Course notes of « Linear Algebraic Groups »

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Ch. I. Basic notions and properties

§1. Definitions and examples

• Def. 1.1. Let $\mathscr{C}$ be a category in which finite products (in particular, a terminal object t) exist. A group object in $\mathscr{C}$ is a quadruple $(G, \bullet, e, \iota)$ consisting of an object $G \in ob(\mathscr{C})$ and morphisms

$$\bullet : G \times G \to G, \quad e : t \to G, \quad \iota : G \to G,$$

such that the following diagrams commute:

(associativity)
$$\begin{tikzcd} G\times G\times G \arrow[r, "\bullet\times id"] \arrow[d, "id\times\bullet"'] & G\times G \arrow[d, "\bullet"] \\ G\times G \arrow[r, "\bullet"] & G \end{tikzcd}$$

(unity)
$$\begin{tikzcd} G\times t \arrow[r, "id\times e"] & G\times G \arrow[d, "\bullet"] & t\times G \arrow[l, "e\times id"'] \\ G \arrow[u, "\simeq"] \arrow[r, "id"] & G & G \arrow[l, "id"'] \arrow[u, "\simeq"'] \end{tikzcd}$$

(inversion)

$$\begin{tikzcd}
 & & t \arrow[r, bend left, "e"] & G \\
 & & G \times G \arrow[ur, phantom] \arrow[r, "\bullet"] & G \\
G \arrow[r, "\Delta"] \arrow[uurr] \arrow[ddrr] & G \times G \arrow[ur, "\mathrm{id} \times \iota"] \arrow[dr, "\iota \times \mathrm{id}"'] & & \\
 & & G \times G \arrow[r, "\bullet"] & G \\
 & & t \arrow[ur, bend right, "e"'] &
\end{tikzcd}$$

.

- Exercise 1.2. Given an object G in $\mathcal{C}$, show that to put a group object structure on G is equivalent to giving a lifting

$$\begin{tikzcd}
 & \underline{\text{Groups}} \arrow[d, "\text{forgetful}"] \\
\mathcal{C}^{op} \arrow[ur, dashed] \arrow[r, "h_G"'] & \underline{\text{Sets}}
\end{tikzcd}$$

of the functor $h_G := \mathrm{Hom}_{\mathcal{C}}(-, G)$. Consequently, a group object in $\mathcal{C}$ can be equivalently defined to be a functor $F : \mathcal{C}^{op} \to \underline{\text{Groups}}$ such that the composition

$$\mathcal{C}^{op} \xrightarrow{F} \underline{\text{Groups}} \longrightarrow \underline{\text{Sets}}$$

is representable. Use Yoneda's lemma to show that this is also equivalent to a group object in the functor category $\mathrm{Func}(\mathcal{C}^{op}, \underline{\text{Sets}})$ that is representable. //.

- Examples 1.3. When $\mathcal{C} = \underline{\text{Sets}}$ (resp. Groups, top. spaces, differentiable manifolds, complex manifolds, S-schemes, for a base scheme S), group objects in $\mathcal{C}$

are abstract groups (resp. abelian groups, top. groups, Lie groups, complex Lie groups, S-group schemes).

Our main object to study in these notes is the case when $S = \mathrm{Spec}(k)$ for a base field k, and G is an <u>affine</u> k-group scheme. For simplicity, we will always assume that k is <u>algebraically closed</u> of <u>characteristic zero</u>, even though it is not necessary for some results in the sequel. By an <u>alg. group</u> over k, we mean an affine k-group scheme of finite type.

As $h_G : k\text{-}\mathcal{S}ch^{op} \longrightarrow \underline{\mathrm{Groups}}$ is determined by its restriction to the full subcategory of affine k-schemes (an exercise), we usually consider

$$\begin{aligned} h_G : k\text{-}\mathcal{A}lg &\longrightarrow \underline{\mathrm{Groups}} \\ R &\longmapsto G(R) := h_G(\mathrm{Spec}\, R) \\ &\qquad =: \{R\text{-points of } G\}. \end{aligned}$$

- <u>Examples 1.4.</u>
  - $\mathbb{G}_{a,k}$: it is defined by $R \mapsto$ additive group of R. As a scheme, it is $\mathrm{Spec}\, k[T] \simeq \mathbb{A}^1_k$; it is called the <u>additive group scheme</u>.

- $\mathbb{G}_{m,k}$: it is defined by $R \mapsto$ mult. group $R^\times$.

As a scheme, it is $\mathrm{Spec}\, k[T, T^{-1}]$; it is called the <u>multiplicative group scheme</u>.

- $GL_{n,k}$: it is defined by $R \mapsto GL_n(R)$. As a scheme, it is

$$\mathrm{Spec}\; k[T_0, T_{ij} \mid 1 \le i, j \le n] \big/ (T_0 \cdot \det(T_{ij}) - 1) \,;$$

it is called the <u>general linear group</u>.

- $\mathbb{T}_{n,k}$: it is defined by

$$R \mapsto \{ M \in GL_n(R) \mid M \text{ is upper-triangular} \}. \quad \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix}.$$

- $\mathbb{D}_{n,k}$: it is defined by

$$R \mapsto \{ M \in GL_n(R) \mid M \text{ is diagonal} \}. \quad \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$$

As a group scheme, it is isomorphic to $\mathbb{G}_{m,k}^n$.

- $\mathbb{U}_{n,k}$: it is defined by

$$R \mapsto \{ M \in GL_n(R) \mid M \text{ takes the form } \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \}.$$

As a scheme, it is an affine space

$$\mathrm{Spec}\; k[T_{ij} \mid 1 \le i < j \le n] \simeq \mathbb{A}_k^{n(n-1)/2} ,$$

but as a group scheme, it is not isomorphic to $\mathbb{G}_{a,k}^{n(n-1)/2}$ unless $n = 2$; it is not commutative for $n > 2$.

- $SL_{n,k}$: it is defined by $R \mapsto SL_n(R)$. As a scheme, it is
$$\operatorname{Spec} k[T_{ij} \mid 1 \le i, j \le n] / (\det(T_{ij}) - 1);$$
it is called the <u>special linear</u> group.

- $O_{n,k}$: it is defined by
$$R \mapsto \{ M \in GL_n(R) \mid M^T \cdot M = I \};$$
it is called the <u>orthogonal</u> group.

- $Sp_{2n,k}$: it is defined by
$$R \mapsto \left\{ M \in GL_{2n}(R) \,\middle|\, M^T \cdot \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} \cdot M = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} \right\};$$
it is called the <u>symplectic</u> group.

> intrinsic: GL_V, SL_V, $O(V, \phi)$, $Sp(V, \phi)$ ...
>
> GL_V, when V is ∞-dim'l, is only a group functor, ~~unless $V = \bigcup V_i$~~ GL_V^{alg} = those auto. that stablize larger and larger f.d. subspaces

- <u>finite constant group</u>: let Γ be an abstract finite group, and let k^Γ be the algebra of set-functions from Γ to k: it is the direct product of copies of k indexed by elements in Γ (note that in general, this algebra is different from the group algebra $k[\Gamma]$). Define $\Delta : k^\Gamma \to k^\Gamma \otimes k^\Gamma \simeq k^{\Gamma \times \Gamma}$ by
$$(\Delta \varphi)(g, h) = \varphi(g \cdot h).$$
Consider the functor

$$(\underline{\Gamma})_k : k\text{-}\mathcal{A}lg \longrightarrow \underline{Sets}$$
$$R \longmapsto \mathrm{Hom}_{k\text{-alg}}(k^\Gamma, R).$$

This set Hom has a group structure, functorial in R, as follows: given $\varphi, \psi : k^\Gamma \to R$, define $\varphi\cdot\psi$ to be

$$k^\Gamma \xrightarrow{\Delta} k^\Gamma \otimes k^\Gamma \xrightarrow{\varphi\otimes\psi} R\otimes R \xrightarrow{m} R;$$

the identity element is

$$k^\Gamma \xrightarrow{ev_{1_\Gamma}} k \longrightarrow R;$$

the inverse of $\varphi : k^\Gamma \to R$ is

$$k^\Gamma \longrightarrow k^\Gamma \xrightarrow{\varphi} R$$
$$(f : \Gamma \to k) \longmapsto (g \mapsto f(g^{-1}) : \Gamma \to k).$$

These make $(\Gamma)_k := \mathrm{Spec}(k^\Gamma)$ into an alg. group, finite of degree $\#\Gamma$ over k.

- $\mu_{n,k}$: it is defined by $R \mapsto \mu_n(R) = \{x \in R^\times \mid x^n = 1_R\}$.

As a scheme, it is $\mathrm{Spec}\, k[T]/(T^n-1)$; it is called the <u>group of n^{th} roots of unity</u>. Note that $k[T]/(T^n-1)$ is isom. to the group algebra $k[\mathbb{Z}/n\mathbb{Z}]$ of the additive group $\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$ as follows:

$$\sum a_{\bar{i}}\cdot[\bar{i}] \longmapsto \sum a_{\bar{i}}\cdot T^i.$$

By **our** assumption on k, the polynomial $T^n - 1$ is separable and splits into linear factors over k:

$$T^n - 1 = (T-1)(T-\zeta_n)\cdots(T-\zeta_n^{n-1}),$$

so by the Chinese remainder theorem,

$$\frac{k[T]}{(T^n-1)} \simeq \frac{k[T]}{(T-1)} \times \frac{k[T]}{(T-\zeta_n)} \times \cdots \times \frac{k[T]}{(T-\zeta_n^{n-1})}$$

$$\simeq k \times \cdots \times k \simeq k^{\mathbb{Z}/n\mathbb{Z}}.$$

In fact, the choice of a ζ_n defines an isom. of alg. groups

$$(\mathbb{Z}/n\mathbb{Z})_k \xrightarrow{\ \sim\ } \mu_{n,k}.$$

Intrinsically, $(\varphi, \sum a_\gamma \cdot \gamma) \mapsto \sum a_\gamma \cdot \varphi(\gamma)$ is a perfect pairing on $k^\Gamma \times k[\Gamma]$, $\forall$ fini. gr. Γ, inducing an isom. $k[\Gamma] \simeq (k^\Gamma)^\vee$ of Hopf alg.

- Def. 1.5. Given two alg. groups G and H, a homomorphism $\varphi: G \to H$ is a natural transformation

$$\begin{tikzcd} k\text{-}\mathcal{A}lg \arrow[r, bend left, "h_G"] \arrow[r, bend right, "h_H"'] \arrow[r, phantom, "\Downarrow\varphi"] & \underline{\text{Groups}} \end{tikzcd},$$

in view of Ex.(1.2). When $H = GL_n$, a homomorphism $\varphi: G \to GL_n$ is called a linear (or rational) representation of ~~degree n~~ G, of degree n. For a hom. $\varphi: G \to H$, its kernel, $\ker(\varphi)$, is the fiber-product

$$\begin{tikzcd} \ker(\varphi) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[d, "\varphi"] \\ \mathrm{Spec}(k) \arrow[r, "e_H"'] & H \end{tikzcd},$$

which represents the <u>kernel functor</u>

$$R \longmapsto \ker(\varphi_R : G(R) \to H(R))$$ (an Yoneda exercise).

§2. Hopf algebras.

<u>Notation</u>: for G an affine group$/k$, let

$$k[G] = \Gamma(G, \mathcal{O}_G).$$

The group structure $(\cdot, e, \iota)$ on G translates into certain diagrams involving $k[G]$, which leads to the following definition.

• <u>Def. 2.1.</u> A k-<u>Hopf algebra</u>, always assumed commutative in these notes, consists of

$$(A, m, e, \Delta, \varepsilon, S)$$

where $(A,\ m: A\otimes_k A \to A,\ e: k \to A)$ is a comm. k-algebra, and $(A,\ \Delta: A \to A\otimes_k A,\ \varepsilon: A \to k)$ is a "co-associative" k-"co-algebra" with "co-identity", i.e. the following diagrams commute

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A\otimes A \arrow[d, "\Delta\otimes id"] \\ A\otimes A \arrow[r, "id\otimes\Delta"] & A\otimes A\otimes A \end{tikzcd}
\qquad
\begin{tikzcd} & A \arrow[dl, "\sim"'] \arrow[d, "\Delta"] \arrow[dr, "\sim"] & \\ k\otimes A & A\otimes A \arrow[l, "\varepsilon\otimes id"] \arrow[r, "id\otimes\varepsilon"'] & A\otimes k \end{tikzcd}$$
,

and $S : A \to A$ is a k-algebra homomorphism, such that

i). both Δ and ε are k-algebra homomorphisms,

ii). the diagram commutes:

$$\begin{tikzcd}
A\otimes A \arrow[r, "S\otimes id"] & A\otimes A \arrow[d, "m"] & & A\otimes A \arrow[ll, "id\otimes S"'] \\
A \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] & k \arrow[r, "e"] & A & k \arrow[l, "e"'] & A \arrow[l, "\varepsilon"'] \arrow[ul, "\Delta"']
\end{tikzcd}$$

A <u>homomorphism</u> of k-Hopf algebras $A \to B$ is a k-linear map preserving **all** structures $(m, e, \Delta, \varepsilon, S)$.

• <u>Th. 2.2</u>. <u>The functor</u>

$$G \mapsto k[G] : \{\text{affine } k\text{-groups}\}^{op} \to \{k\text{-Hopf alg.}\}$$

<u>is an equivalence of categories, with quasi-inverse</u>

$A \mapsto Spec(A)$.

<u>Pf</u>: This is clear, taking into account that

$$\mathrm{Hom}_{k\text{-Sch.}}(Spec(B), Spec(A)) = \mathrm{Hom}_{k\text{-alg.}}(A, B)$$

and $Spec(A) \times_{Spec(k)} Spec(B) = Spec(A \otimes_k B)$. $\square$

• <u>Examples 2.3.</u>

• $\mathbb{G}_{a,k}$ corresponds to the Hopf algebra $k[T]$ with

$$\Delta : k[T] \longrightarrow k[T] \otimes k[T]$$

$$\Delta(T) = 1 \otimes T + T \otimes 1 \quad (\text{so } \Delta(f(T)) = f(\Delta(T)) \cdots),$$

$$\varepsilon : k[T] \longrightarrow k$$

$$\varepsilon(T) = 0 \quad (\text{so } \varepsilon(f(T)) = f(0)),$$

$$S : k[T] \longrightarrow k[T]$$

$$S(T) = -T \quad (\text{so } S(f(T)) = f(-T)).$$

- $\mathbb{G}_{m,k}$ corresponds to

$$\Delta : k[T, T^{-1}] \longrightarrow k[T, T^{-1}] \otimes k[T, T^{-1}]$$

$$\Delta(T) = T \otimes T \quad (= (1 \otimes T) \cdot (T \otimes 1)),$$

$$\varepsilon : k[T, T^{-1}] \longrightarrow k$$

$$\varepsilon(T) = 1 ,$$

$$S : k[T, T^{-1}] \longrightarrow k[T, T^{-1}]$$

$$S(T) = T^{-1}.$$

- $(\Gamma)_k$ corresponds to

$$\Delta : k^\Gamma \longrightarrow k^\Gamma \otimes k^\Gamma \quad \text{described above},$$

$$\varepsilon = ev_{1_\Gamma} : k^\Gamma \longrightarrow k ,$$

$$S : k^\Gamma \longrightarrow k^\Gamma$$

$$S(f)(g) = f(g^{-1}).$$

- <u>Def. 2.4.</u> Let (A, Δ, ε) be a co-assoc. k-coalgebra with co-identity, and let V be a k-vector space. A <u>co-action</u>

of A on V (or a <u>right</u> A-<u>co-module structure</u> on V) is a k-linear map $\sigma : V \to V \otimes_k A$ such that the diagrams commute:

$$\begin{tikzcd} V \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & V \otimes A \arrow[d, "id \otimes \Delta"] \\ V \otimes A \arrow[r, "\sigma \otimes id"] & V \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\sigma"] \arrow[dr, "\sim"'] & V \otimes A \arrow[d, "id \otimes \varepsilon"] \\ & V \otimes k \end{tikzcd}.$$

A <u>homomorphism</u> of right co-modules $V \xrightarrow{\varphi} W$ is a k-linear map such that the diagram commutes

$$\begin{tikzcd} V \arrow[r, "\sigma_V"] \arrow[d, "\varphi"'] & V \otimes A \arrow[d, "\varphi \otimes id"] \\ W \arrow[r, "\sigma_W"] & W \otimes A \end{tikzcd}.$$

When φ is injective, we also say that V is a <u>sub-comodule</u> of (W, σ), i.e. $\sigma(V) \subset V \otimes A$.

