It's the "Même"

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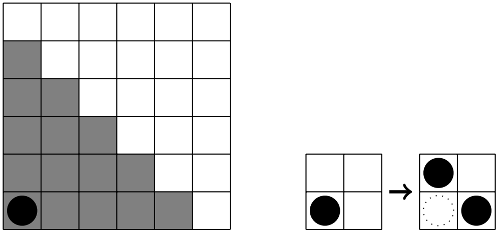
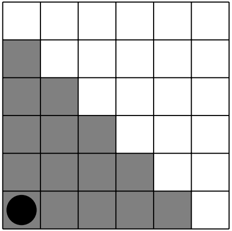
Kontsevich's Puzzle

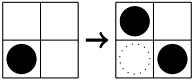
There is a quarter plane (infinite) with integer grid lines drawn. The origin $(0,0)$ is the lower left corner. A stone is placed at origin. You are allowed to take any stone away, and put back two stones, one in the box right to it and one in the box above it. If one of the designations is already occupied, then the whole move cannot happen.



Kontsevich's Puzzle

A part of the corner near the origin is painted gray and your goal is to move all stones out of the gray area.
The gray area has length $n = 5$ in the graph. Is it possible for $n = 1$? $n = 2$? All n?





Solution

It turns out that the only possible n's are 1 and 2. Let's explain why $n \geqslant 3$ is not possible. Number the grids with following numbers.

$\frac{1}{32}$					
$\frac{1}{16}$	$\frac{1}{32}$				
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$			
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$		
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$

What do you observe?
The sum of number of stones is always 1.

Solution

Geometric Summation Formula

If you have a sequence $a, aq, aq^2, \dots$ with $q < 1$, then the total sum is given by $\frac{a}{1-q}$.

$\frac{1}{32}$					
$\frac{1}{16}$	$\frac{1}{32}$				
$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$			
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$		
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$

So the sum for the 3rd row is $\frac{1/4}{1-1/2} = \frac{1/4}{1/2} = 1/2$.

So the sum for the 4th row is $\frac{1/8}{1-1/2} = \frac{1/4}{1/2} = 1/4$.

...

So the sum for the nth row is $\frac{1/2^{n-1}}{1-1/2} = \frac{1/2^{n-1}}{1/2} = 1/2^{n-2}$.

Solution

For gray area of side length n, there are $n+1$ rows with row sum $1/2^{n-1}$ and a sequence of row sums $1/2^n, 1/2^{n+1}, \cdots$. The total sum is $\frac{n+2}{2^{n-1}}$. For $n > 3$, we always have $n+2 < 2^{n-1}$. So $\frac{n+2}{2^{n-1}} < 1$.

How about $n = 3$? It seems that we have total sum $5/4 > 1$. Maybe it is possible?

Keep in mind that we can have at most one stone at x-axis and y-axis respectively. Therefore the total sum is at most $\frac{5}{4} - 2 \times \frac{1}{2} + 2 \times \frac{1}{4} = \frac{3}{4} < 1$. Therefore it is impossible for $n = 3$ as well.

Red and Blue Balls

A bag has 20 blue balls and 14 red balls. Each time you randomly take two balls out. (Assume that each ball in the bag has equal probability of being taken.) You do not put these two balls back. Instead,

- if both balls have the same color, you add a blue ball to the bag;
- if they have different colors, you add a red ball to the bag.

Assume that you have an unlimited supply of blue and red balls, if you keep on repeating this process, what will be the color of the last ball left in the bag? What if the bag has 20 blue balls and 13 red balls instead?

Solution

Let us use R for the number of red balls, and B for the number of blue balls. We also use the pair (B, R) to indicate how many balls we still have in the bag.

There are basically three cases:

- If we take out two blue balls, we have $(B-2, R)$ in the bag. Then we put one blue ball back. So it is $(B-1, R)$ eventually.
- If we take out two red balls, we have $(B, R-2)$ in the bag. Then we put one blue ball back. So it is $(B+1, R-2)$ eventually.
- If we take one blue and one red, then we have $(B-1, R-1)$ in the bag. Then we put one red ball back, and it becomes $(B-1, R)$.

So after one round, the number of balls changes to either $(B-1, R)$ or $(B+1, R-2)$. Note that red balls R either stays the same or reduce by 2. Hence we know that parity of R is an invariant.

When there is only one ball left, if $R = 14$ at the beginning, then it cannot be red. On the other hand, if $R = 13$, then it must be red.

