ is an NZD, then x_1 is clearly NZD. Want to show that x_k is an NZD on $M/(x_1, ..., x_{k-1})M$: Suppose $x_k u \in (x_1, ..., x_{k-1})M$, then $x_1^{m_1-1}x_k u \in (x_1^{m_1}, ..., x_{k-1})M$. So $x_1^{m_1-1}u \in (x_1^{m_1}, ..., x_{k-1})M$.

$$x_1^{m_1-1}u = x_1^{m_1}v_1 + x_2v_2 + \cdots + x_{k-1}v_{k-1}$$
$$x_1^{m_1-1}(u - x_1v_1) = x_2v_2 + \cdots + x_{k-1}v_{k-1}$$

By induction hypothesis, $x_1, ..., x_{k-1}$ is a p.i.r.s. and therefore $x_1^{m_1-1}, x_2, ..., x_{k-1}$ is a p.i.r.s. Now we can apply Lemma 1.3 to conclude that

$$u - x_1v_1 \in (x_1^{m_1-1}, x_2, ..., x_{n-1})M$$

So $u \in (x_1, x_2, ..., x_{n-1})M$, as required. □

Theorem 1.6. *Let $x_1, ..., x_n$ be a regular sequence on the R-module M. Let $I = (x_1, ..., x_n)R$. Suppose that $u \in M$ such that*

$$x_1^{m_1} \cdots x_n^{m_n} u \in (x_1^{m_1+1}, ..., x_n^{m_n+1})M$$

where m_i are integers, then $u \in IM$.

Proof. We can write

$$x_1^{m_1} \cdots x_n^{m_n} u = x_1^{m_1+1}v_1 + ... + x_n^{m_n+1}v_n$$

We prove by induction on the number of nonzero m_j's. If all are zero then the conclusion follows.

If $m_j > 0$, let $x = \prod_{i \neq j} x_j^{m_j}$. Then

$$x_i^{m_i} xu = x_1^{m_1+1}v_1 + ... + x_n^{m_n+1}v_n$$
$$x_i^{m_i}(xu - x_iv_i) = x_1^{m_1+1}v_1 + ... + x_{i-1}^{m_{i-1}+1}v_{i-1} + x_{i+1}^{m_{i+1}+1}v_{i+1} + ... + x_n^{m_n+1}v_n$$

Since the powers of x_i's also form a regular sequence. We can apply Lemma 1.3 and conclude that $xu - x_iv_i \in (x_1^{m_1+1}, ..., x_{i-1}^{m_{i-1}+1}, x_i^{m_i}, x_{i+1}^{m_{i+1}+1}, ..., x_n^{m_n+1})M \Rightarrow xu \in (x_1^{m_1+1}, ..., x_{i-1}^{m_{i-1}+1}, x_i, x_{i+1}^{m_{i+1}+1}, ..., x_n^{m_n+1})M$. Now in x there is one fewer nonzero m_j. The conclusion follows from induction. □

Let I be an ideal of R and let M be an R-module. We can form the associated graded ring $\mathrm{gr}_I(R)$ and associated graded module $\mathrm{gr}_I(M)$. Notice that if $x_1, ..., x_n$ generates I, then the classes $[x_i] \in I/I^2$ generate $\mathrm{gr}_I(R)$ as an (R/I)-algebra. So we have a surjection:

$$\theta : (R/I)[X_1, ..., X_n] \to \mathrm{gr}_I(R)$$
$$X_i \mapsto x_i$$

Similarly we have a surjection

$$\theta_M : (R/I)[X_1, ..., X_n] \otimes_{R/I} (M/IM) \to \mathrm{gr}_I(M)$$
$$X_1^{m_1} \cdots X_n^{m_n} \otimes [u] \mapsto [x_1^{m_1} \cdots x_n^{m_n} u]$$

But in the case when $x_1, ..., x_n$ is a regular sequence and $I = (x_1, ..., x_n)R$, we have following theorem

Theorem 1.7. *Let $x_1, ..., x_n$ be a regular sequence on the R-module M and suppose that $I = (x_1, ..., x_n)R$. Then the map $(R/I)[X_1, ..., X_n] \otimes_{R/I} (M/IM) \to \mathrm{gr}_I(M)$ described above is injection and hence isomorphism.*

Proof. Suppose it's not. Then the kernel is a homogeneous ideal. So we can pick an element of the smallest degree, say k. Thus it's equivalent to say that a polynomial with coefficients in $M - IM$ is in $I^{k+1}M$. Pick a monomial $x_1^{a_1} \cdots x_n^{a_n} u$ where $a_1 + \cdots + a_n = k$ and $u \in M - IM$. Then for any other terms in the polynomial, at least one x_j has exponents bigger than a_j. Therefore

$$x_1^{a_1} \cdots x_n^{a_n} u = \sum_{i=1}^{n} x_i^{a_i+1} u_i$$

Now apply Theorem 1.6 we have that $u \in IM$. A contradiction! □

1.2. **Permutability of regular sequence.** If M is a finitely generated nonzero module over a local ring (R, m, K) and the regular sequence is in m or if M is $\mathbb{N}$-graded module over the $\mathbb{N}$-graded ring R and the regular sequence is of positive degree. Then the regular sequence is permutable.