- <u>Example 2.5.</u> The comultiplication Δ makes A into a right A-co-module, called the <u>regular comodule</u> of A.

- <u>Prop. 2.6.</u> <u>Every</u> A-<u>comodule</u> (V, σ) <u>is a directed union of its finite-dim. sub-comodules</u>.

<u>Pf</u>: It suffices to prove that every finite-dim'l subspace $W_0 \subset V$ is contained in a finite-dim'l sub-comodule $W \subset V$. We may assume $\dim W_0 = 1$, so let $v \in V$,

and let $\sigma(v) = \sum_{i \in I} v_i \otimes a_i$ (finite sum) for a chosen basis $(a_i)_{i \in I}$ of the k-vector space A. Then we claim that $\mathrm{Span}(v, v_i\ (i \in I))$ is a sub-comodule of V. It is finite-dim'l, since all but finitely many v_i's are zero.

We have

$$(id_V \otimes \Delta)\Big(\sum_i v_i \otimes a_i\Big) = (\sigma \otimes id_A)\Big(\sum_i v_i \otimes a_i\Big),$$

so if we let

$$\Delta(a_i) = \sum_j b_{ij} \otimes a_j \ , \quad b_{ij} \in A \ ,$$

then

$$\sum_{i,j} v_i \otimes b_{ij} \otimes a_j = \sum_i \sigma(v_i) \otimes a_i \ ,$$

so $\sigma(v_j) = \sum_i v_i \otimes b_{ij} \in \mathrm{Span}(v, v_i\ (i \in I)) \otimes A$.

One can in fact drop v from the span because

$$\sum_i \varepsilon(a_i) \cdot v_i = v. \qquad \square$$

- <u>Prop. 2.7. Let</u> G <u>be an affine</u> k-<u>group scheme</u>, <u>and let</u> V <u>be a</u> k-<u>vector space. There is a natural 1-1 correspondence between</u> G-<u>actions on</u> V <u>and</u> $k[G]$-<u>coactions on</u> V.

<u>Sketch of proof</u>. Given $\rho: G \to GL_V$, consider

$$\begin{array}{ccc} \rho_{k[G]} : G(k[G]) & \longrightarrow & GL(V \otimes_k k[G]) \\ \| & & \nwarrow (k[G]\text{-linear automorphisms}) \\ \mathrm{Hom}_{k\text{-alg}}(k[G], k[G]) & & \\ \cup\!\!\!| & & \\ id_{k[G]} & \longmapsto & \rho_{k[G]}(id_{k[G]}). \end{array}$$

Then

$$V \simeq V \otimes_k k \hookrightarrow V \otimes_k k[G] \xrightarrow{\rho_{k[G]}(id)} V \otimes_k k[G]$$

gives a coaction by $k[G]$ on V. Conversely, given $\sigma: V \to V \otimes k[G]$, define $\rho: G \to GL_V$ to be that $\forall$ k-algebra R, $\forall$ $g \in G(R) = \mathrm{Hom}_{k\text{-alg}}(k[G], R)$, extend

$$V \xrightarrow{\sigma} V \otimes k[G] \xrightarrow{id_V \otimes g} V \otimes R$$

by linearity to an R-linear morphism denoted $\rho_R(g)$:

$$V \otimes R \longrightarrow V \otimes R. \quad \square$$

Explicitly, for $\dim(V) < \infty$, choose a basis for V, so that we may assume $V = k^n$ with standard basis $(e_i)_{1 \le i \le n}$ and $GL_V = GL_n$; one verifies that

$$\left\{ \begin{array}{l} \rho: G \to GL_n \\ \quad g \mapsto (a_{ij}(g))_{i,j} \\ \text{for } a_{ij} \in k[G] \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \sigma: k^n \to k^n \otimes k[G] \\ \quad e_j \mapsto \sum_{i=1}^n e_i \otimes a_{ij} \end{array} \right\}$$

is a hom. $\Longleftrightarrow$ defines a coaction.

• Cor. 2.8. For an affine k-group scheme G, every linear rep. is a directed union of its finite-dim'l subrepresentations. $\square$.

• Rem. 2.9. Let G be an affine k-group. The regular rep. of G on $k[G]$ is defined as follows. Explicitly it is the "right-translation" action:

$$(g\cdot f)(x) = f(x\cdot g), \quad \text{for "}x, g \in G\text{"}, \ f \in k[G].$$

Formally,

$$\mathrm{Reg.}: G \longrightarrow GL_{k[G]}$$

is defined by: $\forall$ k-alg. R, $\forall g \in G(R)$, the R-morphism

$$\begin{array}{ccccc} G_R \simeq G_R \times_R \mathrm{Spec}(R) & \xrightarrow{id \times g} & G_R \times_R G_R & \xrightarrow{\bullet_R} & G_R \\ \| & & & & \| \\ \mathrm{Spec}(R[G]) & & & & \mathrm{Spec}(R[G]) \end{array}$$

induces an R-algebra (a fortiori, R-linear) homomorphism (in fact, isom.) $R[G] \longrightarrow R[G]$.

• Exercise: Show that the regular rep. of G on $k[G]$ corresponds, via (2.7), to the regular coaction of $k[G]$ on itself (2.5). //

• Th. 2.10. Every k-Hopf algebra $(A, \cdots)$ is a directed union of its k-Hopf subalgebras that are finitely generated as k-algebras. Consequently, every affine k-group scheme is an inverse limit of k-alg. groups.

Pf: It suffices to show that every finite subset of A is contained in some f.g. k-Hopf subalgebra. From the proof of (2.6) one sees that any finite subset is contained in a finite-dim'l sub-comodule V (say with a basis $\{v_i\}$) of the regular comodule A of A; let

$$\Delta(v_i) = \sum_j v_j \otimes a_{ij} \quad , \quad a_{ij} \in A .$$

Then since

$$\begin{array}{ccc} (\Delta \otimes id)(\Delta(v_i)) & = & (id \otimes \Delta)(\Delta(v_i)) \\ \| & & \| \\ \sum_j \Delta(v_j) \otimes a_{ij} & & \sum_j v_j \otimes \Delta(a_{ij}) \\ \| & & \| \\ \sum_{j,k} v_k \otimes a_{jk} \otimes a_{ij} & & \sum_k v_k \otimes \Delta(a_{ik}) , \end{array}$$

we have $\Delta(a_{ik}) = \sum_j a_{jk} \otimes a_{ij}$. So $\mathrm{Span}(v_i, a_{ij})$ is a sub-coalgebra of (A, Δ, ε).

We have $(gh)^{-1} = h^{-1} \cdot g^{-1}$ in a group, so the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\cdot"] \arrow[d, "\text{flip}"'] & G \arrow[r, "\iota = \text{inversion}"] & G \\ G \times G \arrow[r, "\iota \times \iota"'] & G \times G \arrow[ur, "\cdot"'] & \end{tikzcd}$$

commutes, so also does the diagram

$$\begin{tikzcd}
A\otimes A & & A \arrow[ll, "\Delta"'] & \\
& & & A \arrow[ul, "S = \text{inversion}"'] \arrow[dl, "\Delta"] \\
A\otimes A \arrow[uu, "\text{flip}"] & & A\otimes A \arrow[ll, "S\otimes S"] &
\end{tikzcd},$$

i.e., if $\Delta(a) = \sum_i b_i \otimes c_i$, then $\Delta(S(a)) = \sum_i S(c_i)\otimes S(b_i)$.

Let B be the subalgebra of A generated by the

$$v_i,\ a_{ij},\ S(v_i),\ S(a_{ij}).$$ (as well as S)

Then B is of finite type over k, and as Δ is an algebra homomorphism, we see that

$$\Delta(B) \subset B\otimes B,\quad S(B)\subset B.$$

Hence B is a f.g. Hopf subalgebra of A, containing an arbitrarily given finite subset of A. $\square$.

§3. A theorem of Cartier.

In this section, we prove a theorem of P. Cartier that, under our assumption on k, every alg. group over k is smooth.

(see Bourbaki, Alg., V, §15, no. 2).

First, a quick review of some comm. algebra.

- Def. 3.1. A k-algebra R is separable if $\forall$ field ext. L/k, the ring $R\otimes_k L$ is reduced.

One verifies that when R is a field, this agrees with the "classical" notion of separability. Consequently, assuming char$(k)=0$, all k-fields are sep.

- Lem. 3.2. i). If a k-algebra R is reduced, then it is sep.
  ii). If A and B are reduced k-algebras, so is $A\otimes_k B$.

Pf: i). The kernel of the diagonal mapping

$$R \longrightarrow \prod_{\mathfrak{p}} k(\mathfrak{p}) \ , \quad \mathfrak{p}\subset R \text{ prime ideals}$$

is $\bigcap_{\mathfrak{p}} \mathfrak{p} = \mathfrak{Nil}(R)=0$. $\forall$ field ext. L/k, $R\otimes L$ embeds into $(\prod_{\mathfrak{p}} k(\mathfrak{p}))\otimes L$, hence into $\prod_{\mathfrak{p}}(k(\mathfrak{p})\otimes_k L)$. As $k(\mathfrak{p})$ is sep. over k, the rings $k(\mathfrak{p})\otimes_k L$, and hence their product, and hence $R\otimes L$, are reduced.

ii). As in i), we have $A \hookrightarrow \prod_{\mathfrak{p}} k(\mathfrak{p})$, so

$$A\otimes_k B \hookrightarrow (\prod_{\mathfrak{p}} k(\mathfrak{p}))\otimes_k B \hookrightarrow \prod_{\mathfrak{p}} (k(\mathfrak{p})\otimes_k B).$$

By i), B is sep., so each $k(\mathfrak{p})\otimes B$ is reduced, hence $A\otimes B$ is reduced. $\square$.

- Cor. 3.3. For any alg. group G over k, the max. closed reduced subscheme G_{red} inherits an alg. group structure from G, such that the canonical embedding $G_{red}\hookrightarrow G$ is a homomorphism of alg. groups over k.

Pf: By 3.2 (ii), $k[G]_{red} \otimes_k k[G]_{red}$ is reduced, so we have factorizations

$$\begin{tikzcd} k[G] \arrow[r, "\Delta"] \arrow[d] & k[G]\otimes k[G] \arrow[r] & k[G]_{red}\otimes k[G]_{red} \\ k[G]_{red} \arrow[urr, dashed, "\Delta_{red}"'] & & \end{tikzcd}$$

$$\begin{tikzcd} k[G] \arrow[r, "\varepsilon"] \arrow[d] & k \\ k[G]_{red} \arrow[ur, dashed, "\varepsilon_{red}"'] & \end{tikzcd} \qquad \begin{tikzcd} k[G] \arrow[r, "S"] \arrow[d] & k[G] \arrow[d] \\ k[G]_{red} \arrow[r, dashed, "S_{red}"'] & k[G]_{red} \end{tikzcd}$$

which make $k[G]_{red}$ into a k-Hopf algebra, such that $k[G] \to k[G]_{red}$ is a hom. of k-Hopf algebras. $\square$.

As $k = \bar{k}$, G is smooth over k if and only if G is regular, if and only if $\mathcal{O}_{G,e}$ is a regular local ring (by homogeneity). Since any alg. variety (i.e. integral scheme sep. and of f.t. over k) has at least one regular closed point, we see that G is regular if and only if G is reduced. For instance, G_{red} is always regular.

- <u>Lem. 3.4</u>. G <u>is smooth over</u> $k \Leftrightarrow \mathrm{Nil}(k[G]) \subset \mathfrak{m}_e^2$.

Pf: $(\Rightarrow)$ is clear.
$(\Leftarrow)$. Let $\mathcal{N} = \mathrm{Nil}(k[G])$. Let $\mathfrak{m}_{e'}$ be the image of $\mathfrak{m}_e$ in $k[G]_{red}$:

$$\begin{tikzcd} & G_{red} \arrow[d, hook] \\ \mathrm{Spec}(k) \arrow[ur, "e'"] \arrow[r, "e"'] & G \end{tikzcd}, \quad \mathfrak{m}_{e'} = \mathfrak{m}_e/\mathcal{N}.$$

As $\mathfrak{N} \subset m_e^2$, we have $m_{e'}^2 = m_e^2/\mathfrak{N}$, and so

$$m_e/m_e^2 \simeq m_{e'}/m_{e'}^2$$

as k-vector spaces. Since $k[G]_{m_e}$ and $k[G_{red}]_{m_{e'}}$ have the same Krull dim., and G_{red} is regular, we see that

$$\text{Krull. dim. } k[G]_{m_e} = \dim_k(m_e/m_e^2),$$

i.e. $k[G]_{m_e}$ is a regular local ring. $\square$.

• <u>Th. 3.5.</u> (Cartier). <u>Every alg. group</u> G/k <u>is smooth</u>.

<u>Pf</u>. Let $A = k[G]$, $m = m_e$. We start with a lemma.

• <u>Lem. 3.5.1</u>. <u>For any</u> $a \in m$, <u>we have</u>

$$\Delta(a) \equiv 1 \otimes a + a \otimes 1 \pmod{m \otimes m}.$$

<u>Pf. of the Lem</u>. The k-alg. hom. $\varepsilon : A \to k$ gives a decomp.

$$\begin{aligned} A &\xrightarrow{\sim} k \oplus m \\ a &\longmapsto \varepsilon(a) + (a - \varepsilon(a)). \end{aligned}$$

Assume $\Delta(a) = \sum b_i \otimes c_i$. Then

$$\begin{tikzcd} \sum \varepsilon(b_i) \otimes c_i \arrow[d, equal] & \Delta(a) \arrow[l, mapsto, "\varepsilon\otimes id"'] \\ 1 \otimes a & a \arrow[u, mapsto, "\Delta"'] \arrow[l, mapsto] \end{tikzcd} \qquad \begin{tikzcd} k \otimes A & A \otimes A \arrow[l, "\varepsilon\otimes id"'] \\ & A \arrow[u, "\Delta"'] \arrow[ul, "\sim"] \end{tikzcd}.$$

Similarly, $a \otimes 1 = \sum b_i \otimes \varepsilon(c_i)$. Decomposing b_i and c_i as

$$b_i = \varepsilon(b_i) + \underbrace{(b_i - \varepsilon(b_i))}_{\in m} \quad \text{and} \quad c_i = \varepsilon(c_i) + \underbrace{(c_i - \varepsilon(c_i))}_{\in m},$$

and comparing $\Delta(a)$ and $1 \otimes a + a \otimes 1$, we see that

it suffices to show $\sum \varepsilon(b_i)\otimes\varepsilon(c_i)=0$, which follows from
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \; \varepsilon(\sum\varepsilon(b_i)c_i)$
that $a\in m$: $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; \varepsilon(a)=0.$

$$\begin{tikzcd}
k & k\otimes k \arrow[l, "\sim"'] & & \\
& & k\otimes A \arrow[ul, "\mathrm{id}\otimes\varepsilon"'] & A\otimes A \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[ull, bend right, "\varepsilon\otimes\varepsilon"'] \\
& & & A \arrow[u, "\Delta"'] \arrow[ul, "\sim"] \arrow[uulll, bend left, "\varepsilon"]
\end{tikzcd}$$
. qed.

