RESEARCH STATEMENT

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A. Introduction

My main research interests are in commutative algebra and algebraic geometry. I have been mainly working on topics related to closure theory using perfectoid theory in mixed characteristic and descent technique in equal-characteristic 0. But I also enjoy thinking about questions related to singularities, multiplicities, D-modules, and homological properties. Before starting my research career in commutative algebra, I spent two years studying combinatorics. In consequence, I am also fascinated with topics in combinatorial commutative algebra, e.g., Hibi rings, toric varieties, cluster algebras, etc.

Commutative algebra can be described as the study of commutative rings and their ideals and modules. These arise naturally in the study of solution sets to polynomial equations. The solution sets form geometric objects called *algebraic sets* (or *varieties* if they are irreducible). To each algebraic set, we can associate a commutative ring, called a *coordinate ring*. The algebraic set itself can be recovered as the set of maximal ideals of the coordinate ring over an algebraically closed field like $\mathbb{C}$, by Hilbert Nullstellensatz. The coordinate ring can be thought of as a collection of (regular) functions on this algebraic set. Many geometric properties of algebraic sets can be proved by studying their coordinate rings.

Most of the questions I am working on trace back to a certain set of conjectures about homological properties of commutative rings, called *Homological Conjectures* [Hoc75], e.g., the direct summand conjecture, the existence of big Cohen-Macaulay modules/algebras, the improved new intersection conjecture, etc. Many of these conjectures in positive characteristic are resolved due to the development of tight closure, invented by Mel Hochster and Craig Huneke [HH90]. Along with the development of tight closure, there are several unexpected applications. For example, by using *descent techniques* and tight closure, one can show that for $n+1$ polynomials in n variables $f_1,\dots,f_{n+1}\in\mathbb{C}[x_1,\dots,x_n]$, one has $f_1^n\cdots f_{n+1}^n\in(f_1^{n+1},\dots,f_{n+1}^{n+1})$. The statement is elementary, but the proof is by no means obvious, even in the case $n=2$.

Mel Hochster and Craig Huneke also proved various remarkable results [Hoc73, HH92, HH94b, HH94a] using tight closure. Inspired by their results, Raymond C. Heitmann developed four closure operations $\mathsf{ep},\mathsf{r1},\mathsf{epf}$, and $\mathsf{r1f}$ in the mixed characteristic case [Hei01], and proved the direct summand conjecture for rings of dimension at most 3 using the epf closure [Hei02]. Recently, due to the development of perfectoid theory [Sch12], many homological conjectures in mixed characteristic have been resolved by Yves André, Bhargav Bhatt, Raymond Heitmann, and Linquan Ma [And18a, And18b, And20, Bha17, HM18a, Bha20].

A major part of my work focuses on generalizing tight closure theory. Specifically, I have worked on

- A new closure operation called wepf in mixed characteristic (Section B).
- Test element theory for tight closure in equal characteristic 0 (Section C).
- A new numerical notion called size for ideals (Section D).
- Other future projects (Section E)

In the rest of this note, each section will be devoted to explaining one bullet point above.

B. wepf Closure in Mixed Characteristic

B.I. **Motivation.** En route to generalizing the notion of tight closure, Geoffrey D. Dietz developed a set of characteristic-free axioms for closure operations, and discovered the relation between closure operations satisfying his axioms (such a closure is called a *Dietz closure*) and the existence of big Cohen-Macaulay modules [Die10]. Later, Rebecca R.G. added one more axiom and gave comparable results for the existence of big Cohen-Macaulay algebras [R.G18]. Naturally, people started wondering which closure operations in mixed characteristic are Dietz closures. Let us recall the definition of epf for ideals below.

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Definition B.1. Let R be a complete local domain of mixed characteristic p. Let R^+ be the integral closure of R in an algebraic closure of its fraction field $\mathrm{Frac}(R)$. Let $I \subseteq R$ be an ideal and $u \in R$ be an element. We say $u \in I^{\mathsf{epf}}$ if there exists a nonzero element $c \in R$ such that for any rational number $\varepsilon \in \mathbb{Q}^+$ and any integer $N \in \mathbb{N}^+$, we have

$$c^{\varepsilon} u \in (I, p^N) R^+.$$

Raymond Heitmann and Linquan Ma proved that epf closure satisfies the (usual) colon-capturing condition [HM18b]. But whether epf closure is a Dietz closure remains open. Also, as far as I know, no closure operations in mixed characteristic were known to be Dietz closures at the time when Raymond Heitmann and Linquan Ma wrote their paper.

