

I don't know

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This worksheet¹ is for Math Circle. There are 8 questions on 3 pages. The last two question are *really hard*.

1. Duke pick two hats randomly from two black hats and one white hat. He gave one hat to his friend Kevin, and the other one to their common friend Chris. Both Kevin and Chris can see each other's hats, but not themselves'. Duke asked them to guess their hats' colors.

Kevin I don't know.

Chris I didn't know either, but now I know!

Kevin I know, too!

What are the colors of their hats?

2. Kevin and Chris know that Duke's birthday is one of the following 10 dates:

- Mar 4, Mar 5, Mar 8
- Jun 4, Jun 7
- Sep 1, Sep 5
- Dec 1, Dec 2, Dec 8

Duke told Kevin only the month of his birthday, and told Chris only the day, and asked them to guess his birthday.

Kevin I don't know, but I know that Chris doesn't know either.

Chris I didn't know Duke's birthday. But now I know.

Kevin Now I know it, too.

So what is Duke's birthday?

3. Duke chose a card from following cards, told Kevin the suit of the card and Chris the rank of the card. He asked Kevin and Chris to guess which card he chose.

- Spade J, 8, 4, 2
- Heart A, Q, 4
- Diamond A, 5
- Club K, Q, 5, 4

*I'd like to thank Wendy Wang for choosing the topic, checking and typing up solutions

¹You can also download it from http://www-personal.umich.edu/~zoeng/files/Math_Circle/Math_Circle_1.pdf

Kevin and Chris don't know each other's information.

Chris I don't know.

Kevin I knew that you didn't know. Before you say that.

Chris Now I know.

Kevin Now I know it, too.

So what is the card?

4. Duke chose two *possibly equal* integers a, b between 1 and 20 (including 1 and 20). He told Kevin the sum $S = a + b$, and Chris the product $P = ab$. But Kevin and Chris didn't know each other's number.

Kevin I don't know what a and b are.

Chris I don't know, either.

Kevin I know now.

What are a and b ?

5. Duke wrote two consecutive *positive* integers on two separate sheets of paper, and attached them to Chris and Kevin's heads. So Chris and Kevin can see each other's number, but not themselves'.

Duke Who has the larger number?

Kevin I don't know.

Chris I don't know either.

Kevin I don't know.

Chris I don't know.

Kevin After hearing all these, I still don't know.

Chris Neither do I.

Kevin Now I know!

Chris I know, too.

Who has the larger number?

6. Duke generated two *different positive* integers, gave one to Kevin and the other one to Chris. He asked them to guess which one has a larger number.

Kevin I don't know.

Chris Neither do I.

Kevin I still don't know.

Chris I don't know, either.

Kevin I know it!

Chris I know too! And I also know what number you have.

What are these two numbers?

7. Duke came up with two *different* numbers of the form $n - \frac{1}{2^k} - \frac{1}{2^{k+r}}$ where both n, k are *positive integers*, r is a *non-negative integer*. He gave one to Kevin and the other one to Chris. He asked Kevin and Chris to guess who had the larger number.

Kevin I don't know.

Chris Neither do I.

Kevin Indeed, I still do not know.

Chris And still neither do I.

Duke Well, it is no use to continue that way! I can tell you that no matter how long you continue that back-and-forth, you shall not come to know who has the larger number.

Kevin What interesting new information! But alas, I still do not know whose number is larger.

Chris And still also I do not know.

Kevin I continue not to know.

Chris I regret that I also do not know.

Duke Let me say once again that no matter how long you continue truthfully to tell each other in succession that you do not yet know, you will not know who has the larger number.

Kevin Well, thank you very much for saving us from that tiresome trouble! But unfortunately, I still do not know who has the larger number.

Chris And also I remain in ignorance. However shall we come to know?

Duke Well, in fact, no matter how long we three continue from now in the pattern we have followed so far—namely, the pattern in which you two state back-and-forth that still you do not yet know whose number is larger and then I tell you yet again that no further amount of that back-and-forth will enable you to know—then still after as much repetition of that pattern as we can stand, you will not know whose number is larger! Furthermore, I could make that same statement infinitely many times!

Kevin Such powerful new information! But I am very sorry to say that still I do not know whose number is larger.

Chris And also I do not know.

Kevin But wait! It suddenly comes upon me after Chris's last remark, that finally I know who has the larger number!

Chris Really? In that case, then I also know, and what is more, I know both of our numbers!

Kevin Well, now I also know them!

What are Chris' and Kevin's numbers?

8. Duke chose two *distinct* integers a, b between 2 and 99 (including 2 and 99). He told Kevin the sum $S = a + b$, and Chris the product $P = ab$. But Kevin and Chris didn't know each other's number. After careful calculation, Kevin said, "I don't know what a and b are, but I know that Chris doesn't know for sure.". After hearing that, Chris smiled and said, "I didn't either. But now I know." Kevin replied, "I know too". What are a and b ?