By (3.4), it suffices to show $\mathcal{N}il(A)\subset m^2$. Let $a\in\mathcal{N}il(A)$. If $\frac{a}{1}=0$ in A_m, then it maps to 0 in $A_m/(mA_m)^2\simeq A/m^2$, i.e., $a\in m^2$. So we may assume $\frac{a}{1}\neq 0$ in A_m. Let n be such that $\frac{a^n}{1}=0$ but $\frac{a^{n-1}}{1}\neq 0$ in A_m; so $sa^n=0$ in A for some $s\in A-m$. Replacing a by sa (<u>exercise</u>: $sa\in m^2\Rightarrow a\in m^2$), we may assume that $a^n=0$ in A and $\frac{a^{n-1}}{1}\neq 0$ in A_m.

By (3.5.1), $a\in\mathcal{N}il(A)\subset m\Rightarrow \Delta(a)=1\otimes a+a\otimes 1+x$ for some $x\in m\otimes m$. Then

$$0=\Delta(a^n)=\Delta(a)^n=(1\otimes a+a\otimes 1+x)^n=$$

$$\underbrace{1\otimes a^n}_{=0}+n\cdot(1\otimes a^{n-1})\cdot(a\otimes 1+x)+\sum_{i+j\geq 2}\binom{n}{n-i-j\;\; i\;\; j}\, a^i\otimes a^{n-i-j}\cdot x^j,$$

where $a^i\otimes a^{n-i-j}\cdot x^j\in(m^i\otimes A)\cdot(m\otimes m)^j\subset m^{i+j}\otimes m^j\subset m^2\otimes A$.

As $n\in k^\times\subset(A\otimes A)^\times$, we have that in $A/m^2\otimes A$,

$$\bar a\otimes a^{n-1}=-(1\otimes a^{n-1})\cdot\bar x\in m/m^2\otimes a^{n-1}\cdot m.$$

But $a^{n-1}\notin a^{n-1}\cdot m$: if $a^{n-1}=a^{n-1}\cdot y$ for some $y\in m$, then

$1-y \in A_m^\times \Longrightarrow \frac{a^{n-1}}{1} = 0$ in A_m. Now it is an easy exercise (by choosing a basis for the k-vector space $m/m^2 \otimes A/a^{n-1}m$) to see that $\bar{a} = 0$ in m/m^2, i.e. $a \in m^2$. $\square$.

- Cor. 3.6. For a k-alg. group G, if $G(k) = \{1\}$, then G is the trivial alg. group $\mathbb{1}$, i.e. $k[G] = k$.

Pf: This is true for any reduced k-scheme of f. type. $\square$.

- Rem. 3.7. The story is totally different in char. p.

§4. Alg. subgroups.

- Def. 4.1. (not a satisfactory one) An alg. subgroup H of an alg. k-group G is a nonempty constructible subset of G (w.r.t. the Zariski top.) that is closed under multiplication and inversion on G ($\Leftrightarrow H(k)$ is a subgroup of $G(k)$).

- Lem. 4.2. Any alg. subgroup H of G is a closed subvariety.

Pf: Step 1. Its Zariski closure $\overline{H}$ is also an alg. subgroup.

Note that the inversion $\imath : G \to G$, as well as left and right translations $x \mapsto g\cdot x$ or $x \cdot g$, for $g \in G(k)$, are all homeomorphisms, hence preserve taking closure:

$$\imath(\overline{H}) = \overline{\imath(H)} = \overline{H}, \quad g\cdot\overline{H} = \overline{(g\cdot H)} = \overline{H}, \ \forall g \in H(k).$$

Since the closed points (i.e. k-points, under our assumption that $k = \bar{k}$) are dense, we deduce that $H \cdot \overline{H} = \overline{H}$.

$\forall h \in \overline{H}(k)$, we have $H\cdot h \subset \overline{H}$, so $\overline{H}\cdot h = \overline{H\cdot h} \subset \overline{H}$, hence $\overline{H}\cdot\overline{H} = \overline{H}$. One may prove in this way that the closure of a subgroup of a top. group is always a subgroup.

<u>Step 2</u>. $\overline{H} \subset H$.

By definition, H is constructible, so it contains an open dense subset of $\overline{H}$. $\forall h \in \overline{H}(k)$, $H\cdot h$ again contains an open dense subset of $\overline{H}$, so $H \cap Hh \neq \phi$.

Let
$$a \in (H \cap H\cdot h)(k), \quad a = b\cdot h \text{ for some } b \in H(k).$$

Then $h = b^{-1}\cdot a \in H(k)$. $\square$.

• <u>Cor. 4.3.</u> <u>Any alg. subgroup of G is affine and smooth over k.</u> $\square$.

• <u>Exercise 4.4.</u> Show that alg. subgroups in G are in one-to-one, inclusion-reversing correspondence with <u>Hopf ideals</u> $\mathfrak{a}$ of $k[G]$, i.e. those ideals $\mathfrak{a} \subset k[G]$ s.t.
$$\Delta(\mathfrak{a}) \subset \mathfrak{a}\otimes k[G] + k[G]\otimes\mathfrak{a}, \quad \varepsilon(\mathfrak{a}) = 0, \quad S(\mathfrak{a}) \subset \mathfrak{a}. \ //.$$

• <u>Lem. 4.5.</u> <u>Let G and G' be k-alg. groups, and $\varphi: G \to G'$ a homomorphism. Then φ is dominant $\Leftrightarrow$ φ is surjective $\Leftrightarrow$ $\varphi^{\#}: k[G'] \to k[G]$ is injective. In this case, φ is faithfully flat.</u>

Pf: By a theorem of Chevalley, $\varphi(G)$ is constructible, and is an alg. subgroup of G', hence is closed by (4.2), therefore φ is dominant $\Leftrightarrow$ φ is surjective. This is so if and only if $\ker(\varphi^{\#}: k[G'] \to k[G]) \subset \mathrm{Nil}(k[G'])$, which is zero by (3.5).

Now assume φ to be surjective. One can reduce to the case where G (and hence G') is connected (by showing $G^\circ \twoheadrightarrow G'^\circ$), and hence integral. We postpone this reduction step to the next chapter (II, 1.9). Now let ξ_G and $\xi_{G'}$ be their generic points; then $\varphi(\xi_G) = \xi_{G'}$. As $k(\xi_G)$ is the function field $k(G)$, clearly $k(G)/k(G')$ is flat, i.e. φ is flat at ξ_G. By the open nature of the flat locus of φ and homogeneity, we deduce that φ is flat everywhere. $\square$.

A hom. $\varphi: H \to G$ is said to be <u>injective</u> if it is so as a natural transformation, i.e. $\forall R \in k\text{-Alg}.$,

$$\varphi_R : H(R) \hookrightarrow G(R).$$

This is the case when H is an alg. subgroup of G. In fact, as $k[H]$ is a quotient algebra of $k[G]$, any k-alg. hom. $k[H] \to R$ is determined by its pullback to $k[G]$. Also, φ is injective $\Leftrightarrow$ $\varphi_k : H(k) \to G(k)$ is injective (hence $\Leftrightarrow$ φ is injective as a morphism of ~~varieties~~ schemes), because $\ker(\varphi)$

is an alg. subgroup of H, hence by (3.6),

$$\ker(\varphi)(k) = \ker(\varphi_k) = \{1\} \iff \ker(\varphi) \text{ is trivial}$$

$$\iff \ker(\varphi)(R) = \{1\}, \ \forall R \in k\text{-}\mathcal{A}lg.$$

• <u>Prop. 4.6.</u> <u>A hom.</u> $\varphi : H \to G$ <u>of</u> <u>k-alg. groups is</u> <u>injective</u> <u>if</u> <u>and only if</u> φ <u>is a closed immersion</u>, <u>i.e.</u> φ <u>realizes</u> H <u>as an ~~sub~~ alg. subgroup of</u> G.

<u>Pf</u>: Only need to prove ($\Longrightarrow$). By a theorem of Chevalley, $\varphi(H)$ is constructible, hence an alg. subgroup of G, hence is closed in G by (4.2). Being reduced, it is the image scheme of the morph. φ, i.e. it is defined by the ideal $\mathfrak{a} = \ker(\varphi^\#: k[G] \to k[H])$. As φ is a hom., $\varphi^\#$ is a hom. of Hopf algebras, so that $\mathfrak{a}$ is a Hopf ideal (4.4), so that $k[G]/\mathfrak{a}$ is also a k-Hopf algebra (exercise); in fact $k[G]/\mathfrak{a} = k[\varphi(H)]$.
As $\varphi : H \to \varphi(H)$ is surjective, by (4.5),

$$\overline{\varphi^\#} : k[G]/\mathfrak{a} \hookrightarrow k[H]$$

is faithfully flat, so $k[G]/\mathfrak{a}$ is the equalizer of

$$a \mapsto \begin{matrix} a\otimes 1 \\ 1\otimes a \end{matrix} : k[H] \rightrightarrows k[H] \otimes_{k[\varphi(H)]} k[H].$$

These two k-alg. hom. define two $k[H]\otimes_{k[\varphi(H)]} k[H]$-points of H that become equal in $G(k[H]\otimes_{k[\varphi(H)]} k[H])$. Since φ

is an injective natural transformation, we deduce that these two k-alg. hom. coincide, so

$$\overline{\varphi^\#} : k[G]/\mathfrak{a} \xrightarrow{\sim} k[H],$$

i.e. φ is a closed immersion. $\square$

only have "$\mathbb{R}$-points", so inj. is not a strong condition.

• <u>Rem. 4.6.1.</u> Whereas for Lie groups, there is a difference between "Lie subgroups" and "sub-Lie groups."

• <u>Examples 4.7.</u> — Let $\varphi : G \to G'$ be a hom. of alg. groups. Then $\ker(\varphi)$ is an alg. subgroup of G, and is normal (i.e. $\forall$ k-alg. R, $\cdots$). We will see later that each normal alg. subgroup arises in this way. ($\frac{\text{Exercise: } H \lhd G \Leftrightarrow}{H(k) \lhd G(k).}$).

— For an alg. subgroup H of G, the <u>normalizer</u> $N_G(H)$ and the <u>centralizer</u> $C_G(H)$ are two alg. subgroups of G, representing the functors

$$R \mapsto \{ g \in G(R) \mid \forall\ R\text{-alg. } R', \text{ we have } g \cdot H(R') g^{-1} = H(R') \}$$

($\Leftrightarrow$ $g \cdot H_R \cdot g^{-1} = H_R$ in G_R).

$$R \mapsto \{ g \in G(R) \mid \forall\ R\text{-alg. } R', \forall\, h \in H(R'), \text{ we have } g h g^{-1} = h \}$$

respectively. (We omit the proof of their representability.) Note that even though $N_G(H)(k) = N_{G(k)}(H(k))$, in general

$$N_G(H)(R) \neq N_{G(R)}(H(R)),$$

but rather, $N_G(H)$ is a "sheafification" (w.r.t. "fc" or "affine" topology) of the naive functor $R \mapsto N_{G(R)}(H(R))$.

—— The <u>center</u> of G, denoted $Z(G)$, is $C_G(G)$. For instance, $Z(GL_V) = \mathbb{G}_m$, the group of homotheties (or scalar matrices, once a basis for V is chosen). $Z(SL_n) = \mu_n$.

—— Let H_i $(i \in I)$ be a family of alg. subgroups of G. Then $\bigcap_{i\in I} H_i$ is again closed under mult. and inversion, thus is an alg. subgroup, representing the naive functor $R \mapsto \bigcap_I H_i(R)$. As G is a noetherian scheme, we see that $\exists$ a finite subset $J \subset I$ s.t. $\bigcap_{i\in J} H_i = \bigcap_{i\in I} H_i$.

—— Let S be a subset of $G(k)$. The (<u>normal</u>) <u>alg. subgroup of G generated by</u> S, is defined to be the intersection of all (normal) alg. subgroups $H \subset G$ s.t. $S \subset H(k)$.

—— Let S be the commutator subgroup $G(k)^{der} \subset G(k)$. The alg. subgroup generated by S, which is normal by the usual rule $(g \cdot [x,y] \cdot g^{-1} = [gx, y] \cdot [y, g])$, is called the <u>derived group</u> G^{der} of G.

—— Let $\rho : G \to GL_V$ be a linear rep. of G with $\dim(V) < \infty$, and let $W \subset V$ be a subspace. Then the functor

$$R \mapsto \{ g \in G(R) \mid \rho_R(g)(W \otimes R) = W \otimes R \}$$

is represented by an alg. subgroup $Stab_G(W)$ of G, called the <u>stabilizer group</u> of W in G.

Pf: Choose a k-basis $v_1, \cdots, v_\ell$ for W, and extend it to a basis $v_1, \cdots, v_n$ of V. If ρ corresponds to the coaction

$$\sigma : V \longrightarrow V \otimes k[G],$$

let
$$\sigma(v_i) = \sum_{j=1}^{n} v_j \otimes a_{ij} \quad , \quad a_{ij} \in k[G],$$

and let $\mathfrak{a} \subset k[G]$ be the ideal generated by those a_{ij}'s with $1 \le i \le \ell$ and $j > \ell$. Then one sees that

$$\mathrm{Hom}_{k\text{-alg}}(k[G]/\mathfrak{a}, R) = \{ g \in \mathrm{Hom}_{k\text{-alg}}(k[G], R) \mid g(\mathfrak{a}) = 0 \}$$

$$= \left\{ g \in \mathrm{Hom}_{k\text{-alg}}(k[G], R) \;\middle|\; \begin{array}{l} V \xrightarrow{\sigma} V \otimes k[G] \xrightarrow{\mathrm{id} \otimes g} V \otimes R \text{ extends by} \\ R\text{-linearity to } V_R \xrightarrow[\rho_R(g)]{} V_R \text{ , then } \rho_R(g)(W_R) = W_R \end{array} \right\},$$

i.e. $k[G]/\mathfrak{a}$ represents the functor $\mathrm{Stab}_G(W)$. Clearly it is a subgroup functor (hence $\mathfrak{a}$ is a Hopf ideal). □

— Let $\rho : G \to GL_V$ be a linear rep. and $v \in V, v \ne 0$.