Lemma 1.8. *Suppose that we have either of the following two situations:*

(1) (R, m, K) *is a local ring and* M *is a finitely generated module over* R. $x_1, ..., x_n$ *is a sequence in* m
(2) R *is a* $\mathbb{N}$*-graded ring and* M *is a* $\mathbb{N}$*-graded module.* $x_1, ..., x_n$ *is a sequence in* m *which is the unique maximal homogeneous ideal.*

Proof. We only need to show that we can tranpose first two terms. Now assume that x_1 is an NZD on M and x_2 is an NZD on M/x_1M. First we show that x_2 is an NZD on M: Let $N = \operatorname{Ann}_M x_2$. If $u \in N$, $x_2u = 0 \Rightarrow u \in x_1M$. So $u = x_1v$. Then $x_1(x_2v) = 0 \Rightarrow x_2v = 0$. Therefore $v \in N$. So $N = x_1N$. Nakayama's lemma tells us that $N = 0$.

Now we have to show that x_1 is an NZD on M/x_2M: (This time we don't need local of grading hypothesis) Suppose $x_1u = 0$ in M/x_2M, which is $x_1u = x_2v$. Kill x_1 we see that $v \in x_1M$. So $v = x_1\tilde{v}$. Then $x_1(u - x_2\tilde{v}) = 0$ which implies that $u = x_2\tilde{v}$. So $v = 0$ in M/x_2M. □

1.3. **Freeness over regular rings.**

Lemma 1.9. *Let* K *be a field and assume either that*

(1) R *is a regular local ring of dimension* n *and* $x_1, ..., x_n$ *is a system of parameters. Let* M *be a nonzero finitely generated module.*
(2) $R = K[x_1, ..., x_n]$ *is a graded polynomial ring over* K *in which each of the* x_i *is of positive degree. Let* M *be a nonzero finitely generated* $\mathbb{Z}$*-graded module.*

Then M *is free iff* $x_1, ..., x_n$ *is a regular sequence on* M

Proof. The "only if" part is clear, since $x_1, ..., x_n$ is a regular sequence on R and M is a direct sum of R.

Let $m = (x_1, ..., x_n)R$. Then M/mM is a finite-dimensional K-vector space over R. We can find a basis and lift them back to a set of generators of M by Nakayama's lemma. We also notice that in case (2), we can lift them to a set of homogeneous basis.

We prove by induction on n. Assume $u_1, ..., u_k$ is the set of (homogeneous) basis and $n > 0$ ($n = 0$ is clear). Let $N = \{(r_1, ..., r_k) \in R^k : r_1u_1 + \cdots + r_ku_k = 0\}$. By the induction hypothesis, the image of u_i's in M/x_1M are a free basis. It follows that every r_i is zero in R/x_1R. So $r_i = x_1s_i$. Therefore $x_1(s_1u_1 + \cdots + s_ku_k) = 0$. Since x_1 is an NZD, we have $(s_1, ..., s_k) \in N$. So $x_1N = N \Rightarrow N = 0$ by Nakayama's lemma. □

1.4. **Transition from one system of parameters to another.**

Definition 1.10. A **system of parameters** for a local ring (R, m, K) is a sequence of elements $x_1, ..., x_n \in m$ such that m is nilpotent over $(x_1, ..., x_n)$.

Definition 1.11. Suppose R is a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$. A **homogeneous system of parameters** for R is a sequence of homogeneous elements $f_1, ..., f_n$ in R such that $\dim(R/(f_1, ..., f_n)R) = 0$.

We can use Homogeneous Prime Avoidance to show the existence of a homogeneous system of parameters.

Suppose we are in one of the two situations:

- (R, m, K) is local and $f_1, ..., f_n$ and $g_1, ..., g_n$ are two system of parameters
- R is finitely generated $\mathbb{N}$-graded over $R_0 = K$, m is the homogeneous maximal ideal and $f_1, ..., f_n$ and $g_1, ..., g_n$ are two homogeneous system of parameters for R.

In either situation we have a finite sequence of system of parameters starting with $f_1, ..., f_n$ and ending with $g_1, ..., g_n$ such that two consecutive elements of the sequence agree in all but one element.

We do this by induction on n: If $n = 1$ we're done. If $n > 1$, we can choose h to avoid all minimal primes of $(f_1, ..., f_{n-1})$ and all minimal primes of $(g_1, ..., g_{n-1})$. Then the question reduces to find a transition from $f_1, ..., f_{n-1}, h$ to $g_1, ..., g_{n-1}, h$, which has shorter length. So the conclusion follows.

2. Local Case

2.1. Depth.

First we have a lemma

Lemma 2.1. *Let $R \to S$ be a homomorphism of Noetherian rings. Let I be an ideal of R and let M be a finitely generated S-module. Then $IM = M$ iff $IS + \mathrm{Ann}_S M$ is the unit ideal.*

Proof. If $IM = M$, then choose a set of generators $u_1, ..., u_n$. Each is an I-linear combination of u_i's. Therefore $U = AU \Rightarrow (I - A)U = 0$. In particular, $\det(I - A)U = 0$. So $\det(I - A) = 1 + a \in \mathrm{Ann}_S M$ where $a \in IS$. Then $IS + \mathrm{Ann}_S M = S$.

If $IS + \mathrm{Ann}_S M = S$, then $IM = (IS + \mathrm{Ann}_S M)M = SM = M$. ☐

Actually we have a faster prove: $I + J = S$ iff $V(I) \cap V(J) = \emptyset$. But we also know that $\mathrm{Supp}(M/IM) = \mathrm{Supp}(S/IS \otimes M) = \mathrm{Supp}(S/IS) \cap \mathrm{Supp}(M)$.

