Why is there no quintic formula?

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Abel–Ruffini theorem

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A closed formula solution to an equation is a formula of the coefficients only involving additions, subtractions, multiplications, divisions and root extractions such that it always gives a root to the original equation.

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For a non-example, it is obvious that $x = a$ is a root to the quintic equation $x^5 - a^5 = 0$ for any a, but this is *NOT* the closed formula we want to discuss today.

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Figure: Tom and you

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$a = 100, b = \pi, c = 5000, d = -0.001, e = \mathrm{e} = 2.718281828...$, and plug into Tom's formula

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I don't want to do the calculation!

Better Approach

Let us denote all five roots by $x_1, x_2, \cdots, x_5$. Then we have

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$

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Theorem (Vieta's Formula)

If $x_1, x_2, \ldots, x_5$ are roots to the equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e$, then

$$\begin{aligned}
-a &= x_1 + x_2 + \cdots + x_5 \\
b &= x_1x_2 + x_1x_3 + \cdots + x_4x_5 \\
-c &= x_1x_2x_3 + x_1x_2x_4 + \cdots + x_3x_4x_5 \\
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So each coefficient is a *symmetric* function of these five roots.

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But eventually they will come back to where they are. Because they are symmetric in $x_1, \cdots, x_5$. For example, if we make the change $x_1 \to x_2, x_2 \to x_3, \cdots, x_4 \to x_5, x_5 \to x_1$, then $a = -x_1 - \cdots - x_5$ will stay the same when the movement is done.

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- on one hand, x_1 becomes x_2, so Tom's formula must give us x_2 after all movements;

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- on one hand, x_1 becomes x_2, so Tom's formula must give us x_2 after all movements;
- on the other hand, all coefficients, as functions of roots, are unchanged, therefore Tom's formula still gives us x_1.

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Therefore Tom's formula must be wrong!
The talk is over! Thanks!

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Let us go back and look at the quadratic case $x^2 + ax + b = 0$. We know that

$$x_{1,2} = \frac{-a \pm \sqrt{\Delta}}{2}$$

where $\Delta = a^2 - 4b$.

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$$x_{1,2} = \frac{-a \pm \sqrt{\Delta}}{2}$$

where $\Delta = a^2 - 4b$. Using Vieta's formula, we can rewrite it as $\Delta = (x_1 + x_2)^2 - 4x_1x_2 = (x_1 - x_2)^2$.

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From the movie, we see that $\sqrt{\Delta}$ actually changes its value when we switch x_1 and x_2! Therefore the quadratic formula also changes from x_1 to x_2.

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- The number of rounds, that a complex valued function $z = f(t)$ winds around the origin in the complex plane as t changes, is called the *winding number* of z.

Ultimate Approach

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Therefore the same argument works: on the one hand, zero winding number implies no change in outputs; on the other hand, we've moved x_1 to a different solution. So Tom's formula should give us a new solution, where we reach a contradiction!

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If we have a formula, e.g., $\sqrt[26]{a + b^2 - dc^6 + e^{12343}}$, no matter how crazy the formula inside the radical is, the winding number of that part is zero under the action of a commutator.

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The way to get around this issue is to use a commutator of commutators, i.e., to use $[\sigma, \tau]$, where $\sigma = [\sigma_1, \tau_1]$ and $\tau = [\sigma_2, \tau_2]$ are both commutators. Both σ and τ will fix whatever inside the radical sign, and their commutator will fix the whole expression.

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Is this possible?

Finally, some calculation

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Let us do some calculation by .... hand? by a computer!

```
In [1]: S5 = SymmetricGroup(5)

In [2]: A5 = S5.commutator()

In [3]: A5.order()
Out[3]: 60

In [4]: B = A5.commutator()

In [5]: B.order()
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Figure: SageMath

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So each element in A_5 is a commutator of commutators of commutators of ... of commutators of commutators.
So no matter how many times of nested radical expression appears in Tom's formula, we can always find some non-identity element in A_5 such that

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Yeah!

SO WE WIN!

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You are strongly encouraged to learn more about all of these.

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Let us go back and check our conclusion. go to slide

Thank you!

Thanks for listening!