Then the functor
$$R \mapsto \{ g \in G(R) \mid \rho_R(g)(v \otimes 1_R) = v \otimes 1_R \text{ in } V \otimes_k R \}$$
is rep. by an alg. subgroup $\mathrm{Stab}_G(v)$ of G, called the <u>stabilizer group</u> of v in G.

Pf: Choose a k-basis $v_1 = v, v_2, \cdots, v_n$ of V, and let
$$\sigma(v) = \sum_{j=1}^{n} v_j \otimes a_j , \quad a_j \in k[G].$$

Then consider the ideal $\mathfrak{a} = (a_1 - 1, a_2, \cdots, a_n)$ ...... $\square$.

Given $W \subset V$, define the centralizer of W to be $\mathrm{Cent}_G(W) := \bigcap_{v \in W} \mathrm{Stab}_G(v)$.

§5. Faithful rep., quotient groups.

• Def. 5.1. Let G be a k-alg. group, V a k-vector space and $\rho : G \to GL_V$ a linear representation. We say that (V, ρ) is faithful if ρ is injective (as a nat. transf.).

When $\dim(V) < \infty$ so that GL_V is also an alg. group, by (4.6) we see that if ρ is faithful, it realizes G as an alg. subgroup of GL_V, hence the name "linear alg. group".

• Th. 5.2. Let G be a k-alg. group. Then there exists a finite-dim'l faithful subrepresentation V of the regular rep. $k[G]$ of G.

Pf. Let S be a finite set of generators of the k-algebra $k[G]$. From the proof of (2.6), there is a finite-dim'l subrep. V of $(k[G], \mathrm{Reg})$ such that $S \subset V$. Let $f_1, \cdots, f_n$ be a k-basis of V; they also generate the k-algebra $k[G]$. Let

$$\Delta(f_i) = \sum_{j=1}^{n} f_j \otimes a_{ij}, \quad a_{ij} \in k[G].$$

Then $f_i = \sum_j \varepsilon(f_j) \cdot a_{ij}$, so the a_{ij}'s also generate the k-algebra

$k[G]$. The hom. $k[GL_V] \to k[G]$ sents the matrix coordinate function x_{ji} to a_{ij} (see p. 13). In fact, given any $g \in G(k)$,

$$\begin{array}{c} k[G] \xrightarrow{g} k \\ a_{ij} \longmapsto a_{ij}(g) \end{array}, \qquad \begin{tikzcd} V \arrow[r, "\sigma = \Delta|_V"] \arrow[dr, "\rho(g)"'] & V \otimes k[G] \arrow[d, "id \otimes g"] \\ & V \otimes k \simeq V \end{tikzcd} \qquad \begin{array}{c} f_i \longmapsto \sum_j f_j \otimes a_{ij} \\ \downarrow \\ \sum_j a_{ij}(g) \cdot f_j \end{array}$$

and the automorphism $\rho(g)$ corresponds to the matrix $(a_{ji}(g))_{i,j}$, i.e. the composition

$$k[GL_V] \longrightarrow k[G] \xrightarrow{g} k$$

gives rise to $(a_{ji}(g))_{i,j} \in GL(V)$. Therefore, $k[GL_V] \twoheadrightarrow k[G]$ is surjective. $\square$.

In the same way, one can prove the following theorem of Chevalley.

• Th. 5.3. (Chevalley). Let H be an alg. subgroup of an alg. group G. Then there exists a finite-dim'l ~~faithful~~ rep. (V, ρ) of G and a subspace $W \subset V$ such that $H = Stab_G(W)$. Moreover, (V, ρ, W) can be chosen such that $\dim(W) = 1$.

Pf: Let $\mathfrak{a} \subset k[G]$ be the (Hopf) ideal defining H; it is finitely generated, say by a finite set $S \subset \mathfrak{a}$. Let $V \subset k[G]$ be a finite-dim'l subrep. of the regular rep. with $S \subset V$. Let $W = V \cap \mathfrak{a}$. Choose a k-basis $f_1, \cdots, f_\ell$ of W, and extend it to a k-basis $f_1, \cdots, f_n$ of V. As $S \subset W$, we have

$$\mathfrak{a} = (f_1, \cdots, f_\ell).$$

Let $\Delta(f_j) = \sum_{i=1}^{n} f_i \otimes a_{ij}$, $a_{ij} \in k[G]$. Since $\mathfrak{a}$ is a Hopf ideal, $\forall\, 1 \leq j \leq \ell$, we have $\Delta(f_j) \in \mathfrak{a} \otimes k[G] + k[G] \otimes \mathfrak{a}$, so $i > \ell \Leftrightarrow f_i \notin \mathfrak{a} \Rightarrow a_{ij} \in \mathfrak{a} \Longleftrightarrow \overline{a_{ij}} = 0$ in $k[H] = k[G]/\mathfrak{a}$.

<u>Step 1</u>. $H \subset \mathrm{Stab}_G(W)$.

$(V, \rho|_H)$ corresponds to the coaction

$$\overline{\Delta}|_V : V \xrightarrow{\Delta|_V} V \otimes k[G] \longrightarrow V \otimes k[H],$$

and it suffices to show that W is a sub-comodule of V, i.e.

$$\overline{\Delta}|_V (W) \subset W \otimes k[H].$$

This is clear:
for $1 \leq j \leq \ell$, $\overline{\Delta}|_V(f_j) = \sum_{i=1}^{n} f_i \otimes \overline{a_{ij}} = \sum_{i=1}^{\ell} f_i \otimes \overline{a_{ij}} \in W \otimes k[H]$.

<u>Step 2</u>. $H = \mathrm{Stab}_G(W)$.

Since $f_j = \sum_{i=1}^{n} \varepsilon(f_i) \cdot a_{ij} \overset{\varepsilon(\mathfrak{a})=0}{=\!=\!=} \sum_{i>\ell} \varepsilon(f_i) \cdot a_{ij}$, we see that

$$\mathfrak{a} = (a_{ij} \mid j \leq \ell < i).$$

Let $g \in \mathrm{Stab}_G(W)(R)$, i.e. a k-alg. hom. $g: k[G] \to R$ s.t.

$$V \xrightarrow{\Delta|_V} V \otimes k[G] \xrightarrow{id \otimes g} V \otimes_k R$$

maps W into $W \otimes R$. Then $\forall\, j \leq \ell < i$, because $f_i \notin W$, we must have $a_{ij}(g) = 0$ in R, hence $g: k[G] \to R$ vanishes

(image of a_{ij} under $g: k[G] \to R$)

on $\mathfrak{a}$, i.e. it factor through $k[H]$:

$$\begin{tikzcd} k[G] \arrow[r, "g"] \arrow[d] & R \\ k[G]/\mathfrak{a} = k[H] \arrow[ur, dashed, "\bar{g}"'] & \end{tikzcd}$$

namely, $g \in H(R)$.

Now let (V, ρ, W) be a triple with $Stab_G(W) = H$, $\dim(W) = \ell$. Consider $(\Lambda^\ell V, \Lambda^\ell \rho, \Lambda^\ell W)$, in which $\Lambda^\ell W \subset \Lambda^\ell V$ is a line. Clearly $Stab_G(W) \hookrightarrow Stab_G(\det(W))$. Let $f_1, \cdots, f_\ell$ (resp., $f_1, \cdots, f_n$) be a k-basis of W (resp. V) as before, and let $h = f_1 \wedge \cdots \wedge f_\ell \in \Lambda^\ell W$. It is easy to see that for $f \in V_R$, we have

$$f \in W_R \Longleftrightarrow f \wedge h = 0 \quad \text{in } \Lambda^{\ell+1}(V_R).$$

Given $g \in Stab_G(\Lambda^\ell W)(R)$, i.e. $g \in G(R)$ s.t. $\Lambda^\ell \rho_R(g)(\Lambda^\ell W_R) = \Lambda^\ell W_R$, we have

$$\Lambda^\ell \rho_R(g)(h) = \lambda \cdot h, \text{ for some } \lambda \in R^\times.$$

To see $g \in Stab_G(W)(R)$, let $f \in W_R$; then $f \wedge h = 0$, so

$$0 = \Lambda^{\ell+1} \rho_R(g)(f \wedge h) = \rho_R(g)(f) \wedge (\Lambda^\ell \rho_R(g)(h)) = \lambda \cdot (\rho_R(g)(f) \wedge h),$$

therefore $\rho_R(g)(f) \wedge h = 0$ and hence $\rho_R(g)(f) \in W_R$. $\square$.

Now we discuss <u>surjective hom.</u> $\varphi: G \to Q$ of alg. groups. Unlike injective hom., we won't define it to be surjective as a nat. transformation (in other words, as morp. of presheaves of groups on k-Alg.). A hom. φ of alg. groups$/k$ is surjective if it is so as a morphism of schemes ($\Longleftrightarrow \varphi_k : G(k) \to Q(k)$ is surjective $\Longleftrightarrow \varphi$ is surjective as a morphism of fpqc sheaves of groups on k-Alg., for those who know the terminology). It is so if and only if $\varphi^{\#} : k[Q] \hookrightarrow k[G]$ is injective, by (4.5). We also call it a quotient morphism, in view of the following theorem.

• Th. 5.4. Let $\varphi : G \to Q$ be a surjective hom. of alg. groups. For any hom. $\varphi' : G \to G'$ with $\ker(\varphi) \hookrightarrow \ker(\varphi')$, $\exists!$ hom. $\psi : Q \to G'$ such that $\varphi' = \psi \circ \varphi$.

Pf: Let $N = \ker(\varphi)$. $\forall$ k-alg. R, $\forall (g_1, g_2)$ in

$$(G \underset{Q}{\times} G)(R) = G(R) \underset{Q(R)}{\times} G(R),$$

because $\varphi_R(g_1) = \varphi_R(g_2)$ in $Q(R)$, we have $g_1^{-1} \cdot g_2 \in N(R) \hookrightarrow (\ker(\varphi'))(R)$, hence $\varphi'_R(g_1) = \varphi'_R(g_2)$. This means that the two compositions

$$G \underset{Q}{\times} G \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} G \xrightarrow{\varphi'} G',$$

and hence

$$k[G] \underset{k[Q]}{\otimes} k[G] = k[G \underset{Q}{\times} G] \underset{i_2}{\overset{i_1}{\leftleftarrows}} k[G] \xleftarrow{\varphi'^{\#}} k[G'],$$

are **equal**. As $k[G]$ is faithfully flat over $k[Q]$ by (4.5), the equalizer of (i_1, i_2) is the subalgebra $k[Q] \subset k[G]$, therefore $\varphi'^{*}: k[G'] \to k[G]$ factors through $k[Q]$:

$$\begin{tikzcd}
k[G'] \arrow[rr, "\varphi'^{*}"] \arrow[dr, dashed] & & k[G] \\
& k[Q] \arrow[ur, "\varphi^{*}"'] &
\end{tikzcd}$$

$\psi: Q \to G'$. $\square$.

By the universal property, $\forall$ normal alg. subgroup N of G, the **quotient morphism** $\varphi: G \to Q$ with kernel N, if exists, is unique, and is denoted by $G \to G/N$.

- Def. 5.5. Let (V, ρ) and (V', ρ') be two linear rep. of an alg. group G. Their tensor product, $(V\otimes V', \rho\otimes\rho')$, is defined by

$$(\rho\otimes\rho')_R = \rho_R \otimes \rho'_R : G(R) \longrightarrow GL(V_R \otimes_R V'_R).$$
$$g \longmapsto \rho_R(g) \otimes \rho'_R(g)$$

Similarly, the internal hom, $(\mathrm{Hom}_k(V, V'), \mathrm{Hom}_k(\rho, \rho'))$, is defined by

$$G(R) \longrightarrow GL(\mathrm{Hom}_R(V_R, V'_R))$$
$$g \longmapsto (f \mapsto \rho'_R(g)\cdot f(\rho_R(g^{-1})\cdot \square),\ \square \in V_R),$$

and the dual rep. of (V, ρ), denoted by $(V^{\vee}, \rho^{\vee})$, is defined as the internal hom $(\mathrm{Hom}_k(V, k), \mathrm{Hom}_k(\rho, \rho_{triv}))$.

The subspace of invariant vectors, V^G, is defined to be

$$V^G = \{ v \in V \mid \rho_R(g)(v \otimes 1_R) = v \otimes 1_R \text{ in } V_R, \ \forall R \in k\text{-Alg}, \ \forall g \in G(R) \}.$$

- Exercise 5.5.1. Show that the usual properties of $\otimes$ and hom of rep. of an abstract group hold in this context, e.g.

$\boxed{\text{also describe them in terms of comodules}}$

$$\mathrm{Hom}_k(\rho \otimes \rho', \rho'') = \mathrm{Hom}_k(\rho, \mathrm{Hom}_k(\rho', \rho'')),$$

$$\mathrm{Hom}_k(\rho, \rho') = \rho^\vee \otimes \rho', \quad \text{etc.}$$

If $\sigma : V \to V \otimes k[G]$ is the coaction corresp. to ρ, then

$$V^G = \{ v \in V \mid \sigma(v) = v \otimes 1 \} \overset{\text{also}}{\equiv} V^{G(k)}. \quad //$$

- Lem. 5.6. Let (V, ρ) be a linear rep. of an alg. group G, and $W \subset V$ a subspace. If

$$\rho_k(g) \cdot W = W, \ \forall g \in G(k),$$

then W is a subrepresentation of (V, ρ).

Pf: For $\mathrm{Stab}_G(W) \hookrightarrow G$, we have $\mathrm{Stab}_G(W)(k) = G(k)$. $\square$.

- Th. 5.7. Let N be a normal alg. subgroup of an alg. group G. Then $\exists$ linear rep. (V, ρ) of G with $\ker(\rho) = N$. Consequently, $G \to \mathrm{Im}(\rho) \subset GL_V$ is a quotient map with kernel N.

Pf: By (5.3), $\exists$ a fini. dim'l linear rep. (V, ρ) of G and a line $L \subset V$ s.t. $\mathrm{Stab}_G(L) = N$. Let $W = \sum_{g \in G(k)} \rho_k(g) \cdot L \subset V$; by

(5.6), it is a subrepresentation of (V, ρ). Clearly, W is a finite direct sum of lines $\rho_k(g_i)\cdot L$, $g_0 = 1, g_1, \cdots, g_\ell \in G(k)$, each of which is N-stable (by (5.6) and the normality of N). When viewed as a rep. of N (but not of G), we have

$$W^\vee = \bigoplus_i (\rho_k(g_i)\cdot L)^\vee \supset L^\vee .$$

In the G-rep. $(W \otimes W^\vee, \rho \otimes \rho^\vee)$, $L \otimes L^\vee$ is a line on which N acts trivially. By (5.6) and the normality of N, the space of N-invariants $(W \otimes W^\vee)^N$ is G-stable; let η be this subrep.:

$$\eta : G \longrightarrow GL_{(W\otimes W^\vee)^N} .$$

We claim that $\ker(\eta) = N$. Clearly $N \hookrightarrow \ker(\eta)$. For the converse inclusion, it suffices to check their k-points (similar for R-points, if you don't assume $k = \bar{k}$ so that k-points is not enough). If $g \in G(k)$ acts trivially on $(W \otimes W^\vee)^N$, it acts trivially also on the line $L \otimes L^\vee \subset (W \otimes W^\vee)^N$. Now an easy exercise (on choosing basis for tensor product) shows that this happens if and only if g stabilizes both L and $L^\vee$ and acts via scalars that are inverse to each other. Since $N = \mathrm{Stab}_G(L)$, we deduce that $g \in N(k)$. $\square$.

