I don't know

Zhan Jiang*

January 30, 2020

This worksheet[1] is for Math Circle. There are 8 questions on 6 pages. The last two question are *really hard*.

1. Duke pick two hats randomly from two black hats and one white hat. He gave one hat to his friend Kevin, and the other one to their common friend Chris. Both Kevin and Chris can see each other's hats, but not themselves'. Duke asked them to guess their hats' colors.

   Kevin I don't know.

   Chris I didn't know either, but now I know!

   Kevin I know, too!

   What are the colors of their hats?

   Solution: If Chris was wearing the only white hat, then Kevin would immediately know that his is black. Since Kevin doesn't know, Chris must be wearing a black hat. Same logic applies to Chris' words. Since he couldn't recognize his hat directly, Kevin must also have a black hat.

2. Kevin and Chris know that Duke's birthday is one of the following 10 dates:

   - Mar 4, Mar 5, Mar 8
   - Jun 4, Jun 7
   - Sep 1, Sep 5
   - Dec 1, Dec 2, Dec 8

   Duke told Kevin only the month of his birthday, and told Chris only the day, and asked them to guess his birthday.

   Kevin I don't know, but I know that Chris doesn't know either.

   Chris I didn't know Duke's birthday. But now I know.

   Kevin Now I know it, too.

   So what is Duke's birthday?

*I'd like to thank Wendy Wang for choosing the topic, checking and typing up solutions

[1]You can also download it from http://www-personal.umich.edu/~zoeng/files/Math_Circle/Math_Circle_1_solutions.pdf

Solution: If the day is unique, i.e. 2 or 7, then Kevin knew that Chris would immediately know. But Kevin was sure that he didn't know. So Chris knew that Kevin's month cannot be Jun or Dec. Then if Chris was given day 5, he still couldn't figure out the month. Therefore possible answers must be Mar 4, Mar 8 and Sep 1. If Kevin's month was Mar, then Kevin couldn't figure out which day. Hence it must be **Sep 1**.

3. Duke chose a card from following cards, told Kevin the suit of the card and Chris the rank of the card. He asked Kevin and Chris to guess which card he chose.

   - Spade J, 8, 4, 2
   - Heart A, Q, 4
   - Diamond A, 5
   - Club K, Q, 5, 4

   Kevin and Chris don't know each other's information.

   Chris I don't know.

   Kevin I knew that you didn't know. Before you say that.

   Chris Now I know.

   Kevin Now I know it, too.

   So what is the card?

Solution: Since Chris knows the rank but doesn't know the card, the rank must appear in more than one suit, which must be among $\{\text{A,Q,4,5}\}$. If the suit contains ranks that are unique ($\{2, 8, \text{J,K}\}$), then Kevin would be unsure of whether Chris knows. Therefore, Kevin must have either Heart or Diamond – both of which have every possible rank appearing in other suits. Knowing this, Chris immediately says he knows the card, which means the rank can't be Ace, as it appears in both Heart and Diamond. Lastly, if Kevin has Heart, then he still wouldn't know the card as he has no way to tell between Queen of Hearts or 4 of Hearts. Since Kevin knows immediately after Chris, Kevin must have Diamond, and the card must be **5 of Diamond**.

4. Duke chose two *possibly equal* integers a, b between 1 and 20 (including 1 and 20). He told Kevin the sum $S = a + b$, and Chris the product $P = ab$. But Kevin and Chris didn't know each other's number.

   Kevin I don't know what a and b are.

   Chris I don't know, either.

   Kevin I know now.

   What are a and b?

Solution: The sum S ranges from $1 + 1 = 2$ to $20 + 20 = 40$. But $2 = 1 + 1, 3 = 1 + 2, 39 = 19 + 20, 40 = 20 + 20$ have unique decomposition. So S must be between 4 and 38. Chris said he didn't know. So P cannot be a prime number. Then Kevin claimed that he knew. Consider the product $a'b'$ from all possible decomposition of $S = a' + b'$. There should be exactly one composite number, so that Kevin can figure out the two numbers. If S is larger than 6, then we always have $2 + (S - 2)$ and $4 + (S - 4)$ two ways to decompose S, and their product are both composite. So $4 \leqslant S \leqslant 6$.

For $S = 6$, we have $2 + 4$ and $3 + 3$. Both give composite products. For $S = 5$, we have $1 + 4$ and $2 + 3$. Both give composite products. Hence the only possibility is that $S = 4$ and since the product cannot be a prime, it has to be $2 * 2 = 4$ as well. So **both numbers are 2**.