B.II. **Contribution.** I developed a new closure operation wepf in mixed characteristic, defined below.

Definition B.2. Let R be a complete local domain of mixed characteristic p. Let $I \subseteq R$ be an ideal. Then

$$I^{\mathsf{wepf}} = \bigcap_{n=1}^{\infty} (I, p^n)^{\mathsf{epf}}.$$

I showed that

Theorem B.3. wepf *is a Dietz closure satisfying the algebra axiom.*

This gives a different proof of the existence of big Cohen-Macaulay algebras (modules). This result is achieved by showing a strong property about epf closure of ideals generated by part of system of parameters, which I call *p-colon-capturing*, with help from perfectoid theory.

Theorem-Definition B.4 (p-colon-capturing)**.** *Let R be a d-dimensional complete local domain of mixed characteristic p (with F-finite residue field). Let $x_1, \ldots, x_n$ be part of a system of parameters of R. Then $x_1, \ldots, x_n$ satisfies what we call* p-colon-capturing. *That is, there is some fixed positive integer N_0 such that for all integers $N \geqslant N_0$ we have*

$$\left(x_1, \ldots, x_{n-1}, p^N\right) :_{R^+} x_n \subseteq \left(\left(x_1, \ldots, x_{n-1}, p^{N-N_0}\right) R^+\right)^{\mathsf{epf}}.$$

Let me also point out that this property can be used to prove that $\mathsf{r1f}$ is a Dietz closure satisfying the algebra axiom. The general problem of whether epf is a Dietz closure remains open.

There is a notion called "phantom extension" for any module-finite extension S over a noetherian local ring R, where R has a closure operation cl. One first resolves S/R by free R-modules as follows

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] & S \arrow[r] & S/R \arrow[r] & 0, \\ & R^{\oplus m} \arrow[u, "\phi"] \arrow[r, "\nu"'] & R^{\oplus n} \arrow[u] \arrow[r] & S/R \arrow[u] \arrow[r] & 0 \end{tikzcd}$$

where ν is a $n \times m$ matrix, ϕ is a m-dimensional vector. Let $(-)^{\vee} = \mathrm{Hom}_R(-, R)$.

Definition B.5 (cl-phantomness)**.** The extension $R \to S$ is cl-phantom if $\phi^{\vee} \in \left((\nu^{\vee}) R^{\oplus m}\right)^{\mathsf{cl}}$.

I also proved the following theorem.

Theorem B.6. *If $R \to S$ is a module-finite extension of complete local domains of mixed characteristic p with an F-finite residue field, then this map is* epf*-phantom.*

This result, together with Raymond Heitmann's and Linquan Ma's result saying that epf closure on regular local rings [HM18b], implies the direct summand conjecture. We make a great deal of use of techniques from Yves André's [And18c] and Bhargav Bhatt's results [Bha17].

I also prove a property similar to p-colon-capturing in the positive characteristic case. This is a completely new phenomenon about tight closure.

Theorem B.7. *Suppose R is a d-dimensional complete local domain of prime characteristic p. Let R^+ be its absolute integral closure. Suppose $x_1, \ldots, x_d$ is a system of parameters of R. Then for any $1 \leqslant n \leqslant d$ and any $y \in R$, there is some positive integer N_0 such that for all $N \geqslant N_0$,*

$$(x_1, \ldots, x_n, y^N) R^+ :_{R^+} x_{n+1} \subseteq (x_1, \ldots, x_n, y^{N-N_0}) R^+.$$

B.III. **Future Objectives.** I want to dig deeper into the closure theories in mixed characteristic. Bhargav Bhatt proved that the p-adic completion of the absolute integral closure in mixed characteristic serves as a big Cohen-Macaulay algebra [Bha20]. There are many questions about closure theory that can be answered using this breakthrough, as well as the prismatic cohomology theory developed by Bhargav Bhatt and Peter Scholze [BS19]. I am particularly interested in developing a theory about test elements for these closure operations in mixed characteristic. It is notable that Linquan Ma and Karl Schwede developed a test ideal theory in mixed characteristic using perfectoid spaces and proved the uniform bound for symbolic powers on regular rings [MS18]. Whether their test ideals arise from a test element theory for some closure operation remains to be explored.