• <u>Rmk. 5.8.</u> For the existence of homogeneous space G/H, where H is any alg. subgroup of G (not necessarily normal), much more alg. geometry will be involved and we postpone its discussion. Here is a sketch: by Chevalley's theorem (5.3), $\exists\ (V, \rho, L)$ with $\mathrm{Stab}_G(L) = H$. Then G acts on the projective space $\mathbb{P}V$, and H is the stabilizer group of $[L] \in (\mathbb{P}V)(k)$. The orbit $\mathrm{Orb}_G([L])$, i.e. the image of

$$G \longrightarrow \mathbb{P}V, \quad \text{“} g \longmapsto g\cdot[L] \text{”},$$

is a constructible subset of $\mathbb{P}V$ (by another thm. of Chevalley) and hence gets the structure of a k-scheme, quasi-projective. This will be identified with "G/H", an object we shall define intrinsically first, so that it is independent of the choice of (V, ρ, L).

• Def. 5.9. An alg. group G is adjoint if $Z(G)$ is trivial.

For any alg. group G, let G^{ad} be $G/Z(G)$, called the adjoint group associated with G.

• Example. $PGL_n := GL_n/\mathbb{G}_m$,
$PSL_n := SL_n/\mu_n$

Show that $SL_n \hookrightarrow GL_n$ induces an isomorphism

$$PSL_n \xrightarrow{\sim} PGL_n.$$

Course notes of "Linear Algebraic Groups"

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Ch. II. Various types.

A ~~general~~ linear alg. group G/k is built up from some special types of groups:

- $$1 \to \underset{\text{connected alg. gr.}}{G^\circ} \to G \to \underset{\text{finite alg. group (étale over } k)}{\pi_0(G)} \to 1$$

- $$1 \to \underset{\text{solvable alg. gr.}}{R(G^\circ)} \to G^\circ \to \underset{\text{semi-simple alg. gr.}}{G^\circ/R(G^\circ)} \to 1$$

- $$1 \to \underset{\text{unipotent alg. gr.}}{R_u(G^\circ)} \to R(G^\circ) \to \underset{\text{torus}}{R(G^\circ)/R_u(G^\circ)} \to 1$$

- $$\left(G^\circ/R(G^\circ)\right)^\sim \xrightarrow[\text{finite covering}]{} G^\circ/R(G^\circ)$$

  simply-connected semi-simple alg. group, closely related to the Lie algebra of $G^\circ/R(G^\circ)$ $\leftrightarrow$ root system ...

We study these special types of alg. groups individually.

Again, $k = \bar{k}$, $\mathrm{char}(k) = 0$.

§1. Finite alg. groups.

• Def. 1.1. An alg. group G is *finite* if G is a finite k-scheme, i.e. $k[G]$ is finite dim'l over k. In this case, the *rank* of G, $\mathrm{rank}(G)$, is $\dim_k(k[G])$.

For a finite group Γ (ordinary), we have defined its associated constant alg. group $(\Gamma)_k$ in (I, 1.4), where

$$k[(\Gamma)_k] = k^\Gamma .$$

So $(\Gamma)_k$ is a finite alg. group, whose rank is the order of Γ.

• Lem. 1.2. The functor $\Gamma \mapsto (\Gamma)_k$ defines an equivalence between the category of finite abstract groups and that of finite alg. groups over k, with quasi-inverse $G \mapsto G(k)$.

Pf: In fact, similarly defined functors $\left(\Sigma \mapsto (\Sigma)_k ,\ X \mapsto X(k)\right)$ give an equiv. between the category of finite sets and that of finite reduced k-schemes. Reason: $\forall$ finite dim'l reduced k-alg. A, it is artinian (since ideals are also k-subspaces), hence it has only finitely many max. ideals $m_1, \cdots, m_r$, and

$$\bigcap_{i=1}^{r} m_i = \mathrm{Nil}(A) = 0 .$$

By Chinese remainder thm.,

$$A \xrightarrow{\sim} \prod_i A/m_i ,$$

and $A/m_i \simeq k$ because it is a finite extension of k; thus $A \simeq k^\Sigma$

with $\Sigma = \{1, \cdots, r\}$. Conversely, $\forall$ finite set Σ, we have

$$\Sigma = \mathrm{Hom}_{k\text{-alg}}(k^{\Sigma}, k)$$

because any max. ideal in $k^{\Sigma} = \prod_{\Sigma} k$ takes the form

$$k \times \cdots \times k \times \{0\} \times k \times \cdots \times k,$$

i.e. any k-alg. hom. $k^{\Sigma} \to k$ is the projection onto a factor.

Moreover, these two functors preserve categorical products:

$$k^{\Sigma \times \Sigma'} \simeq k^{\Sigma} \otimes_k k^{\Sigma'}, \quad (X \times_k Y)(k) = X(k) \times Y(k),$$

so they induce an equiv. between their group objects. $\square$.

• Rem. 1.2.1. For a general base field k, not necessarily alg. closed or of char. 0, these two functors induce an equiv. between

$$\left\{\begin{array}{l}\text{finite sets with a} \\ \text{continuous action} \\ \text{by } \mathrm{Gal}(k^s/k)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{finite étale} \\ k\text{-schemes}\end{array}\right\},$$

and

$$\left\{\begin{array}{l}\text{finite groups with a} \\ \text{cont. action of } \mathrm{Gal}(k^s/k) \\ \text{by group automorphisms}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{finite étale} \\ k\text{-group schemes}\end{array}\right\}.$$

• Def. 1.3. For a k-algebra A of finite type, define $\pi_0(A)$

to be the subalgebra generated by all k- subalgebras of finite dimension that are reduced. ($\underset{\text{ex.}}{\cong} k\times\cdots\times k$).

• Lem. 1.4. $\forall$ A of finite type/k, $\pi_0(A)$ is of finite dim. over k.

Pf. $\forall$ subalg. $B\subset A$, the map $\mathrm{Spec}(A)\to\mathrm{Spec}(B)$ is dominant. As A is noetherian, Spec(A) has only finitely many connected components (or irred.). When $\dim_k(B)<\infty$ and B is reduced, so that Spec(B) is a finite discrete space, we see that

$$\#(\mathrm{Spec}(B)) = \#\{\text{max. ideals in } B\} = \dim_k(B)$$

is bounded by $\#\pi_0(\mathrm{Spec}(A))$ (or just $\#$(irred. comp. of Spec(A))).

Let $B_1,\dots,B_n$ be reduced k-subalgebras of A of finite dim.. Then their composite is the image of the mult. map

$$B_1\otimes_k\cdots\otimes_k B_n \longrightarrow A,$$

hence has finite dim.. Also, $B_1\otimes\cdots\otimes B_n$ takes the form $\prod k$, so is its quot. $B_1\cdots B_n$. □.

• Lem. 1.5. $\forall$ k-alg. A of finite type, the natural map $\mathrm{Spec}(A)\to\mathrm{Spec}(\pi_0(A))$ induces a bijection

$$\pi_0(\mathrm{Spec}(A)) \xrightarrow{\sim} \mathrm{Spec}(\pi_0(A)).$$

Pf. The map being well-defined and surjective, it suffices to show that both sides have the same cardinality. The ways of decomposing Spec(A) into a disjoint union of r closed subsets correspond bijectively to the ways of decomposing $1 \in A$ into a ~~finite~~ sum of r orthogonal idempotents (i.e. $1 = e_1 + \cdots + e_r$, $e_i \cdot e_j = 0$, $e_i^2 = e_i$), which correspond bijectively to reduced k-subalgebras $B \subset A$ of dim. r ($B = k\langle e_1, \ldots, e_r\rangle$; given B, we have $B \xrightarrow{\sim} \prod_{\text{max. ideals}} B/m$, giving rise to $e_i = (\ldots 0, 1, 0, \ldots)$ in $B \subset A$). So the number of connected comp. of $Spec(A)$ is equal to $\dim_k(\pi_0(A))$. $\square$

• Lem. 1.6. Let A and B be k-algebras of f. type. Then $\pi_0(A \otimes B) = \pi_0(A) \otimes \pi_0(B)$.

Pf. Clearly $\pi_0(A) \otimes \pi_0(B) \subset A \otimes B$ is finite dim'l and reduced, being of the form $\prod k$, so $\pi_0(A) \otimes \pi_0(B) \subset \pi_0(A \otimes B)$. It suffices to compare their dim., so we shall show that

$$\# \pi_0(Spec(A \otimes B)) = \# \pi_0(Spec(A)) \cdot \# \pi_0(Spec(B)).$$

We reduce to the case when $\pi_0(A) \simeq \pi_0(B) \simeq k$.

Assume that $f = \sum_i a_i \otimes b_i$ is an idempotent in $A \otimes B$, $f \neq 0, 1$ (so that f is neither a unit nor nilpotent). Then $V(f) \neq \phi$; let $(m_0, n_0) \in V(f)$, $(m_1, n_1) \in V(1-f)$, so

$$\sum_i a_i(m_0) \cdot b_i(n_0) = 0, \quad \sum_i a_i(m_1) \cdot b_i(n_1) = 1,$$

where $a(m)$ is the image of $a \in A$ under $A \to A/m \simeq k$. The "restriction" $f|_{n_0} := \sum_i b_i(n_0) \cdot a_i \in A$, being the image of f under $A \otimes B \to A \otimes B/n_0 \simeq A \otimes k \simeq A$, is an idempotent, hence is either

0 or 1 ; since $f|_{n_0} \in m_0$, it is 0; in particular, $f|_{n_0}(m_1) = 0$.
Similarly, $f|_{m_1} := \sum_i a_i(m_1)\cdot b_i$ in B is 1, so $f|_{m_1}(n_0) = 1$.
But $f|_{n_0}(m_1) = f|_{m_1}(n_0)$:

$$\begin{tikzcd}
 & A/m_1 \otimes B \arrow[dr] & \\
A \otimes B \arrow[ur] \arrow[dr] & & A/m_1 \otimes B/n_0 \\
 & A \otimes B/n_0 \arrow[ur] &
\end{tikzcd},$$

a contradiction. Therefore, $\pi_0(A\otimes B) \simeq k$. □.

• <u>Th. 1.7.</u> <u>Let G be a k-alg. group.</u> <u>Then $\pi_0(G) :=$ $\mathrm{Spec}(\pi_0(k[G]))$ also has an alg. group structure such that the canonical morp. $G \to \pi_0(G)$ is a homomorphism.</u>

<u>Pf</u>. Clearly, $\pi_0(A)$ is functorial in A. The comultiplication

$$\Delta : k[G] \longrightarrow k[G] \otimes k[G]$$

is a k-alg. hom., so by (1.6), it induces

$$\pi_0(\Delta) : \pi_0(k[G]) \longrightarrow \pi_0(k[G]\otimes k[G]) = \pi_0(k[G]) \otimes \pi_0(k[G]).$$

One sees easily that together with $\pi_0(\varepsilon) : \pi_0(k[G]) \subset k[G] \to k$ and $\pi_0(S) : \pi_0(k[G]) \to \pi_0(k[G])$, this makes $\underset{\pi_0}{k[G]}$ a sub-Hopf k-algebra of $k[G]$. □.

Let G^0 be $\ker(G \twoheadrightarrow \pi_0(G))$, called the <u>identity</u> (or <u>neutral</u>) <u>component</u> of G. By (1.5), it is the connected component of G containing the identity element $e \in G(k)$.

One could have first proved X, Y connected $\Rightarrow X \underset{k}{\times} Y$ connected, defined G° to be the conn. comp. containing e, so that $G^\circ \cdot G^\circ \subset G^\circ$ and $\iota(G^\circ) \subset G^\circ$, deducing that G° is a (normal) alg. subgroup of G, and then took $\pi_0(G)$ to be G/G°. Lemma (1.6) is again crucial in this approach.

• <u>Exercises 1.8</u>. i). Every hom. from a conn. alg. group to G factors through $G^\circ \hookrightarrow G$;

ii). G° is the unique conn. normal alg. subgroup of G such that G/G° is finite;

iii). $\pi_0(G \underset{k}{\times} G') = \pi_0(G) \underset{k}{\times} \pi_0(G')$, and

$(G \underset{k}{\times} G')^\circ = G^\circ \underset{k}{\times} G'^\circ$;

iv). if $1 \to H \to G \to Q \to 1$ is an exact sequence of k-alg. groups (i.e. $H \triangleleft G$ and $Q \simeq G/H$), then the sequence

$$\pi_0(H) \to \pi_0(G) \to \pi_0(Q) \to 1$$

is exact; in particular, if H and Q are connected, so is G;

v). $\pi_0(O_n) = (\mathbb{Z}/2\mathbb{Z})_k$; denote $(O_n)^\circ$ by SO_n. //.

For G an affine k-group scheme, $G = \varprojlim_I G_i$ with G_i quotients of G of f. t., one can define

$$G^\circ := \varprojlim_I G_i^\circ \text{ and } \pi_0(G) := \varprojlim_I \pi_0(G_i).$$

They are well-defined, because any two choices of limit presentations are cofinal.

• Prop. 1.9. If $\varphi : G \twoheadrightarrow Q$ is a surjective hom. of alg. groups, so is $\varphi^\circ : G^\circ \to Q^\circ$.

Pf. Let $H = \ker(\varphi)$. Surjectivity can be checked on k-points. By (1.8, iv) we have

$$\begin{tikzcd} & & G^\circ(k) \arrow[r, "\varphi^\circ_k"] \arrow[d] & Q^\circ(k) \arrow[d] & \\ 1 \arrow[r] & H(k) \arrow[r] \arrow[d] & G(k) \arrow[r, "\varphi_k"] \arrow[d] & Q(k) \arrow[r] \arrow[d] & 1 \\ & \pi_0(H)(k) \arrow[r] & \pi_0(G)(k) \arrow[r] & \pi_0(Q)(k) \arrow[r] & 1 \end{tikzcd},$$

and a routine diagram-chasing will finish the proof:

$\forall q \in Q^\circ(k)$, $\exists\, g \in G(k)$ s.t.

$$\begin{tikzcd} & g \arrow[r, mapsto, "\varphi_k"] \arrow[d, mapsto] & q \arrow[d, mapsto] \\ H(k) \ni h \arrow[d, mapsto] & & \\ \exists [h] \arrow[r, mapsto] & {[g]} \arrow[r, mapsto] & {[q] = 1 \text{ in } \pi_0(Q)(k)} \end{tikzcd}$$

then $h^{-1}\cdot g \xmapsto{\varphi_k} q$ and $[h^{-1}\cdot g] = 1$ in $\pi_0(G)(k)$, so $h^{-1}\cdot g \in G^\circ(k)$. $\square$.

§2. Diagonalizable groups.