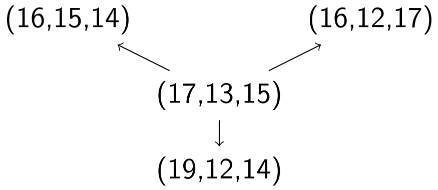
Red, Blue and Green Balls

A bag has three types of balls in it: 13 red, 15 green and 17 blue. Each time you take out two balls with different colors, then you paint both of them to the third color and put it back. For example, if the two balls taken out are green and red, you will paint both of them to blue. Is it possible for all balls to become the same color? Why or why not?

Solution

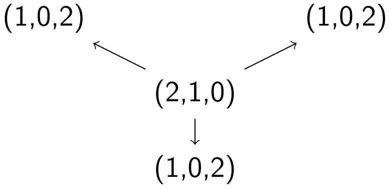
Let us use R for red balls, B for blue balls and G for greens balls. If two balls are red and blue, then we will get two more green balls. So it changes from $(B, R, G) \to (B-1, R-1, G+2)$. Similarly we have $(B, R, G) \to (B+2, R-1, G-1)$ and $(B, R, G) \to (B-1, R+2, G-1)$. Seems like nothing stays unchanged.

Let's draw a picture of that



Solution

How about we take the remainder of each number dividing by 3?



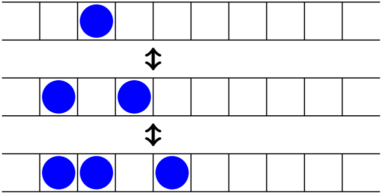
So the state is always a re-order of $(2, 1, 0)$. This is invariant no matter how you take balls out.
If you end with only one color, then the sequence must be of the form $(0, 0, ?)$, or $(0, ?, 0)$ or $(?, 0, 0)$, which is not a re-order of $(2, 1, 0)$. So it is impossible.

Splitting and Merging

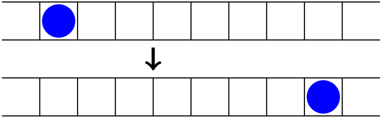
There is an infinitely long grids with a blue dot somewhere.

- You can remove one dot, split it into two and put one in the left box and the other in the right box.
- You can reverse what you did above.

Multiple dots in one spot are allowed.



The question is that, can you shift the dot 7 blocks to the right?



Solution

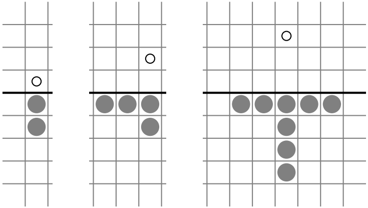
Let's number the line by consecutive powers of x, then

	x^{-1}	1	x	x^2	x^3	x^4	x^5	x^6	x^7

Each step we are removing x^a, x^{a+2} and get x^{a+1} or vice versa. So the change in sum is $x^a(-1 - x^2 + x)$ or its opposite. Therefore we want to make $-1 - x^2 + x = 0$, which gives us two roots $\omega = e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and ω^{-1}. Then since we start with 1, and each step doesn't change the sum. We should end at sum 1. Notice that $\omega^6 = 1$ but $\omega^7 = 1 \times \omega \neq 1$. So it is impossible.

Conway's Soldiers

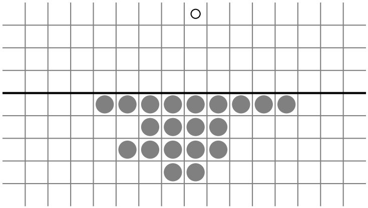
There is an infinitely large chessboard with some soldiers on it. The only way that a soldier can move is by jumping over another soldier and kill the soldier it jumped over. There is a horizon marked black. You are only allowed to place soldiers below the horizon and your goal is to go as high as you can.



Above are solutions to $n = 1, 2, 3$. Can you come up with a solution for $n = 4$? How about $n = 5$?

Solution

For $n = 4$, we have



Next we will demonstrate that why $n \geqslant 5$ is not possible.

Solution

Let's label the chessboard by the powers of a variable φ, then for those three moves, we analyze the change in sums.

φ^6	φ^5	φ^4	φ^3	φ^2	φ	1	φ	φ^2	φ^3	φ^4	φ^5
φ^7	φ^6	φ^5	φ^4	φ^3	φ^2	φ	φ^2	φ^3	φ^4	φ^5	φ^6
φ^8	φ^7	φ^6	φ^5	φ^4	φ^3	φ^2	φ^3	φ^4	φ^5	φ^6	φ^7
φ^9	φ^8	φ^7	φ^6	φ^5	φ^4	φ^3	φ^4	φ^5	φ^6	φ^7	φ^8
φ^{10}	φ^9	φ^8	φ^7	φ^6	φ^5	φ^4	φ^5	φ^6	φ^7	φ^8	φ^9

Once we reached 1. The total sum must be at least 1.