C. Tight Closure in Characteristic 0

C.I. **Motivation.** The notion of the tight closure is generalized to equal-characteristic 0 by Mel Hochster and Craig Huneke [HH99]. This notion has a lot of fruitful applications, e.g., the generalized Briançon-Skoda theorem and the vanishing theorem for maps of Tor. However, there is no satisfactory test element theory for complete domains over a field.

C.II. **Contribution.** I began by proving a theorem about test elements for tight closure in characteristic p, stated as follows.

Theorem C.1. *Let K be a field of characteristic p and let R be a d-dimensional geometrically reduced domain over K of the form $R = K[\![x_1, \dots, x_n]\!]/(f_1, \dots, f_r)$. Then the $(n-d)\times(n-d)$ minors of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ are contained in the test ideal of R, and remain so after localization and completion. Thus, any element of the Jacobian ideal generated by all these minors that is in R° is a completely stable test element.*

My main contribution is proving a similar theorem in equal characteristic 0.

Theorem C.2. *Suppose that R is a reduced, equidimensional, complete local domain of dimension d over k, i.e., we have a presentation $R = k[\![x_1, \dots, x_n]\!]/(f_1, \dots, f_r)$ where each f_i is a power series in x_i's. Let δ be a $(n-d)\times(n-d)$ minor of the Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$. For any $u \in I^*$ where $u \in R$ and $I \subseteq R$, we have $\delta u \in I$. Therefore the complete Jacobian ideal multiplies any element in I^* into I for any ideal $I \subseteq R$.*

The notion of tight closure in equal-characteristic 0 is defined using a technique called "descent". Let R be a noetherian K-algebra and $N \subseteq M$ finitely generated R-modules. Roughly speaking, a descent of the triple (R, M, N) to some finitely generated $\mathbb{Z}$-subalgebra A of K is a triple (R_A, M_A, N_A) such that we recover (R, M, N) when tensoring with K. An element is in the tight closure of a submodule if it is so in for some descent data modulo the maximal ideals in a dense open subset of $\operatorname{Max}\operatorname{Spec}(A)$.

The key idea of proving Theorem C.2 is to descend relevant data while preserving the desired form so that we have some control over the Jacobian ideal in the descent data. I made very substantial use of the following Artin-Rotthaus theorem [AR88], which is a special case of the Néron-Popescu desingularization theorem [Swa98].

Theorem C.3 (Artin-Rotthaus). *Let K be a field. Then the power series ring $K[\![x_1, \dots, x_n]\!]$ is a direct limit of smooth $K[x_1, \dots, x_n]$-algebras.*

Granted this theorem, I was able to construct a direct system of descent data preserving the presentation and the height (pure height) of the Jacobian ideal. Then I use characteristic p results to finish the proof.

C.III. **Future Objectives.** There are a variety of interesting questions about the tight closure notion in equal-characteristic 0. For example, when we take R_A to be a finitely generated $\mathbb{Z}$-algebra model for the K-algebra R, instead of studying tight closure on closed fibers R_A/μ where $\mu \in \operatorname{Max}\operatorname{Spec}(R_A)$, one may also consider the mixed-characteristic ring $(R_A)_\mu$ obtained by localizing at maximal ideals and study closure operations in mixed characteristic.

When studying big Cohen-Macaulay algebras and modules in characteristic 0, the method of "ultraproducts" is used by Dietz Geoffrey and Rebecca R.G. [DR16]. It is fascinating to study the connections between this method and the descent method.

D. Size

My third project is about a new numerical invariant of ideals, called "size".

D.I. **Motivation.** The notion of "size" depends on the notion of quasilength, which was introduced by Mel Hochster and Craig Huneke in their joint paper [HH09]. They used the notion of quasilength to define two nonnegative real numbers that are intended heuristically as "measures" of a highest local cohomology module $\mathrm{H}^d_{(f_1,\dots,f_d)R}(M)$.