Let M be a commutative (abstract) group, written multiplicatively, and let $k[M]$ be its group algebra. Define

$\Delta : k[M] \to k[M] \otimes k[M]$, $\varepsilon : k[M] \to k$, $S : k[M] \to k[M]$

by $\Delta(m) = m \otimes m$, $\varepsilon(m) = 1$, $S(m) = m^{-1}$ $(\forall m \in M)$, extended by linearity. One verifies that $k[M]$ becomes a k-Hopf algebra. Let $D(M)$ be the associated affine group scheme:

$$D(M) = \mathrm{Spec}(k[M]).$$

• <u>Prop. 2.1.</u> $D(M)$ <u>represents the</u> functor

$$R \longmapsto \mathrm{Hom}_{\text{group}}(M, R^\times) \; : \; k\text{-}\mathcal{A}lg \longrightarrow \underline{\mathcal{G}roups}.$$

<u>Pf</u>. This is the usual adjunction between $R \mapsto R^\times$ and $M \mapsto k[M]$:

$$\mathrm{Hom}_{k\text{-alg}}(k[M], R) \simeq \mathrm{Hom}_{\text{gr}}(M, R^\times),$$

which is easy to prove. It remains to verify that this bijection of sets is an isom. of groups, which is also straightforward:

$$k[M] \underset{\tilde\psi}{\overset{\tilde\varphi}{\rightrightarrows}} R \;\;\leftarrow\!\shortmid\;\; M \underset{\psi}{\overset{\varphi}{\rightrightarrows}} R^\times$$

$$k[M] \xrightarrow{\Delta} k[M]\otimes k[M] \xrightarrow{\tilde\varphi\otimes\tilde\psi} R\otimes R \xrightarrow{\cdot} R$$
$$m \longmapsto m\otimes m \longmapsto \varphi(m)\otimes\psi(m) \mapsto \varphi(m)\cdot\psi(m) = (\varphi\cdot\psi)(m). \qquad \square.$$

• <u>Example. 2.2</u>. If M is infinite cyclic, $M \approx T^{\mathbb{Z}}$, we have $k[M] = k[T, T^{-1}]$ with $\Delta(T) = T\otimes T$, so $D(\mathbb{Z}) \approx \mathbb{G}_m$.

If M is cyclic of order n, $M = \langle T \mid T^n = 1\rangle$, we have $k[M] = k[T]/(T^n - 1)$ with $\Delta(\overline{T}) = \overline{T}\otimes\overline{T}$, so $D(\mathbb{Z}/n\mathbb{Z}) \approx \mu_n$.

If M is finitely generated, then $D(M)$ is of finite type. Clearly, $k[M\times M'] \simeq k[M]\otimes k[M']$ as k-Hopf algebras, so

$$(m, m') \longleftrightarrow m\otimes m'$$

$D(M)$ is a finite product of $\mathbb{G}_m$'s and μ_n's.

Note that $k[M]$ is spanned as a k-vector space by elements x satisfying $\begin{cases} \Delta(x) = x \otimes x \\ \varepsilon(x) = 1 \end{cases}$ (namely, those $x \in M$).

• Def. 2.3. Let $(A, \cdots)$ be a k-~~Hopf~~ co-algebra. An element $a \in A$ is group-like if $\Delta(a) = a \otimes a$ and $\varepsilon(a) = 1$. An ~~alg.~~ affine group G is diagonalizable if $k[G]$ is spanned as a k-vector space by its group-like elements. A torus is a connected diagonalizable alg. group. A character of an affine group G is a hom. $\chi : G \to \mathbb{G}_m$. Let $X(G)$ be the group $\mathrm{Hom}_{k\text{-gr}}(G, \mathbb{G}_m)$.

~~The main theorem of this section is that~~

• Th. 2.4. $M \mapsto D(M) : \{\text{Ab. Gr}\}^{op} \longrightarrow \{\text{diag. groups}\}$ is an equiv. of categories with quasi-inverse $G \mapsto X(G)$.

• Lem. 2.4.1. Let $(A, \cdots)$ be a k-~~Hopf~~ coalgebra. Then the group-like elements are linearly independent. (analogous to Artin's thm. on linear indep. of characters of a monoid.)

Pf. Assume that

$$y = a_1 x_1 + \cdots + a_n x_n,$$

where the x_i's and y are group-like, is a linear relation of min. length. In particular, the x_i's are linearly indep. Applying ε on both sides, we obtain $\sum a_i = 1$. Applying Δ on both sides, we obtain

$$\begin{aligned} y \otimes y &= a_1 x_1 \otimes x_1 + \cdots + a_n x_n \otimes x_n \\ \| & \\ (\textstyle\sum a_i x_i) \otimes (\sum a_i x_i) &= \sum_{i,j} a_i a_j \, x_i \otimes x_j . \end{aligned}$$

Since the $x_i \otimes x_j$'s are linearly indep. in $A \otimes A$, we see that

$a_i a_j = 0 \quad (\forall i \neq j)$ and $a_i^2 = a_i$. So one of them is 1 and the rest are 0: $y = x_i$. $\square$

Note that a group-like element x (in a Hopf algebra) is invertible:

$$\begin{tikzcd}
x \arrow[r, mapsto, "\Delta"] \arrow[d, "\varepsilon"] & x\otimes x \arrow[d, "S\otimes \mathrm{id}"] \\
1 \arrow[d, "e"] & Sx\otimes x \arrow[dl, "m"] \\
1_A = (Sx)\cdot x &
\end{tikzcd}$$

Also, if $\Delta x = x\otimes x$, then

$$\begin{tikzcd}
x \arrow[r, mapsto, "\Delta"] \arrow[d] & x\otimes x \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\
x\otimes 1 & x\otimes \varepsilon(x)
\end{tikzcd}$$

$$x\otimes 1 = x = \varepsilon(x)\cdot x \quad (x\otimes\varepsilon(x) = \varepsilon(x)\cdot x),$$

so if x is a unit, then $\varepsilon(x) = 1$ and hence x is group-like.

If $\chi : G \to \mathbb{G}_m$ is a character, then as T is group-like in $k[T, T^{-1}] = k[\mathbb{G}_m]$, we see that $\chi^*(T) \in k[G]$ is group-like. Conversely, given a group-like element x in $k[G]$, since x is a unit, $T \mapsto x$ defines a k-alg. hom.

$$k[T, T^{-1}] \longrightarrow k[G],$$

which is a k-Hopf alg. hom. So

$$\underbrace{\{\text{group-like elements in } k[G]\}}_{\text{a subgroup of } k[G]^\times} \xrightarrow{\sim} X(G).$$

<u>Pf. of (2.4).</u> Let G be a diagonalizable group, and let $M = X(G)$ be the group of group-like elements in $k[G]$. By definition and (2.4.1), M is a basis of the k-vector space $k[G]$, so we have an isom. of k-vector spaces

$$k[M] \xrightarrow{\sim} k[G].$$

It preserves mult. and comult., which is easily checked on the basis. Hence $G \simeq D(M)$. It is also easy to verify that both mappings

$$\mathrm{Hom}(M_1, M_2) \to \mathrm{Hom}(D(M_2), D(M_1)) \to \mathrm{Hom}(X(D(M_1)), X(D(M_2)))$$

and

$$\mathrm{Hom}(G_1, G_2) \to \mathrm{Hom}(X(G_2), X(G_1)) \to \mathrm{Hom}(D(X(G_1)), D(X(G_2))),$$

with M and G being identified with $X(D(M))$ and $D(X(G))$ resp., are identities. $\square$

• Exercise 2.4.2. $M \mapsto D(M)$ is exact; alg. subgr. and quot. of a diag. gr. are diag. //.

~~Note also that $x \in k[M]$~~ One can also characterize the diagonalizability of G in terms of linear rep. of G.

• Def. 2.5. Let G be an affine group, (V, ρ) a linear rep. of G. We say that (V, ρ) is diagonalizable, if it is a (direct) sum of 1-dim'l subrep. Let σ be the corresp. coaction on V. $\forall\, \chi \in X(G)$, let

$$V_\chi = \{ v \in V \mid \sigma(v) = v \otimes \chi \},$$

here χ is identified with $\chi^{\#}(T) \in k[G]$.

called the χ-eigenspace of (V, ρ).

• Exercise 2.6. Show that V_χ is indeed a subspace of V, (hence a subrep.) and $V_\chi = \{ v \in V \mid \rho_k(g)(v) = \chi_k(g) \cdot v \}$. //.

• Th. 2.7. Let G be an affine group. TFAE:

i). G is diagonalizable ;

ii). every linear rep. of G is diagonalizable ;

ii bis). every f. dim'l rep. of G is diag. ; $\Rightarrow$ all simple obj. in $\mathrm{Rep}(G)$ is 1-dim'l.

iii). $\forall$ rep. (V, ρ) of G, we have $V = \bigoplus_{\chi \in X(G)} V_\chi$;

iv). G is commutative and $\mathrm{Hom}_{k\text{-gr.}}(G, \mathbb{G}_a) = 0$.

Pf: i) $\Rightarrow$ ii). $G = D(M)$, $X(G) \simeq M$

$\chi_m \longleftrightarrow m \ : \ \chi_m^{\#}(T) = m$.

$\forall (V, \rho) \leftrightsquigarrow \sigma : V \to V \otimes k[G]$, need to show that $\forall v \in V$,

$$v \in \sum_{\text{fini}} (\text{lines stable by } G).$$

Let $\sigma(v) = \sum v_i \otimes m_i$, $m_i \in M$. Then

$$\begin{array}{ccc} v & \overset{\sigma}{\longmapsto} & \sigma(v) = \sum v_i \otimes m_i \\ & \searrow & \downarrow \mathrm{id}_V \otimes \varepsilon \\ & & v \otimes 1 \ = \ \sum v_i \otimes 1 \end{array} \quad \Rightarrow v = \sum v_i .$$

Moreover, $v_i \in V_{\chi_{m_i}}$:

$$\begin{array}{ccccc} v & \overset{\sigma}{\longmapsto} & \sum v_i \otimes m_i & \overset{\mathrm{id}_V \otimes \Delta}{\longmapsto} & \sum v_i \otimes m_i \otimes m_i \\ \sigma \downarrow & & & & \| \\ \sum v_i \otimes m_i & \overset{\sigma \otimes \mathrm{id}_A}{\longmapsto} & \sum \sigma(v_i) \otimes m_i & & \end{array} \quad \Rightarrow \sigma(v_i) = v_i \otimes m_i .$$

So each line $\langle v_i \rangle$ is a sub-comodule of V, and $v \in$ sum of these lines.

ii) $\Leftrightarrow$ ii bis). Clear, by (I, 2.8).

ii) $\Rightarrow$ iii). For ~~$\chi \neq \chi'$, we have $V_\chi \cap V_{\chi'} = 0$~~ that $\sum_\chi V_\chi$ is a direct sum, let $v_1 + \cdots + v_n = 0$ be a relation of min. length, with $v_i \in V_{\chi_i}$ and that the χ_i's are distinct.

Then $\exists\, g \in G(k)$ s.t. $\chi_1(g) \neq \chi_2(g)$. Applying $\rho_k(g)$, we obtain

$$\chi_1(g)\cdot v_1 + \chi_2(g)\cdot v_2 + \cdots + \chi_n(g)\cdot v_n = 0,$$

so $$(\chi_2(g) - \chi_1(g))\, v_2 + \cdots + (\chi_n(g) - \chi_1(g))\cdot v_n = 0,$$

contradicting with minimality. It remains to see $V \subset \sum_\chi V_\chi$. A 1-dim'l rep. is nothing but a character, so by ii), $\forall v \in V$, it is contained in a finite sum of lines L_i on which G acts via characters χ_i, so $v \in \sum L_i \subset \sum V_{\chi_i}$.

<u>iii) $\Rightarrow$ iv)</u>. Let (V, ρ) be a faithful rep. of G (necessarily ∞-dim'l if G is not of f. type, e.g. the regular rep.). To see that G is commutative, it suffices to show that

$$\rho_R(g) \circ \rho_R(h) = \rho_R(h) \circ \rho_R(g), \ \forall\, g, h \in G(R).$$

This is clear on each $(V_\chi)_R$:

$$\chi_R(g)\cdot \chi_R(h) = \chi_R(h)\cdot \chi_R(g) \quad \text{in } R^\times,$$

therefore by iii), it holds on V_R.

Let $\varphi : G \to \mathbb{G}_a$ be a hom. Then

$$\rho : G \xrightarrow{\varphi} \mathbb{G}_a \underset{(\mathrm{I}, p.4)}{\simeq} U_2 \hookrightarrow GL_2 \qquad \left(g \mapsto \begin{pmatrix} 1 & \varphi(g) \\ & 1 \end{pmatrix} \right)$$

does not satisfy iii) unless $\varphi = 0$: if $\varphi_k(g) \neq 0$, then $\left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \middle| \, x \in k \right\}$ is the only line in k^2 stable under $\rho_k(g)$.

<u>iv) $\Rightarrow$ i)</u>. It amounts to showing that, if A is a

co-commutative Hopf algebra, such that

$$\Delta f = f \otimes 1 + 1 \otimes f \Rightarrow f = 0,$$

then A is spanned by group-like elements. Although being a pure algebra problem (or rather, a coalgebra problem), it is difficult to be proved directly (at least for me). We shall pass to the dual world, and the structure thm. of Artinian rings will play an essential role.

Consider the regular coaction of A on A. If $B \subset A$ is a sub-comodule (i.e. $\Delta(B) \subset B \otimes A$), then by co-commutativity, it is also a sub-coalgebra (i.e. $\Delta(B) \subset B \otimes B$; this is an easy exercise, by taking a basis for B and extending it to a basis for A). As A is the directed union of its f. dim'l sub-coalgebras B, it suffices to show that B is spanned by group-like elements. ~~As $\Delta(1) = 1 \otimes 1$, we may assume $1 \in B$.~~

Then the dual space $B^\vee$ is a f. dim'l comm. k-algebra, hence artinian, hence equals a direct product of local Artin rings. It then suffices to show that all these factors are k ($\Leftrightarrow B^\vee$ is reduced). Reason: note that if $\Delta(a) = \sum b_i \otimes c_i$, $a \in B$, then for $h_1, h_2 \in B^\vee$, we have

$$\langle h_1 h_2, a \rangle = \sum \langle h_1, b_i \rangle \cdot \langle h_2, c_i \rangle.$$

If $B^\vee \simeq k \times \cdots \times k$, with e_i the corresponding idempotents $(\cdots, 0, 1, 0, \cdots)$, and let e_i^* be the dual basis for B, then $\Delta e_i^* = e_i^* \otimes e_i^*$ (exercise). Also, $\left.\begin{array}{l} e_i^* = \varepsilon(e_i^*) \cdot e_i^* \\ e_i^* \neq 0 \end{array}\right\} \Rightarrow \varepsilon(e_i^*) = 1.$

Now assume $B^\vee \simeq C \times \cdots$, where (C, m_c) is a local artin k-alg. with $m_c \neq 0$. Then $C/m_c^2 \to C/m_c \simeq k$ gives a splitting $C/m_c^2 \simeq k \oplus m_c/m_c^2$, and any k-linear map $m_c/m_c^2 \to k\cdot\varepsilon_1 \simeq k$ gives a k-algebra hom. (exercise)

$$\begin{array}{ccc} C/m_c^2 & \longrightarrow & k[\varepsilon_1], \text{ ring of dual numbers } (\varepsilon_1^2=0). \\ \wr\| & & \| \\ k & \xrightarrow{\ id_k\ } & k \\ \oplus & & \oplus \\ m_c/m_c^2 & \xrightarrow{\text{any } k\text{-linear}} & k\cdot\varepsilon_1. \end{array}$$

Therefore we obtain a k-algebra hom. $B^\vee \to C \to C/m_c^2 \to k[\varepsilon_1]$, which is surjective if we take $m_c/m_c^2 \to k\cdot\varepsilon_1$ to be so. This map takes the form $\langle -, a\rangle + \langle -, b\rangle\cdot\varepsilon_1$ for some $a, b \in B$, $a, b \neq 0$. That it preserves multiplication means that $\forall\, h_1, h_2 \in B^\vee$, we have

$$\begin{cases} \langle h_1h_2, a\rangle = \langle h_1, a\rangle\cdot\langle h_2, a\rangle, \\ \langle h_1h_2, b\rangle = \langle h_1, a\rangle\cdot\langle h_2, b\rangle + \langle h_1, b\rangle\cdot\langle h_2, a\rangle, \end{cases}$$

i.e.

$$\begin{cases} \Delta a = a\otimes a \\ \Delta b = a\otimes b + b\otimes a. \end{cases}$$

(note that $S(a)$ may not live in B.)