Next we can prove following

Theorem 2.2. *Every maximal regular sequence on M in I is of the same finite length.*

Proof. First we show that every sequence if finite: suppose we have a regular sequence $x_1, ..., x_n, ...$. Let $I_n = (x_1, ..., x_n)$, then $I_n \subseteq I_{n+1}$ must terminate. Thus $I_{n+1} = I_n$, then $x_{n+1} \in I_n$ which means the action of x_{n+1} on M/I_nM is zero. A contradiction!

Now assume that we have two sequences $x_1, ..., x_n$ and $y_1, ..., y_m$. Assume WLOG that $n \leq m$. Let Q be an associated prime of $M/(x_1, ..., x_n)M$ contaning I and let P be its contraction to R. We can replace R be R_P, S by S_Q and M by M_Q. Therefore assume WLOG that $R \to S$ is local.

If $n = 0$, then P consists entirely of zerodivisors. Therefore $m = 0$. If $n = 1$, since P is an associated prime of M/x_1M. There is an element $u \in M/x_1M$ of which the annihilator is P. Since $y_1 \in P$, we see $y_1u \in x_1M$. Then $y_1u = x_1v$. Now we prove that P is the associated prime of M/y_1M by proving that $\mathrm{Ann}_{M/y_1M}\, v = P$. If $r \in R$ kills v, i.e. $rv = y_1w$. Since x_1, y_1 are NZD. This holds iff $x_1rv = x_1y_1w \Leftrightarrow y_1ru = y_1x_1w \Leftrightarrow ru = x_1w$, which implies that $r \in P$. Meanwhile any other element in R is an unit.

Now suppose that $n > 1$. Since $x_1, ..., x_{n-1}$ and $y_1, ..., y_{m-1}$ are both regular sequences that are not maximal. Then I is not contained in any associated prime of $M/(x_1, ..., x_{n-1})M$ nor in any associated prime of $M/(y_1, ..., y_{m-1})M$. Thus there is an NZD z on both module. Since z is NZD on $M/(x_1, ..., x_{n-1})M$ and x_n is already a maximal regular sequence. By the case case $n = 1$, we see that z is also maximal. Therefore $(x_1, ..., x_{n-1}, z)$ is maximal. The same argument proves that $(y_1, ..., y_{m-1}, z)$ is also maximal. Now by permutability of regular sequence, $(z, x_1, ..., x_{n-1})$ and $(z, y_1, ..., y_{m-1})$ are regular sequence. Now kill z and by induction we see $n - 1 = m - 1$. Thus $n = m$. ☐

Now we can give following definition

Definition 2.3. Let $R \to S$ be a homomorphism of Noetherian rings and $I \subseteq R$ an ideal. Let M be a finitely generated S-module. If $IM \neq M$ we define the **depth** of M on I to be the maximal length of any maximal regular sequence in M on I. If $IM = M$, we use the convention that the **depth**$= \infty$.

Proposition 2.4. *Let R be a local ring and let M be a finitely generated R-module. Let $I \subseteq R$ be an ideal such that $IM \neq M$. If $(x_1, ..., x_d)$ is a regular sequence on M, then $\dim(M/(x_1, ..., x_d)M) = \dim(M) - d$ and $\mathrm{depth}_I(M/(x_1, ..., x_d)M) = \mathrm{depth}_I(M) - d$.*

Proof. We only need to prove the case $d = 1$. Once we proved it, every time x_i is an NZD on $M/(x_1, ..., x_{i-1})M$, and the statement follows.

The depth statement is easy as every maximal regular sequence is of the same length. For the dimension formula: replace R by $R/\operatorname{Ann}_R M$ then $\dim(M) = \dim(R)$. Since x_1 is an NZD on M, it avoids all associated primes of M, hence it avoids all minimal primes. So $\dim(R/x_1R) = \dim(M/x_1M) \leq \dim(R)-1$. On the other hand, the minimal primes over x_1R has height at most 1. So $\dim(R/x_1R) \geq \dim(R) - 1 \Rightarrow \dim(R/x_1R) = \dim(R) - 1$. □

Corollary 2.5. *We have*

(1) $\operatorname{depth}_I(M) = \inf_P \operatorname{depth}_{IR_P}(M_P)$
(2) $\operatorname{depth}_I(M) = \inf_P \operatorname{depth}_{PR_P}(M_P)$

where P runs through all prime ideals in the support of M/IM.

Proof. (1):First of all, we know that localing at P will only increase the depth: a regular sequence on M will continue to be regular sequence. We only need to show that at some prime P, $\operatorname{depth}_I M = \operatorname{depth}_{IR_P} M_P$.

Fix a regular sequence $x_1, ..., x_n$, since $M_P/(x_1, ..., x_n)M_P \cong (M/(x_1, ..., x_n)M)_P$, we only need to show that if $\operatorname{depth}_I(M) = 0$, then $\operatorname{depth}_{IR_P}(M_P) = 0$ for some P. Now since $\operatorname{depth}_I(M) = 0$, I consists of zerodivisors on M. Therefore I is contained in some associated prime P. The inclusion still holds $IR_P \subseteq PR_P$. Therefore the depth is still zero. On the other hand, if $(M/IM)_P = M_P/IM_P = 0$, then $M_P = IM_P \subseteq PM_P \subseteq M_P \Rightarrow M_P = 0$. This is impossible. So P has to be in the support of M/IM.