Solution

		φ^6	φ^5	φ^4	φ^3	φ^2	φ	1	φ	φ^2	φ^3	φ^4	φ^5		
		φ^7	φ^6	φ^5	φ^4	φ^3	φ^2	φ	φ^2	φ^3	φ^4	φ^5	φ^6		
		φ^8	φ^7	φ^6	φ^5	φ^4	φ^3	φ^2	φ^3	φ^4	φ^5	φ^6	φ^7		
		φ^9	φ^8	φ^7	φ^6	φ^5	φ^4	φ^3	φ^4	φ^5	φ^6	φ^7	φ^8		
		φ^{10}	φ^9	φ^8	φ^7	φ^6	φ^5	φ^4	φ^5	φ^6	φ^7	φ^8	φ^9		

There are three cases.

1. Jump up: we remove φ^{n+1} and φ^n, and get back φ^{n-1}. So the change is $\varphi^{n-1}(1-\varphi-\varphi^2)$.
2. Jump down: we remove φ^{n-1} and φ^n, get back φ^{n+1}. The change is $\varphi^{n-1}(\varphi^2-\varphi-1)$.
3. Jump horizontally: it is essentially the same as jumping up or down. Unless we jump over the middle line, in which case we lose φ^k for some k.

Solution

Let us choose appropriate φ to make jumping up no change to the sum. That is, we want $\varphi^2 + \varphi - 1 = 0$, which gives us $\varphi = \frac{-1\pm\sqrt{5}}{2}$. If you happen to know, this number $\varphi = \frac{-1+\sqrt{5}}{2}$ is so-called *golden ratio*.
With this ratio, jumping down reduces the sum by 2φ and jumping across the middle line reduces the sum by φ^k.

Solution

Let us calculate the total sum of the spots under the horizon.

		φ^6	φ^5	φ^4	φ^3	φ^2	φ	1	φ	φ^2	φ^3	φ^4	φ^5		
		φ^7	φ^6	φ^5	φ^4	φ^3	φ^2	φ	φ^2	φ^3	φ^4	φ^5	φ^6		
		φ^8	φ^7	φ^6	φ^5	φ^4	φ^3	φ^2	φ^3	φ^4	φ^5	φ^6	φ^7		
		φ^9	φ^8	φ^7	φ^6	φ^5	φ^4	φ^3	φ^4	φ^5	φ^6	φ^7	φ^8		
		φ^{10}	φ^9	φ^8	φ^7	φ^6	φ^5	φ^4	φ^5	φ^6	φ^7	φ^8	φ^9		

In the picture above, the first row below the horizon has sum $\frac{\varphi}{1-\varphi} + \frac{\varphi^2}{1-\varphi} = \frac{1}{1-\varphi}$. The second row is the first row multiplied by φ, hence it has sum $\frac{\varphi}{1-\varphi}$. The row sums form a geometric sequence and hence the total sum is $\frac{1/(1-\varphi)}{1-\varphi} = 5 + 3\varphi \approx 9.8541... > 1$.

Solution

If we shift the horizon down by 1, then we basically multiply the whole graph by φ.
So for $n = 2$, the total sum is $(5 + 3\varphi)\varphi = 5\varphi + 3\varphi^2 = 3 + 2\varphi \approx 6.2360... > 1$.
Similarly we have

- $n = 3$, the sum is $(3 + 2\varphi)\varphi = 2 + \varphi \approx 3.6180 > 1$.
- $n = 4$, the sum is $(2 + \varphi)\varphi = 1 + \varphi \approx 2.6180 > 1$.
- $n = 5$, the sum is $(1 + \varphi)\varphi = 1$.
- $n \geqslant 6$, the sum is $\leqslant \varphi \approx 0.6180 < 1$.

So the remaining question is that, is $n = 5$ possible? Note that we are summing over all boxes below the horizon. But we only have finitely many soldiers to start with, therefore for $n = 5$, the initial total sum is strictly smaller than 1. Hence it is impossible.

Thanks

Thanks for listening.
I also like to thank Wendy Wang for helping choosing this topic and reviewing the slides, to thank Chris Chen for typing up some questions.