As $a \neq 0$, $a = \varepsilon(a)\cdot a \Rightarrow \varepsilon(a) = 1 \Rightarrow a\cdot S(a) = 1$, so a is invertible. Then $\Delta(a^{-1}b) = 1\otimes(a^{-1}b) + (a^{-1}b)\otimes 1$, $a^{-1}b \neq 0$, a contradiction. $\square$.

• <u>Cor. 2.8.</u> <u>Let</u> G <u>be a comm. affine k-group.</u> <u>Then</u> G <u>is diag. if and only if each f. dim. rep. of</u> G <u>is semisimple.</u>

Pf: ($\Rightarrow$). Use (2.7), i) $\Rightarrow$ ii bis).

($\Leftarrow$). We have $\mathrm{Hom}_{k\text{-gr.}}(G, \mathbb{G}_a) = 0$, otherwise

$$g \mapsto \begin{pmatrix} 1 & \varphi(g) \\ & 1 \end{pmatrix}$$

is not a semisimple rep. Then apply (2.7), iv) $\Rightarrow$ i). $\square$.

Let G and H be k-alg. groups. An <u>action of H on the alg. group G</u> is an action on the alg. variety G by group automorphisms: $\rho: H \times_k G \longrightarrow G$, s.t. $\forall\, h \in H(R)$,

$$G_R \simeq \mathrm{Spec}(R) \times_R G_R \xrightarrow{h \times id} H_R \times_R G_R \xrightarrow{\rho_R} G_R$$

is a ~~x~~ hom. of R-group schemes (equivalently, the induced map $G(R) \to G(R)$ is a group hom.; this is because $\forall$ R-algebra R', the induced map $G_R(R') \to G_R(R')$ is also induced by the ~~$h \in H(R)$~~ image $h_{R'}$ of h under $H(R) \to H(R')$).

• <u>Prop. 2.9.</u> (Rigidity). <u>Let G be a diagonalizable k-alg. group, and let H be a connected k-alg. group. Then any action of H on the alg. group G is trivial, i.e. ρ is the projection morphism.</u>

Pf: <u>Step 1</u>. It suffices to show that $H(k) \times G(k) \to G(k)$ is the projection map.

Let A be a reduced k-algebra of f.t. (e.g. $A = k[H] \otimes_k k[G]$).

Then $\{$all k-alg. hom. $A \to k\}$ is conservative for A, i.e., the natural ~~mapping~~ homomorphism

$$A \longrightarrow \prod_{\substack{k\text{-alg. hom.} \\ A \to k}} k$$

is injective (because its kernel is $\bigcap\{$max. ideals in $A\}$ by Nullstellensatz, which is $\mathrm{Nil}(A)$ since A is Jacobson, which is 0 since A is reduced).

To see $\rho^\#: k[G] \to k[H] \otimes k[G]$ is the map $f \mapsto 1 \otimes f$ (which is equiv. to the action being trivial), it suffices to check that

$$\rho^\#(f)\,(h, g) = f(g), \ \forall\, (h,g) \in (H \times_k G)(k).$$

If $H(k) \times G(k) \to G(k)$ is the projection, then the equation above holds:

$$\begin{tikzcd} k[G] \arrow[r, "\rho^\#"] \arrow[dr, "g"'] & k[H] \otimes k[G] \arrow[d, "(h,g)"] \\ & k \end{tikzcd}.$$

<u>Step 2</u>. This is true when G is finite. (not necessarily diag.)

Let $g \in G(k)$, and consider

$$H \simeq H \times \{g\} \xrightarrow{\ \rho\ } G.$$

As $\rho(1_H, g) = g$, the image of ρ contains $\{g\}$, which is a connected component of G. Since H is connected, $\rho(H) = \{g\}$.

Since $\{g\}$ is open in G, the morphism factors through $\{g\}$:

$$\begin{tikzcd} H \simeq H \times_k \mathrm{Spec}(k) \arrow[r, "\mathrm{id}_H \times g"] \arrow[dr, dashed] & H \times_k G \arrow[r, "\rho"] & G \\ & \mathrm{Spec}(k) \arrow[ur, "g"'] & \end{tikzcd}$$

In particular, $\forall h \in H(k)$, $\rho(h, g) = g$.

<u>Step 3</u>. Now we are ready to prove the general case.

$\forall\, n \geq 1$, $g \mapsto g^n : G \to G$ is a hom., and let $G[n]$ be its kernel

$$1 \to G[n] \to G \xrightarrow{g \mapsto g^n} G.$$

Under the exact contravariant equivalence (2.4), this sequence becomes

$$1 \leftarrow M/M^n \leftarrow M \xleftarrow{m^n \leftarrow\!\shortmid\, m} M,$$

with $G = D(M)$, M a (multiplicatively written) f.g. comm. group.

An action ρ of H on the alg. group G also induces an action on $G[n]$, which must be trivial as $G[n] = D(M/M^n)$ is a finite alg. group.

$\forall\, h \in H(k)$, $\rho(h, -)$ is an automorphism of G, hence induces an auto. of $M = X(G)$, which becomes the identity automorphism when passing to M/M^n, $\forall\, n \geq 1$. It then must be identity on M because

$$M \hookrightarrow \prod_{n \geq 1} M/M^n.$$

So, $G \simeq \{h\} \times G \to H \times G \xrightarrow[\rho]{} G$ is the identity morp. $\square$.

• Rmk. 2.10. For a general base field k, there is a "Galois" issue. An affine k-group G is said to be of multiplicative type, if $G_{k^{sep}}$ is diagonalizable (i.e. $k^{sep}[G_{k^{sep}}]$ is spanned by group-likes). Then the equiv. (2.4) can be extended to an equiv. between

$$\left\{\begin{matrix}\text{groups } G \text{ of} \\ \text{mult. type}/k\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{comm. groups } M \text{ with a} \\ \text{cont. action by } Gal(k^{sep}/k)\end{matrix}\right\},$$

with $G \longmapsto M = \mathrm{Hom}_{k^{sep}\text{-gr.}}(G_{k^{sep}}, \mathbb{G}_{m,k^{sep}})$ on which $\sigma \in$ Gal. gr. acts by $\chi \mapsto \sigma\cdot\chi\cdot\sigma^{-1}$. The subcategory of the left side, consisting of diag. groups over k (also called the split ones), corresponds to comm. groups M with trivial action by Galois group; this is nothing but (2.4), whose proof does not make use of the assumption that $k=\bar{k}$ of char. 0.

For an affine group G of mult. type, f. dim. rep. are again semisimple, but simple rep. may not be 1-dim'l. Given $\chi \in X(G) := \mathrm{Hom}_{k^{sep}\text{-gr.}}(G_{k^{sep}}, \mathbb{G}_{m,k^{sep}})$, it is fixed by an open subgroup of $Gal(k^{sep}/k)$, so its orbit $\{\chi_1, \cdots, \chi_n\}$ is a finite set. The rep. $\bigoplus_{i=1}^{n} \chi_i$ of $G_{k^{sep}}$ over k^{sep} is stable under Galois, hence descends to a rep. of G over k, which is irreducible. All irrep. of G arise in this way, so there is a one-to-one correspondence between irreps. of G and orbits of $X(G)$ under Galois group.

G is of mult. type $\Longleftrightarrow$ G_L is diag. for some field L/k.

§3. Jordan decomposition.

In this section, we prove the Jordan decomposition in an alg. group, in the framework of Tannakian duality.

Let k be any field, V a f. dim. k-v.s., α an endomorphism of V. We say that α is diagonalizable if V has a k-basis of eigenvectors for α, semisimple (abbreviated s.s.) if $\alpha_{\bar{k}}$ on $V_{\bar{k}}$ is diagonalizable, nilpotent if $\alpha^n = 0$ for some $n > 0$ ($\Leftrightarrow$ all eigenvalues of $\alpha_{\bar{k}}$ are 0), unipotent if $1_V - \alpha$ is nilpotent ($\Leftrightarrow$ all eigenvalues of $\alpha_{\bar{k}}$ are 1).
Although all results in this section are valid for any ~~base~~ perfect field k, we still make the simplifying assumption that $k = \bar{k}$ so that all eigenvalues live in k.

• Lem. 3.1. Let V be a f. dim. k-v.s., $\alpha \in GL(V)$. Then $\exists!\ \alpha_s, \alpha_u \in GL(V)$, with α_s semisimple and α_u unipotent, such that $\alpha = \alpha_s \circ \alpha_u = \alpha_u \circ \alpha_s$. Moreover, both α_s and α_u are polynomials in α : $\alpha_s, \alpha_u \in k[\alpha]$.

Pf. The existence is a consequence of the Jordan normal of a matrix:

$$\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} = \underset{\text{s.s.}}{\begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}} \underset{\text{unip.}}{\begin{pmatrix} 1 & \lambda^{-1} & & \\ & \ddots & \ddots & \\ & & \ddots & \lambda^{-1} \\ & & & 1 \end{pmatrix}} = \begin{pmatrix} 1 & \lambda^{-1} & & \\ & \ddots & \ddots & \\ & & \ddots & \lambda^{-1} \\ & & & 1 \end{pmatrix} \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix},$$

or else one can define directly α_s to be scalar mult. on each generalized eigenspace of α, and put $\alpha_u = \alpha_s^{-1} \circ \alpha$.
Uniqueness is also clear: if $\alpha_s \circ \alpha_u = \alpha_s' \circ \alpha_u'$, then

$$\underbrace{\alpha_s^{-1} \circ \alpha_s'}_{\text{s.s.}} = \underbrace{\alpha_u \circ \alpha_u'^{-1}}_{\text{unip.}} \Rightarrow \alpha_s^{-1} \circ \alpha_s' = 1_V = \alpha_u \circ \alpha_u'^{-1}.$$

Let

$$P(t) = (t-\lambda_1)^{n_1} \cdots (t-\lambda_\ell)^{n_\ell}, \quad \lambda_i\text{'s distinct}$$

be the characteristic polynomial of α (and of α_s); then the generalized eigenspace is

$$V_{\lambda_i}^{gen} = \ker (\alpha - \lambda_i \cdot 1_V)^{n_i}.$$

By the Chinese Remainder thm., $\exists\ Q(t) \in k[t]$ with

$$Q(t) \equiv \lambda_i \mod (t-\lambda_i)^{n_i}, \quad \forall i = 1, \dots, \ell.$$

So on each $V_{\lambda_i}^{gen}$, $Q(\alpha) = \lambda_i \cdot 1_V$, therefore $Q(\alpha) = \alpha_s$.
As $P(\alpha_s) = 0$, and $P(0) = \pm \det(\alpha) \neq 0$, we see that

$$\alpha_s \cdot (\underbrace{\text{-----}}_{\text{in } k[\alpha_s]}) \pm \det(\alpha) = 0,$$

so $\alpha_s^{-1} \in k[\alpha_s] \subset k[\alpha]$, hence $\alpha_u = \alpha_s^{-1} \circ \alpha \in k[\alpha]$. □

This is called the (<u>multiplicative</u>) <u>Jordan decomposition</u> of the linear automorphism α on V. If $W \subset V$ is a subspace invariant under α, then it is inv. under α_s and α_u (since they are in $k[\alpha]$), and $\alpha|_W = \alpha_s|_W \circ \alpha_u|_W$ is the Jordan decom. of $\alpha|_W$. ~~Moreover~~ More generally, we have the following functoriality.

• **Lem. 3.2.** Given a comm. diagram of linear maps between f. dim k-v.s.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \alpha\downarrow & & \downarrow\beta \\ V & \xrightarrow[f]{} & W \end{array}$$

with α and β invertible, we have $\beta_s \circ f = f \circ \alpha_s$, $\beta_u \circ f = f \circ \alpha_u$.

Pf: $\forall \lambda \in k$, α_s and β_s are scalar mult. by λ on V_λ^{gen} and W_λ^{gen}, resp. Note that $\forall$ polynomial $Q(t) \in k[t]$, we have

$$f \circ Q(\alpha) = Q(\beta) \circ f.$$

Hence $f(V_\lambda^{gen}) \subset W_\lambda^{gen}$, $\forall \lambda \in k$. Since $V = \bigoplus_i V_{\lambda_i}^{gen}$, it suffices to verify the identity $\beta_s \circ f = f \circ \alpha_s$ on each $V_{\lambda_i}^{gen}$, where it is clear. The other identity follows. $\square$.

• **Lem. 3.3.** Let $\alpha \in GL(V)$, $\beta \in GL(W)$. Then $(\alpha \otimes \beta)_s = \alpha_s \otimes \beta_s$ and $(\alpha \otimes \beta)_u = \alpha_u \otimes \beta_u$ on $V \otimes W$.

Pf: From the tensor product of matrices (in particular, the Jordan normal forms of α and β), we see that

$$V_\lambda^{gen} \otimes W_\mu^{gen} \subset (V \otimes W)_{\lambda\mu}^{gen}.$$

Since $V \otimes W = \bigoplus_{i,j} (V_{\lambda_i}^{gen} \otimes W_{\mu_j}^{gen})$, and clearly $(\alpha \otimes \beta)_s = \lambda_i \mu_j =$

$\alpha_s \otimes \beta_s$ on $V_{\lambda_i}^{gen} \otimes W_{\mu_j}^{gen}$, we are done. $\square$

Now we come to Jordan decom. in an alg. group G. Let $g \in G(k)$, and let (V, ρ) be a f. dim. faithful rep. of G. Then we have

$$\rho(g) = \rho(g)_s \circ \rho(g)_u \quad \text{in } GL(V),$$

but it is not clear whether $\rho(g)_s$ and $\rho(g)_u$ live in $\mathrm{Im}(\rho)$. Even if they do, say $\rho(g)_s = \rho(g_s)$ for $g_s \in G(k)$, it is not clear whether g_s and g_u depends on the choice of (V, ρ) or not. These facts **follow formally** from (3.2) and (3.3).