(2):It's easy to see that $\operatorname{depth}_{IR_P}(M_P) \leq \operatorname{depth}_{PR_P}(M_P)$. So again we only need to show equality at some P. But this is immediate from the proof of part (1): the P we choose consists of only ZD on M, so $\operatorname{depth}_{PR_P}(M_P) = 0$. □

The Corollary 2.5 enables us to pass a question about depth to a local case, in which case we have permutability and Propsition 2.4.

Now we are ready to prove that

Proposition 2.6. *Let $R \to S$ be Noetherian rings. Let I be an ideal of R and let M be a finitely generated S-module.*

(1) $\operatorname{depth}_I(M) \leq \dim(M) \leq \dim(S)$
(2) $\operatorname{depth}_I(M) \leq \dim(R)$

Proof. (1): The first part $\dim(M) \leq \dim(S)$ is obvious. In the local case, after killing a regular sequence, M is nonzero. So $\dim(M_P) \geq \operatorname{depth}_{IR_P}(M_P)$. Since this is true for any P in the support of M/IM. But any localization only decreases the dimension. So $\dim(M_P) \leq \dim(M)$.

(2): The dimension will only decrease after localization. We can pass to a local ring with depth unchanged. So assume R, S are local. Now kill the annihilator of M in both rings. Now we want to use the induction on depth of M. Let x be an NZD in I on M. Then M/xM has depth 1 less while R/xR and S/xS has dimension 1 less. When we reaches $\dim(R) = 0$, then I consists of nilpotents therefore the depth of M is automatically zero. □

Corollary 2.7. *Let R be a Noetherian ring and let M be a finitely generated R-module, then* $\operatorname{depth}_I(R) \leq \operatorname{ht}(P)$ *where P is in* $\operatorname{Supp}(M) \cap \min(I)$.

2.2. **Cohen-Macaulay rings.** A system of parameters for a local ring (R, m, K) of Krull dimension n is n elements $x_1, ..., x_n$ in m such that $\sqrt{(x_1, ..., x_n)R} = m$.

A local ring is **Cohen-Macaulay** if some (equivalently, every) system of parameters is a regular sequence.

Proposition 2.8. *Let (R, m, K) be a local ring. If one system of parameters is a regular sequence, then every system of parameters are regular sequences.*

Proof. We can choose a chain of system of parameters starting with the given one and ending with an arbitrary one. So we only need to show that when two system of parameters differ by one element, if one is a regular sequence, then so is the other.

Both system of parameters and regular sequences are permutable. We can assume WLOG that the first $\dim(R) - 1$ terms are the same and we kill them. We know that $x^n = cy$ and $y^m = dx$. If x is an NZD, so is y and the converse holds. □

Proposition 2.9. *If (R, m, K) is a CM local ring, then $\mathrm{ht}(I) = \mathrm{depth}_I(R)$ for any ideal I*

Proof. Let $n = \mathrm{ht}(I) = \mathrm{depth}_I(R)$. Choose a maximal sequence of elements of I that is part of a system of parameters, say $x_1, ..., x_k$ where $k < n$. I cannot be contained in any minimal primes of $(x_1, ..., x_k)$: otherwise I will be contained in some minimal prime of $(x_1, .., x_k)$, which has height at most k, a contradiction! So we can choose x_{k+1} not in any minimal prime, therefore $x_1, ..., x_k, x_{k+1}$ is part of a system of parameters for R, contradicting the maximality of the sequence $x_1, ..., x_k$. □

We note that every regular sequence could be extended to a system of parameters:

- A sequence $x_1, ..., x_n$ is a regular sequence if x_i is not in the union of associated primes of $(x_1, ..., x_{i-1})R$
- A sequence $x_1, ..., x_n$ is a system of parameters if x_i is not in the union of minimal primes of $(x_1, ..., x_{i-1})R$

So any regular sequence is automatically part of a system of parameters, and after killing this regular sequence of length d. The dimension drops exactly by d. So we can still choose more elements to make the sequence a system of parameters.

Corollary 2.10. *If (R, m, K) is a CM local ring, then so is R_P for any prime P.*

Proof. By Proposition 2.9, we know that $\dim(R_P) = \mathrm{ht}(P) = \mathrm{ht}(PR_P) = \mathrm{depth}_p(R)$. Choose a regular sequence on R in P, they continue to be a regular sequence on R_P in PR_P. Therefore R_P is Cohen-Macaulay. □

Now it's reasonable to give following definition

Definition 2.11. A Noetherian ring R is **Cohen-Macaulay** if R_P is Cohen-Macaulay local ring for any prime P of R.

By Corollary 2.10 we know that it's equivalent to R_m is Cohen-Macaulay local for any maximal prime m.

Proposition 2.12. *R is CM iff $\mathrm{ht}(I) = \mathrm{depth}_I(R)$ for every ideal I of R.*

Proposition 2.13. *If R is CM, let $x_1, ..., x_n$ be a regular sequence and let $I = (x_1, ..., x_n)R$. Then every associated prime P of I has height n. In particular, I has no mebedded primes.*

Proof. We can localize at P and PR_P is still an associated prime of IR_P, then $x_1, ..., x_n$ could be extended to a system of parameters which is also a regular sequence. But PR_P/IR_P consists entirely of zerodivisors on R_P/IR_P. So it's impossible to extend. Thus R_P/IR_P has dimension 0, which implies that P has height n. □

The Cohen-Macaulay condition is increasingly resetrictive as the Krull dimension increases.