Definition D.1. Let R be a ring, M an R-module, and I a finitely generated ideal of R. If there is a (finite) length h filtration of M in which the factors are cyclic modules killed by I, and no shorter such filtrations, then we say that M has finite I-quasilength h. We denote the quasilength by $\mathcal{L}_I(M) = h$. If there is no such finite filtration, then we say $\mathcal{L}_I(M) = +\infty$.

I define the *size* of an ideal I in a ring R to be

Definition D.2.

$$\mathsf{size}_R(I) = \inf\{n \mid \limsup_{t\to\infty} \frac{\mathcal{L}_I(R/I^t)}{t^n} < \infty\}.$$

We also write $\mathsf{size}(I) = \mathsf{size}_R(I)$ if the ambient ring R is clear in the context.

D.II. **Contribution.** I showed that

Proposition D.3. *Let R be a ring and let $I \subseteq R$ be an ideal. Then*

- $\mathsf{size}(I)$ *is invariant up to taking radicals.*
- $\mathrm{ht}(I) \leqslant \mathsf{size}(I) \leqslant \nu(I)$ *where $\nu(I)$ is the least number of generators of I.*
- *If R is noetherian, then $\mathsf{size}_R(I) = \mathsf{size}_{R_{\mathrm{red}}}(IR_{\mathrm{red}})$.*

From this proposition, one sees that the size of an ideal is bounded above by the arithmetic rank of the ideal, and bounded below by the height of the ideal. Let me point out that the size of an ideal being zero is stronger than the height being zero. The latter one only implies that the ideal is contained in some minimal prime. For the former one, I proved the following proposition.

Proposition D.4. *Let $I \subseteq R$ be a finitely generated ideal. Then I has size 0 if and only if I is nilpotent.*

Regarding the relation between the height and the size of a prime ideal, I also proved the following result.

Theorem D.5. *Let R be a local ring and P a prime ideal of R such that $\dim R/P = 1$. Suppose that there is some c such that $P^{(cn)} \subseteq P^n$ for all sufficiently large n and R/P is module-finite over a regular local ring A (e.g., if R/P is complete). Then $\mathsf{size}(P) = \mathrm{ht}(P)$.*

Given this theorem, I conjecture that

Conjecture D.6. *Let R be a regular local ring and $P \subseteq R$ a prime ideal. Then $\mathsf{size}(P) = \mathrm{ht}(P)$.*

This conjecture is not known even for the ideal generated by 2×2 minors of a 2×3 matrix of indeterminates over complex numbers.

D.III. **Future Objectives.** The notion of size was intended to attack problems about set-theoretic intersections. But it itself turns out to be an exciting subject to study. Under mild assumptions, one may use symbolic powers to compute the size of a prime ideal, the consequences of which are yet to be further explored.

Often, to study size, one has to look deeper into the notion of quailength. However, one difficulty with quasilength is that it is not additive, even in the case of the direct sum of two modules [HZ17]. It is fascinating to study when the additive property fails. It is also intriguing to consider the asymptotic additive property, e.g., whether for a fixed ideal I and a class of R-modules M, there exists some $c > 0$ independent of M such that $cn \cdot \mathcal{L}_I(M) \leqslant \mathcal{L}_I(M^{\oplus n}) \leqslant n \cdot \mathcal{L}_I(M)$. There is always such a constant if the modules in the class are killed by a fixed power of I.

E. Other Future Projects

Apart from these projects I am currently working on, there are other projects that I want to explore further. For example, in a fantastic joint paper by Angélica Benito, Greg Muller, Jenna Rajchgot and Karen E. Smith [BMRS15], it is proved that a large family of cluster algebras (called *locally acyclic cluster algebras*) over fields of positive characteristic are strongly F-regular. However, this direction has not been fully explored, and there are a lot of interesting questions, e.g., what is the F-signature [Tuc12] of such rings?

There is also a related topic that I'm excited to explore. It has been proven that the homogeneous coordinate rings of the Grassmannians $\mathrm{Gr}(2, n)$ defined over an algebraically closed field k of characteristic $p \geqslant \max\{n-2, 3\}$ have finite F-representation type (FFRT). The method used to prove this result in [RŠdB17, RŠdB19] comes from representation theory. It would be of great interest to give a different proof that could extend this result. Since the homogeneous coordinate rings of the Grassmannians $\mathrm{Gr}(2, n)$ have a cluster algebra structure, it is very interesting to study the relation between FFRT and cluster algebras.

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