5. Duke wrote two consecutive *positive* integers on two separate sheets of paper, and attached them to Chris and Kevin's heads. So Chris and Kevin can see each other's number, but not themselves'.

   Duke Who has the larger number?

   Kevin I don't know.

   Chris I don't know either.

   Kevin I don't know.

   Chris I don't know.

   Kevin After hearing all these, I still don't know.

   Chris Neither do I.

   Kevin Now I know!

   Chris I know, too.

   Who has the larger number?

   Solution: Given that the two numbers are consecutive positive integers, Kevin and Chris' numbers must be greater than or equal to 1. If Chris had 1, then Kevin would see this and know that he must have 2. Since Kevin said he didn't know, Chris knows that he must not have 1. If Kevin had 2, then Chris, knowing that he didn't have 1, must be able to conclude that he has 3. However, Chris said he didn't know either, which means that Kevin doesn't have 2. Using the same logic, Kevin and Chris' denial in the next two rounds show that Kevin and Chris both know that Kevin's number is greater than 6 and Chris' number is greater than 5. At this moment, Kevin claims that he knows his number. If Chris has any number larger than 7, then Kevin, knowing his number is at east 7, couldn't figure out his number. So Chris must have either 6 or 7. If Chris has 6, then Kevin must have 7 since their numbers are consecutive. If Chris has 7, then Kevin can only have 8. In both cases, **Kevin has the larger number.**

6. Duke generated two *different positive* integers, gave one to Kevin and the other one to Chris. He asked them to guess which one has a larger number.

   Kevin I don't know.

   Chris Neither do I.

   Kevin I still don't know.

   Chris I don't know, either.

   Kevin I know it!

   Chris I know too! And I also know what number you have.

   What are these two numbers?

Solution: If Kevin has 1, then he knows for sure that Chris' number is larger. Since Kevin doesn't know at first, Kevin's number must be greater than 1. Knowing this and the fact that Kevin and Chris have different numbers, if Chris has 1 or 2, he must know that Kevin's number is larger. Since Chris doesn't know, Chris' number must be greater than 2. Likewise, the next round of denials shows that both Kevin and Chris know that Kevin's number is greater than 3 and Chris' number is greater than 4. At this point, Kevin claims that he knows his number, which shows that he must have either 4 or 5 because he knows that Chris' number is greater than 4 and different from his. Chris immediately says that he knows too, which means that he is able to tell whether Kevin has 4 or 5. In order to claim this, **Chris must have 5**, which means that **Kevin must have 4**.

7. Duke came up with two *different* numbers of the form $n - \frac{1}{2^k} - \frac{1}{2^{k+r}}$ where both n, k are *positive integers*, r is a *non-negative integer*. He gave one to Kevin and the other one to Chris. He asked Kevin and Chris to guess who had the larger number.

   Kevin I don't know.

   Chris Neither do I.

   Kevin Indeed, I still do not know.

   Chris And still neither do I.

   Duke Well, it is no use to continue that way! I can tell you that no matter how long you continue that back-and-forth, you shall not come to know who has the larger number.

   Kevin What interesting new information! But alas, I still do not know whose number is larger.

   Chris And still also I do not know.

   Kevin I continue not to know.

   Chris I regret that I also do not know.

   Duke Let me say once again that no matter how long you continue truthfully to tell each other in succession that you do not yet know, you will not know who has the larger number.

   Kevin Well, thank you very much for saving us from that tiresome trouble! But unfortunately, I still do not know who has the larger number.

   Chris And also I remain in ignorance. However shall we come to know?

   Duke Well, in fact, no matter how long we three continue from now in the pattern we have followed so far—namely, the pattern in which you two state back-and-forth that still you do not yet know whose number is larger and then I tell you yet again that no further amount of that back-and-forth will enable you to know—then still after as much repetition of that pattern as we can stand, you will not know whose number is larger! Furthermore, I could make that same statement infinitely many times!

   Kevin Such powerful new information! But I am very sorry to say that still I do not know whose number is larger.

   Chris And also I do not know.

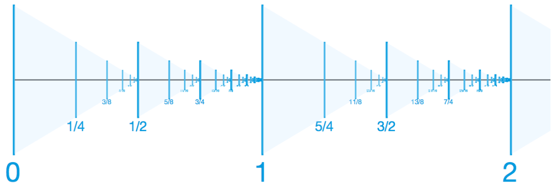
   Kevin But wait! It suddenly comes upon me after Chris's last remark, that finally I know who has the larger number!

   Chris Really? In that case, then I also know, and what is more, I know both of our numbers!