- In dimension 0, every local ring is Cohen-Macaulay.
- In dimension 1, it is sufficient but not necessary that the ring be reduced. The precies characterization is that the maximal ideal not be an embedded prime ideal of (0).
- In dimension 2, it suffices, but is not necessary, that the ring R be normal. (normality implies that (0) is of pure height 1, therefore no embedded prime ideals).

The two dimensional domains $K[[x^2, x^3, y, xy]]$ and $K[[x^4, x^3y, xy^3, y^4]]$ are not CM. But $K[[x^2, x^3, y^2, y^3]]$ is not normal, it is CM.

Theorem 2.14. *Let R be a module-finite local extension of a regular local ring A. Then R is Cohen-Macaulay iff R is A-free*

Proof. If R is Cohen-Macaulay, every regular sequence on A is also a system of parameters of A and continues to be a system of parameters of R. Hence it's regular on R. So R is A-free

If R is A free, some system of parameters of A is a regular sequence on R. So R is Cohen-Macaulay. □

We also notice that

Proposition 2.15. *A local ring R is Cohen-Macaulay iff $\widehat{R}$ is Cohen-Macaulay.*

2.3. Height of ideals.

Proposition 2.16. *Let R be a Noetherian ring and let $x_1, ..., x_d$ generate a proper ideal I of height d. Then there exists elements $y_1, ..., y_d \in R$ such that for every i*

- $y_i \in x_i + (x_{i+1}, ..., x_d)R$
- *$y_1, ..., y_i$ generates an ideal of height i*

and moreover, $(y_1, ..., y_d)R = I$ and $y_d = x_d$.

If R is CM, then $y_1, ..., y_d$ is a regular sequence.

Proof. First we see that $x_1 + (x_2, ..., x_d)R$ is not contained in the union of all minimal primes of R by the coset form of prime avoidace lemma: otherwise $\mathrm{ht}(I) = 0$. So we can choose $x_1 + \delta_1$ not in any minimal prime. Hence $y_1 = x_1 + \delta_1$ is a NZD in R. So $\mathrm{ht}(y_1) = 1$ and $(y_1, x_2, ..., x_d)R = I$. Now we apply induction to R/y_1R. □

Proposition 2.17. *Let R be a Noetherian ring and let P be a minimal prime of R. Let $x_1, ..., x_d$ be elements of R such that $(x_1, ..., x_i)(R/P)$ has height i. Then there exists $\delta_i \in P$ such that let $y_i = x_i + \delta_i$, then $(y_1, ..., y_i)R$ has height i.*

Proof. We construct δ_i recursively: Let $\delta_1, ..., \delta_t$ be chosen. If $t < d$, we cannot have $x_{t+1} + P$ contained in the union of minimal primes of the ideal $(y_1, ..., y_t)R$. Otherwise by prime avoidance we have $x_{t+1} + P \subseteq Q$. Then on one hand $\mathrm{ht}(Q) \leq t$. On the other hand, after modulo P, we get $(x_1, ..., x_{t+1})R/P \subseteq QR/P \Rightarrow \mathrm{ht}(QR/P) \geq t + 1$, a contradiction! Then we can choose $\delta_{t+1} \in x_{t+1} + P$ not in any minimal prime of $(y_1, ..., y_t)R$. □

3. Homogeneous Case

3.1. Preliminary Results.

Proposition 3.1. *Let M be an $\mathbb{N}$(or $\mathbb{Z}$)-graded module over an $\mathbb{N}$(or $\mathbb{Z}$)-graded Noetherian ring R, then every associated prime of M is homogeneous.*

Proof. Any associated prime P of M is the annihilator of some element $u \in M$. Fix P and u, any multiple of u could be identified as an element of $R/P \to Ru \subseteq M$, hence has annihilator P as well.

Let u_i be a nonzero homogeneous component of u of degree i. Its annihilator J_i is easily seen to be a homogeneous ideal of R. If $J_i \neq J_k$ then we can choose an element $r \in J_i - J_k$ then ru is nonzero with fewer homogeneous componenets. We can choose an multiple of u such that it has the fewest homogeneous components. Then $J_i = J_k$, call them J. We have $J \subseteq P$.

If $J \neq P$, choose $r \in P - J$. We can make the choice such that no components of r is in J. Let r_a be homogeneous component of r with lowest degree a and let u_b be homogeneous component of u with lowest degree b. Then $r_a u_b$ is of the lowst degree and hence is zero. So $r_a \in J$, a contradiction! □

Corollary 3.2. *Let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$. Let $m = \oplus_{d=1}^{\infty} R_d$ be the homogeneous maximal ideal of R. Then $\dim(R) = \mathrm{ht}(m) = \dim(R_m)$.*

Proof. Let P be a minimal ideal of R, then P is homogeneous by Proposition 3.1. So $P \subseteq m$. But after killing P, the dimension of the domain R/P equals to height of maximal ideal, which doesn't change after localizing at m.

Therefore

$$\dim(R) = \dim(R/P) = \dim((R/P)_m) \leq \dim(R_m) \leq \dim(R)$$

Now the result follows. □

3.2. **Homogeneous system of parameters.** Recall the definition 1.11 of a homogeneous system of parameters, here we give more characterization on them.