   Kevin Well, now I also know them!

   What are Chris' and Kevin's numbers?

Solution: Below is a graph of all possible numbers that Duke could generate.



Note that we have an infinite sequence between 0 and 1/2, another infinite sequence between 1/2 and 3/4, ... In fact, we have infinitely many infinite sequences between 0 and 1.

The first round that Chris and Kevin telling each other they don't know, they are excluding the numbers in the sequence $0, 1/4, 3/8, \ldots$. But then Duke reminded them that their pattern is useless no matter how many times they repeat it. So this tells us that the number cannot be in the sequence starting with 0. We have to start the game from the sequence $1/2, 5/8, \ldots$.

The next round of saying "I don't know" is happening on the new sequence from 1/2. Duke reminded them again that this pattern is useless. So they have to start from 3/4. But then Duke said that the pattern is useless even if they repeat this "useless pattern" infinitely many times. So that excludes infinitely many sequences between 0 and 1.

So last round of saying "don't know" is happening on the sequence from 1. Now we can do a similar logic induction: Kevin's denial implies that his number is at least 5/4. Then Chris's number must be at least 11/8. But then Kevin knows, so his number must be either 5/4 or 11/8. Chris not only knows whose number is larger, but also what number they have. So **Chris must have** 11/8 **and Kevin has** 5/4.

8. Duke chose two *distinct* integers a, b between 2 and 99 (including 2 and 99). He told Kevin the sum $S = a + b$, and Chris the product $P = ab$. But Kevin and Chris didn't know each other's number. After careful calculation, Kevin said, "I don't know what a and b are, but I know that Chris doesn't know for sure.". After hearing that, Chris smiled and said, "I didn't either. But now I know." Kevin replied, "I know too". What are a and b?

Solution: Kevin knows that Chris doesn't know. There are some possibilities for P where Chris would know the answer,

1. P is a product of two odd primes.
2. P is a product of 2 and a prime.
3. P is a product of a prime $p > 50$ and a number.

Therefore S cannot be decompose into any of above cases. Note that by Goldbach's conjecture, which is true for all even numbers less or equal to 200, S must be an odd number.

Also S cannot exceed 54. As otherwise, $S = 53 + (S - 53)$ and Chris will be able to figure out if P happens to be $53 * (S - 53)$. So S is an odd number less or equal to 53. Finally S cannot be the sum of 2 and an odd prime. So S can only be $\{11, 17, 23, 27, 29, 35, 37, 47, 51, 53\}$.

Chris, on the other hand, didn't know at first. But he knew after hearing what Kevin said. So Chris' product, after decomposed into two factors, can only sum up to one number out of the ten possibilities. Kevin figured that out after he heard from Chris.

For example, if Kevin has 17, and he decomposes into 14+3. Then Chris must have 14*3=42=2*21=6*7. Then Chris could have either 14*3, which sums up to 17, or 2*21, which sums up to 23. Thus Chris cannot figure out which sum Kevin has. So knowing that Chris can figure out the numbers, Kevin must exclude such decomposition. The remaining possible decomposition are

11	(4, 7), (3, 8), (2, 9)
17	(4, 13)
23	(10, 13), (7, 16), (4, 19)
27	(13, 14), (11, 16), (10, 17), (9, 18), (8, 19), (7, 20), (5, 22), (4, 23), (2, 25)
29	(13, 16), (12, 17), (11, 18), (10, 19), (8, 21), (7, 22), (6, 23), (4, 25), (2, 27)
35	(17, 18), (16, 19), (14, 21), (12, 23), (10, 25), (9, 26), (8, 27), (6, 29), (4, 31), (3, 32)
37	(17, 20), (16, 21), (10, 27), (9, 28), (8, 29), (6, 31), (5, 32)
41	(19, 22), (18, 23), (17, 24), (16, 25), (15, 26), (14, 27), (13, 28), (12, 29), (10, 31), (9, 32), (7, 34), (4, 37), (3, 38)
47	(23, 24), (22, 25), (20, 27), (19, 28), (18, 29), (17, 30), (16, 31), (15, 32), (13, 34), (10, 37), (7, 40), (6, 41), (4, 43)
53	(26, 27), (25, 28), (24, 29), (23, 30), (22, 31), (21, 32), (20, 33), (19, 34), (18, 35), (17, 36), (16, 37), (15, 38), (13, 40), (12, 41), (10, 43), (8, 45), (6, 47), (5, 48)

Given that Kevin is able to figure out what a, b are. The sum must be 17 and these two numbers are **4 and 13**.