Theorem 3.3. *Let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$ such that $\dim(R) = n$. Let $f_1, ..., f_n$ be a sequence of homogeneous elements of positive degree, TFAE:*

(1) $f_1, ..., f_n$ is a homogeneous system of parameters.
(2) m is nilpotent in $R/(f_1, ..., f_n)R$
(3) $R/(f_1, ..., f_n)R$ is finite-dimensional as a K-vector space
(4) R is module-finite over the subring $K[f_1, ..., f_n]$

Moreover, when above conditions hold, $f_1, ..., f_n$ are algebraically independent over K. So $K[f_1, ..., f_n]$ is a polynomial ring.

Proof. The basic idea behind this is that $R = K \oplus m$.

(1)⇒(2): If $f_1, ..., f_n$ is a homogeneous system of parameters, we have that $\dim(R/(f_1, ..., f_n)R) = 0$. Then every maximal ideal is minimal and hence homogeneous. So there is only one $m/(f_1, ..., f_n)$ and it follows that $m/(f_1, ..., f_n)$ is nilpotent.

(2)⇒(3): After killing $(f_1, ..., f_n)$, $m/(f_1, ..., f_n)$ is nilpotent. So $m/(f_1, ..., f_n)$ is finite-dimensional as a K-vector space, which implies that $R/(f_1, ..., f_n)R$ is a finite-dimensional K-vector space.

(3)⇒(4): This is obvious by Nakayama's lemma.

(4)⇒(1): R is module-finite over $K[f_1, ..., f_n]$. This is preserved after killing $(f_1, ..., f_n)$ so $R/(f_1, ..., f_n)R$ is module-finite over K. Therefore it's zero-dimensional.

Finally, since R is module-finite over $K[f_1, ..., f_n]$. The Krull dimension of $K[f_1, ..., f_n]$ is necessarily n, which implies that $f_1, ..., f_n$ must be algebraically independent. □

3.3. **Cohen-Macaulay rings.** Now we can prove following:

Theorem 3.4. *Let R be a finitely generated graded algebra of dimension n over $R_0 = K$. Let m denote the homogeneous maximal ideal of R. TFAE:*

(1) Some homogeneous system of parameters is a regular sequence.
(2) Every homogeneous system of parameters is a regular sequence.
(3) For some homogeneous system of parameters $f_1, ..., f_n$, R is a free module over $K[f_1, ..., f_n]$.
(4) For every homogeneous system of parameters $f_1, ..., f_n$, R is a free module over $K[f_1, ..., f_n]$.
(5) R_m is Cohen-Macaulay.
(6) R is Cohen-Macaulay.

Proof. (1)⇔(2): This is the same as the local case.

(1)⇔(3) and (2)⇔(4): This is because Lemma 1.9.

So now we have that the first four equivalent. Clearly (6)⇒(5).

(5)⇒(2): Let $x_1, ..., x_n$ be a homogeneous system of parameters for R. It continues to be a system of parameters of R_m. Hence it's a regular sequence. We claim that they are a regular sequence on R: If not, x_{k+1} is contained in some associated prime of $(x_1, ..., x_k)$, which is homogeneous. This is preserved after localization, a contradictino!

(1)⇒(6): Let $f_1, ..., f_n$ be a homgoeneous system of parameters for R. Then R is a free module over $S = K[f_1, ..., f_n]$. Let Q be a maximal ideal of R and let P be its contraction to S. Then $S_P \to R_Q$ is faithfully flat. Notice that they have the same dimension. Choose a regular sequence in S_P, they will continue to be a regular sequence in R_Q. Then R_Q has a system of parameters which is also a regular sequence. So R_Q is Cohen-Macaulay. □

4. Examples

4.1. **A normal non-C-M domain.** Let $S = K[X,Y,Z]/(X^3 + Y^3 + Z^3) = K[x,y,z]$ where $\text{char}(K) \neq 3$. Let $R = K[s,t]$ then the **Segre product** is $T = K[xs, ys, zs, xt, yt, zt]$, which is a direct summand of $R \otimes_K S$. Since S is normal, so is $S \otimes_K R = S[s,t]$. Therefore T is normal. In fact T is regular except at the origin. Both T and T_m are normal but not C-M.

Let $D = K[xs, ys, xt, yt] \subseteq T$ be a subring, we have

$$(zs)^3 + ((xs)^3 + (ys)^3) = 0$$
$$(zt)^3 + ((xt)^3 + (yt)^3) = 0$$

So T is module-finite over D and D has a homogeneous system of parameters $ys, xt, xs - yt$. But this is not a regular sequence on T: because

$$(zs)(zt)(xs - yt) = (zs)^2(xt) - (zt)^2(ys)$$

If $(zs)(zt) \notin (xt, ys)$ then we're done: suppose not, then $(zs)(zt) \in (xt, ys)$. Consider the K-linear map $T \to S$ sending $s \mapsto 1$ and $t \mapsto 1$. Then $z^2 \in (x,y)$, but this is impossible because $S/(x,y) \cong K[z]/(z^3)$.

5. Cohen-Macaulay Modules

5.1. Definition.

Definition 5.1. If (R, m, K) is local and M a finitely generated R-module of Krull dimension d, a system of parameters for M is a sequence of elements $x_1, ..., x_d \in m$ such that, equivalently:

- $\dim(M/(x_1, ..., x_d)M) = 0$.
- The image of $x_1, ..., x_d$ form a system of parameters in $R/\operatorname{Ann}_R M$.

Definition 5.2. A fintiely generated module M over a Noetherian local ring R is Cohen-Macaulay if some (equivalently, every) system of parameters is a regular sequence.

Definition 5.3. A fintiely generated module M over a Noetherian ring R is Cohen-Macaulay if its localization at maximal (equivalently, prime) ideals in its support are Cohen-Macaulay.

Theorem 5.4. *Let (R, m, K) be a local ring and $M \neq 0$ a finitely generated Cohen-Macaulay R-module of Krull dimension d. Then every nonzero submodule N of M has Krull dimension d.*

Proof. We can replace R by $R/\operatorname{Ann}_R(M)$. Then every system of parameters for R is a regular sequnce on M. We use induction on d. If $d = 0$, then there is nothing to prove.

If $d > 0$ and the result holds for smaller d. Let N be a submodule such that $\dim(N) < d$. We can choose N to be the maximal one with this property, then for any $N' \subseteq M$ with Krull dimension $< d$. We have $N' \subseteq N$: Since $\dim(N) < d$, there is an element $x \in \operatorname{Ann}_R(N)$ avoids all minimal primes of R. There is a similar element y in $\operatorname{Ann}_R(N')$. Then $xy \in \operatorname{Ann}_R(N + N')$ avoids all minimal primes. So $\dim(N + N') < d \Rightarrow N + N' \subseteq N$.

Let $x \in m$ be an NZD on M, then x is automatically an NZD on N. We claim that x is an NZD on M/N: If $xu \in N$, then $Rxu \subseteq N$ is a submodule of Krull dimension $< d$. But this submodule is isomorphic to $Ru \subseteq M$. So $Ru \subseteq N \Rightarrow u \in N$. Therefore multiply by x induces an isomorphism of sequences $0 \to N \to M \to M/N \to 0$ and $0 \to xN \to xM \to x(M/N) \to 0$. Therefore the latter one is also exact. Thus by the 9-lemma we have an injection $N/xN \to M/xM$, but then $\dim(M/xM) = d - 1$ and $\dim(N/xN) = \dim(N) - 1 < d - 1$. This contradicts with the induction hypothesis. □

An immediate corollary is that

Corollary 5.5. *If (R, m, K) is C-M then R is equidimensional, i.e. for every minimal prime p we have* $\dim(R/p) = \dim(R)$.

Proof. If p is minimal, it is an associated prime of R hence $R/p \hookrightarrow R$. Since all nonzero submodules of R has the same dimension $\dim(R)$ by THM 5.4, the results follows. □

5.2. **Regular local case.** We need a preliminary result from [Tor-and-Ext]

Proposition 5.6. *Let $x_1, ..., x_n \in R$ and let M be an R-module. Suppose that $x_1, ..., d_n$ be a possibly improper regular sequence on R and M. Let $I_k = (x_1, ..., x_k)R$, $0 \leq k \leq n$. Then*

$$\mathrm{Tor}_i^R(R/I_k, M) = 0$$

for $i \geq 1$ and $0 \leq k \leq n$

Using this, we can prove following

Theorem 5.7. *Let (R, m, K) be a regular local ring and let M be an R-module. Then M is a big Cohen-Macaulay module over R iff M is faithfully flat over R.*

Proof. We first notice that both conditions imply that $mM \neq M$: both conditions implies that a system of parameters is a regular sequence on M. Let I be the ideal generated by the system of parameters, then m is nilpotent over I so there is some power of m such that $m^n \subseteq I$. If $mM = M$, then $m^n M = M \subseteq IM \subseteq M$, a contradiction!

If M is faithfully flat, then obviously, any system of parameters, which is a regular sequence on R, is a regular sequence on M. Now assume that M is a big CM module, we have to show that M is faithfully flat.

Note that if M is flat, then $mM \neq M \Rightarrow M$ is faithfully flat. So we only need to prove that M is flat, which suffices to show that $\mathrm{Tor}_i^R(N, M) = 0$ for any R-module N. Actually we shall prove $\mathrm{Tor}_i^R(N, M) = 0$ for every $i \geq 1$. We prove this by reverse induction on i.

Since Tor commutes with direct limit, we can reduce to the case where N is finitely generated. Since finitely generated modules over a regular local ring has finite projective resolution (the length is at most $\dim(R)$), we have $\mathrm{Tor}^R_{\dim(R)+1}(N, M) = 0$. Now assume $\mathrm{Tor}_j^R(N, M) = 0$ for all $j \geq i$, we want to show that $\mathrm{Tor}_i^R(N, M) = 0$.

We first consider the case when $N = R/P$ is prime cyclic. Since R is CM (it's even regular), we have a regular sequence of the length $\mathrm{ht}(P)$, say $x_1, ..., x_d$. Then P is a minimal prime of $I = (x_1, ..., x_d)$, which is also an associated prime. So we have an exact sequence:

$$0 \to R/P \to R/I \to Q \to 0$$

for some R-module Q. Now apply $\mathrm{Tor}^R(\_, M)$ we get

$$\cdots \to \mathrm{Tor}^R_{i+1}(Q, M) \to \mathrm{Tor}_i^R(R/P, M) \to \mathrm{Tor}_i^R(R/I, M) \to \cdots$$

By Proposition 5.6 above, we see that $\mathrm{Tor}_i^R(R/I, M) = 0$. By induction hypothesis (all Tor_{i+1} vanish), $\mathrm{Tor}^R_{i+1}(Q, M) = 0$, so $\mathrm{Tor}_i^R(R/P, M) = 0$.

Now we do induction on the least number of factors in the finite filtration of N by prime cyclic modules. Suppose $N' \subseteq N$ is a prime cyclic submodule, then

$$0 \to N' \to N \to C \to 0$$

Again apply $\mathrm{Tor}^R(\_, M)$ we get

$$\cdots \to \mathrm{Tor}_i^R(N', M) \to \mathrm{Tor}_i^R(N, M) \to \mathrm{Tor}_i^R(C, M) \to \cdots$$

This time C has one less factor, so the Tor on both sides are zero, which implies that the middle one is also zero. □

6. Properties of Cohen-Macaulay Rings

6.1. **Algebras finitely generated over a CM ring.** Trivially we see that if R is CM, then so is $W^{-1}R$: note that for any maximal ideal $W^{-1}m$ of $W^{-1}R$, we have $(W^{-1}R)_{W^{-1}m} \cong R_m$. Meanwhile the quotient of R is not always CM. However, we do have following:

Theorem 6.1. *If a Noetherian ring R is CM, then so are $R[x]$ and $R[[x]]$.*

Proof. In the case of $R[[x]]$, note that x is in every maximal ideal Q: Kill Q we see that x is invertible, so there is $1 = xy + z$ for some $z \in Q$. But then $1 - yz$ is not invertible, which is a contradiction! It follows that Q has the form $xR[[x]] + PR[[x]]$ for some maximal ideal P. Since $R[[x]]_Q/(x)R[[x]] \cong R_P$, any system of parameters containing x will be a regular sequence on $R[[x]]_Q$.

In the case of $R[x]$, let Q be a maximal ideal and let P be its contradiction to R. Then we have a map $R_P \to (R[x])_Q \cong (R_P[x])_Q$. Therefore we can assume that (R, P) is local. Let $f_1, ..., f_n$ be a maximal regular sequence on R, then they continue to be a regular sequence on $R[x]_Q$. So killing them won't affect any issue: Thus we may assume that P is nilpotent. After killing m, we have $R[x]_Q/mR[x]_Q \cong R/P[x]$, which is a PID. So $QR[x]_Q$ is generated by m and a monic polynomail f. Now $R[x]_Q$ has dimension 1 and f is an NZD on $R[x]_Q$ (A monic polynomial is always an NZD on any polynomial ring). So the result follows. □

6.2. **Catenary and universally catenary rings.**

Definition 6.2. A Noetherian ring is called **catenary** if for any two prime ideals $P \subseteq Q$, any two saturated chains of primes joining P to Q have the same length. In this case, the common length will be the same as the dimension of the local domain R_Q/PR_Q.

A localization or homomorphic image of a catenary ring is automatically catenary. However, a catenary ring adjoint an indeterminant is no longer catenary. So we have following defnition:

Definition 6.3. R is **universally catenary** if every polynomial ring over R is catenary. This implies that every algebra essentially of finite type over R is catenary

By Theorem 5.4, we have following corollary:

Corollary 6.4. *If (R, m, K) is Cohen-Macaulay, then R is equidimensional: every minimal prime P is such that* $\dim(R/P) = \dim(R)$

Proof. Since P is a minimal prime of R, we have an injection $R/P \to R$. Now by Theorem 5.4, we know that $\dim(R/P) = \dim(R)$. □

Now we are ready to prove following:

Theorem 6.5. *A Cohen-Macaulay ring R is universally catenary*

Proof. We only need to show following: Given two prime ideals $P \subseteq Q$ in R, then every saturated chain of primes from P to Q has length $\mathrm{ht}(Q) - \mathrm{ht}(P)$. Once this is shown, the universal catenarity follows by Theorem 6.1.

First of all, the issue is not affected if we localize at Q. Next we observe: let $x_1, ..., x_d$ be a maximal regular sequence in P on R, then killing $x_1, ..., x_d$ changes nothing: R is still Cohen-Macaulay. So we may assume that (R, Q) is local while P is minimal in R. We know that $\dim(R) = \dim(R/P)$ and so at least one saturated chain from P to Q has length $\mathrm{ht}(Q) - \mathrm{ht}(P)$. To complete the proof, it sufficies to show that all saturated chains from P to Q has the same length.

We use induction on $\dim(R)$: If $\dim(R) = 1$ then there is nothing to prove. Now assume that it's true for any smaller dimensions: Choose two saturated chains from P to Q, let their second element in the chain be P_1 and P_1' respectively. Then $\mathrm{ht}(P_1) = \mathrm{ht}(P_1') = 1$. Otherwise, say $\mathrm{ht}(P_1) > 1$, then we must have $\mathrm{ht}(P_1) < \mathrm{ht}(Q) = \dim(R)$. Therefore R_{P_1} is a Cohen-Macaulay ring with smaller dimension, the saturated chain $PR_{P_1} \subseteq P_1 R_{P_1}$ contradicts our induction hypothesis (every saturated chain should have the length of $\dim(R)$).

Now we have $\mathrm{ht}(P_1) = \mathrm{ht}(P_1') = 1$. Choose $x \in P_1$ and $y \in P_1'$ such that none of these two is in any minimal prime. Then xy is an NZD on R and P_1 and P_1' are both minimal primes of xyR. Now pass to $R/(xy)$ we see that the saturated chain from $P_1/(xy)$ to $Q/(xy)$ is of the same length as the saturated chain from $P_1'/(xy)$ to $Q/(xy)$. □