• <u>Th. 3.4.</u> (Tannakian duality, one way). <u>Let</u> G <u>be an affine</u> k-<u>group</u>. <u>Suppose we are given</u>, <u>for each finite dim'l rep.</u> (V, ρ) <u>of</u> G, <u>an isom.</u> $\varphi_\rho \in GL(V)$, <u>satisfying</u>

a). $\forall$ <u>hom.</u> $f : (V, \rho) \to (W, \pi)$ <u>of</u> G-<u>rep</u>, <u>we have</u>

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ {\scriptstyle \varphi_\rho}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \varphi_\pi} \\ V & \xrightarrow{f} & W \end{array},$$

b). $\forall$ <u>rep.</u> (V, ρ) <u>and</u> (W, π), <u>we have</u>

$$\varphi_\rho \otimes \varphi_\pi = \varphi_{\rho \otimes \pi} \quad \underline{\text{on}}\ V \otimes W,$$

c). <u>for the trivial rep.</u> $(k, \mathbb{1})$, <u>we have</u>

$$\varphi_{\mathbb{1}} = id. \quad \underline{\text{on}}\ k.$$

<u>Then</u>, $\exists!\, g \in G(k)$ <u>s.t.</u> $\rho_k(g) = \varphi_\rho$, $\forall$ <u>f. dim.</u> G-<u>rep.</u> (V, ρ).

• Cor. 3.5. Let G be an alg. group, $g \in G(k)$. Then $\exists!\ g_s, g_u \in G(k)$ s.t. $g = g_s \cdot g_u = g_u \cdot g_s$, and that $\forall$ f. dim. G-rep. (V, ρ), we have

$$\rho(g_s) = \rho(g)_s, \quad \rho(g_u) = \rho(g)_u.$$

Pf: Apply (3.4) to $\{\varphi_\rho = \rho(g)_s\}_{(V,\rho)}$ and $\{\varphi_\rho = \rho(g)_u\}_{(V,\rho)}$. $\square$.

This is called the (mult.) Jordan decom. in $G(k)$.

We say that $g \in G(k)$ is semisimple (resp. unipotent) if $g = g_s$ (resp. $g = g_u$).

• Prop. 3.6. Let $f: G \to H$ be a hom. of alg. groups, $g \in G(k)$. Then $f_k(g_s) = f_k(g)_s$, $f_k(g_u) = f_k(g)_u$.

Pf: To verify $f_k(g_s) = f_k(g)_s$, it suffices to take **one** f. dim faithful rep. (V, ρ) of H, and see if

$$\rho_k(f_k(g_s)) = [\rho_k \circ f_k(g)]_s,$$

and this is clear by definition of g_s. $\square$.

Proof of (3.4). Note that the data $\{\varphi_\rho\}_{(V,\rho)}$ can be extended to all G-rep., even infinite dim'l. This is because **any** G-rep is a directed union of its f. dim. subrep. V_i, for which φ_{ρ_i} is given, with

$$\varphi_{\rho_j}|_{V_i} = \varphi_{\rho_i}$$

whenever $V_i \subset V_j$, by a). So they glue to a $\varphi_\rho \in GL(V)$. Given $f : (V,\rho) \to (W,\pi)$, to verify $\varphi_\pi \circ f = f \circ \varphi_\rho$ on V, it suffices to check this on each V_i, where it follows from condition a) applied to

$$f|_{V_i} : V_i \longrightarrow f(V_i).$$

Similarly, as

$$V \otimes W = \mathrm{colim}\,\{V_i \otimes W_j\},$$

to verify $\varphi_\rho \otimes \varphi_\pi = \varphi_{\rho\otimes\pi}$, it suffices to check this on each $V_i \otimes W_j$, where it is clear. Therefore, a) – c) continue to hold for the extended data.

Now the "universal" (in some sense) rep., the regular rep. $(A = k[G], \text{reg.})$, is included, and we have $\varphi = \varphi_{reg} \in GL(A)$. We will first find a **candidate** $g \in G(k)$ (i.e. a k-alg. hom. $g : A \to k$), and then prove **that** it meets the requirement. In particular, $\varphi : A \longrightarrow A$ should be "right translation by g" : $f(x) \mapsto f(xg)$, so the composite

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & A & \xrightarrow{\varepsilon} & k \\ f(x) & \longmapsto & f(xg) & \longmapsto & f(g) \end{array}$$

is "evaluation at g", namely **the** candidate that we are looking for. But first we **need** to show **that** the linear auto. φ is a k-alg. isom..

We claim that

$$\begin{tikzcd} A\otimes A \arrow[r, "m"] \arrow[d, "\varphi\otimes\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "\varphi"] \\ A\otimes A \arrow[r, "m"] & A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "\varphi"] \\ k \arrow[ur, "e"] \arrow[dr, "e"'] \arrow[r, phantom, "\circlearrowright"] & A \\ & \end{tikzcd} .$$

By b), $\varphi\otimes\varphi = \varphi_{reg.\otimes reg.}$, so the first diagram commutes if m is G-equivariant, by a). Translating into coactions by A, we are asking for the commutativity of

$$\begin{tikzcd} A\otimes A \arrow[r, "\Delta\otimes\Delta"] \arrow[dd, "m"'] & A\otimes A\otimes A\otimes A \arrow[r, "id\otimes\tau\otimes id"] & A\otimes A\otimes A\otimes A \arrow[d, "id\otimes id\otimes m"] \\ & & A\otimes A\otimes A \arrow[d, "m\otimes id"] \\ A \arrow[rr, "\Delta"] & & A\otimes A \end{tikzcd} ,$$

$\tau: A\otimes A \to A\otimes A$ is transposition $x\otimes y \mapsto y\otimes x$

comodule structure for $A\otimes A$, reg.$\otimes$reg.

namely,

$$\begin{tikzcd} A\otimes A \arrow[r, "\Delta\otimes\Delta"] \arrow[dr, "m"'] & A^{\otimes 4} \arrow[r, "id\otimes\tau\otimes id"] & A^{\otimes 4} \arrow[r, "m\otimes m"] & A\otimes A \\ & A \arrow[urr, "\Delta"'] & & \end{tikzcd} ,$$

which is nothing but the fact that Δ, being a k-alg. hom., preserves multiplication. For the second diagram, note that the subspace $e: k \hookrightarrow A$ is G-invariant, on which G acts trivially ("right translations of constant functions are still constant"), i.e., $e: (k, \mathbb{1}) \to (A, reg.)$ is G-equiv., hence the commutativity follows from a) and c).

Now we have our candidate

$$g : A \xrightarrow{\varphi} A \xrightarrow{\varepsilon} k .$$

$\forall$ G-rep. (V, ρ), not necessarily f. dim'l, to show $\rho_k(g) = \varphi_\rho$, we pass to coaction $\sigma : V \to V \otimes A$, and ask for

$$\begin{tikzcd} V \arrow[r, "\sigma"] \arrow[dr, "\varphi_\rho"'] & V\otimes A \arrow[d, "id \otimes g"] \arrow[dl, phantom, "\circlearrowright"] \\ & V \simeq V \otimes k \end{tikzcd},$$

namely,

$$\begin{tikzcd} V \arrow[r, "\sigma"] \arrow[d, "\varphi_\rho"] & V \otimes A \arrow[d, "id \otimes \varphi"] \\ V \arrow[r, "\sigma"] \arrow[dr, "can."', "\simeq"] & V \otimes A \arrow[d, "id \otimes \varepsilon"] \\ & V \otimes k \end{tikzcd}.$$

The bottom triangle commutes by def'n, and the top square commutes if $\sigma : (V, \rho) \to (V \otimes A, \text{triv.} \otimes \text{reg.})$ is G-equiv. by a) ~~and~~ c). This is so, as we pass to A-coactions:

$$\begin{tikzcd} V \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & V \otimes A \arrow[d, "\sigma \otimes id"] \\ V \otimes A \arrow[r, "id \otimes \Delta"] & V \otimes A \otimes A \end{tikzcd}.$$

As for the uniqueness of g, it follows either from that $(A, \text{reg.})$ is faithful (exercise), so that its f. dim'l subrep. form a faithful family, or from that G is an inverse limit of alg. groups G_i, each of which has a f. dim'l faithful rep., and they together form a faithful family for G. $\square$.

• <u>Rem. 3.7</u>. In fact, one can prove, verbatim, a more general result: $\forall$ k-alg. R, if we are given, for each f. dim'l G-rep. (V, ρ), an R-linear automorphism φ_ρ of V_R, satisfying similar conditions a), b), c), then $\exists!$ $g \in G(R)$ s.t. $\rho_R(g) = \varphi_\rho$, $\forall$ f. dim. G-rep. (V, ρ).

Put in more categorical terms, let $\underline{Aut}(\omega^G)(R)$ be the group of natural automorphisms of the functor

$$\begin{array}{ccccc} Rep(G) & \xrightarrow{\omega^G} & Vect_k & \xrightarrow{\phi_R} & Mod_R \\ (V, \rho) & \longmapsto & V & \longmapsto & V_R \end{array},$$

and let $\underline{Aut}^{\otimes}(\omega^G)(R)$ be the subgroup consisting of those natural transformations compatible with tensor product structure (i.e. satisfying b) and c) for R). Then

$$R \mapsto \underline{Aut}^{\otimes}(\omega^G)(R) : k\text{-}Alg \longrightarrow \underline{Groups}$$

is a functor, and one has a nat. transf. $G \longrightarrow \underline{Aut}^{\otimes}(\omega^G)$. Theorem (3.4), in the strong version, says that this nat. transf. is an isom.. In particular, the affine group G can be recovered from its rep. category $Rep(G)$ endowed with $\otimes$, and the "fibre functor" $\omega^G : Rep(G) \longrightarrow Vect_k$. We will come back to this later, and characterize such categories. (see IV, 2.8).

§4. Unipotent Groups.

Note that, products of unip. matrices may no longer be unipotent, e.g. $\begin{pmatrix}1&1\\&1\end{pmatrix}\cdot\begin{pmatrix}1&\\1&1\end{pmatrix}$. We could define an alg. group G to be unip. if all elements in $G(k)$ (or $G(\bar{k})$, if k is not assumed to be alg. closed) are unip., but in char. p, such a group may not be embeddable into any $\mathbb{U}_n$, which is a property that we would like any unip. group to have (think of μ_p). So we will adopt a def'n that works in any characteristic, even though we have assumed char.$(k)=0$ for simplicity.

• Def. 4.1. An affine k-group G is unipotent, if $\forall$ ~~f. dim'l~~ nonzero G-rep. (V,ρ), $\exists\ v\in V, v\neq 0$, that is a fixed vector: $\rho_R(g)\cdot(v\otimes 1_R)=v\otimes 1_R$, $\forall$ k-alg. R and $g\in G(R)$ ($\Longleftrightarrow$ $\sigma(v)=v\otimes 1$, where σ is the corresp. coaction by $k[G]$).
(exercise)

• Lem. 4.2. Let $G\xrightarrow{f}H$ be a surjective hom. of alg. groups. If G is unipotent, so is H. In particular, $\pi_0(G)=\mathbb{1}$.

Pf: $\forall$ nonzero H-rep. (V,ρ), we have a G-rep. $(V,\rho\circ f)$, so $\exists\ v\neq 0$ in V such that

$$\begin{array}{ccccc} V & \longrightarrow & V\otimes k[H] & \xrightarrow{id\otimes f^{\#}} & V\otimes k[G] \\ v & \longmapsto & ? & \longmapsto & v\otimes 1 \end{array}.$$

As $id_V \otimes f^\#$ is injective (I, 4.5), we deduce that "?" $= v \otimes 1$. $\square$.

• <u>Lem. 4.3.</u> <u>An alg. group</u> G <u>is unipotent</u> $\Leftrightarrow \forall$ <u>f. dim'l</u> G-<u>rep.</u> (V, ρ), $\exists$ <u>a</u> k-<u>basis of</u> V <u>with respect to which</u> $\rho: G \to GL_n$ <u>factors through</u> $\mathbb{U}_n$ (<u>with</u> $n = \dim(V)$).

<u>Pf.</u> ($\Leftarrow$). If $e_1, \cdots, e_n$ is such a basis, then e_1 is a fixed vector.

($\Rightarrow$). Given $(V, \rho) \leftrightarrow \sigma: V \to V \otimes k[G]$, let $e_1 \neq 0$ be a fixed vector. Then G acts on $V/\langle e_1 \rangle$. Applying induction on $\dim(V)$, we may assume that $\exists$ k-basis $\bar{e}_2, \cdots, \bar{e}_n$ of $V/\langle e_1 \rangle$ such that, denoting by $\bar{\sigma}: V/\langle e_1 \rangle \to V/\langle e_1 \rangle \otimes k[G]$ the quotient comodule structural map,

$$\bar{\sigma}(\bar{e}_2) = \bar{e}_2 \otimes 1,$$

$$\bar{\sigma}(\bar{e}_3) = \bar{e}_3 \otimes 1 + \bar{e}_2 \otimes a_{23},$$

$$\cdots \quad \cdots$$

$$\bar{\sigma}(\bar{e}_n) = \bar{e}_n \otimes 1 + \bar{e}_{n-1} \otimes a_{n-1,n} + \cdots + \bar{e}_2 \otimes a_{2n},$$

for some $a_{ij} \in k[G]$. **Let** $e_2, \cdots, e_n$ be liftings in V of $\bar{e}_2, \cdots, \bar{e}_n$, resp. Then $\sigma(e_2)$ and $e_2 \otimes 1$ have the same image in $V/\langle e_1 \rangle \otimes k[G]$, so they differ by something in

$$\ker(V \otimes k[G] \to V/\langle e_1 \rangle \otimes k[G]) = \langle e_1 \rangle \otimes k[G],$$

and we can write

$$\sigma(e_2) = e_2 \otimes 1 + e_1 \otimes a_{12},$$

and similarly,

$$\sigma(e_3) = e_3 \otimes 1 + e_2 \otimes a_{23} + e_1 \otimes a_{13},$$

$$\cdots \quad \cdots$$

$$\sigma(e_n) = e_n \otimes 1 + \cdots + e_1 \otimes a_{1n}.$$

This means that ρ is given by

$$\begin{aligned} G(R) &\longrightarrow GL_n(R) \\ g &\longmapsto \begin{pmatrix} 1 & a_{12}(g) & \cdots & a_{1n}(g) \\ & 1 & \cdots & \cdots \\ & & \ddots & \cdots \\ & & & 1 \end{pmatrix}, \end{aligned}$$

i.e. the nat. transf. ρ factors through the ~~sub~~ subfunctor $\mathbb{U}_n$.

The starting case $\dim(V) = 1$ is clear: if $\exists$ a fixed vector $v \neq 0$, then $\sigma(v') = v' \otimes 1$ $(\forall v' \in V)$, namely (V, ρ) is the trivial rep., factoring through $\mathbb{U}_1 = \mathrm{Spec}(k) \overset{e}{\hookrightarrow} \mathbb{G}_m$. $\square$.

• Th. 4.4. An alg. group G is unipotent $\Longleftrightarrow$ $\exists$ injective hom. $G \hookrightarrow \mathbb{U}_n$, for some n.

Pf: ($\Longrightarrow$). It follows from (4.3) and (I, 5.2).

($\Longleftarrow$). $k[G]$ is a quotient of $k[\mathbb{U}_n]$, so $k[G] = k[a_{ij}\,;\, i<j]$ with

$$\Delta(a_{ij}) = a_{ij} \otimes 1 + \sum_{i<l<j} a_{il} \otimes a_{lj} + 1 \otimes a_{ij}.$$

Let $\sigma : V \to V \otimes k[G]$ be a f. dim'l comodule. $\forall d \geqslant 0$, define $A_d \subset A = k[G]$ to be those $f \in A$ that can be expressed as a polynomial $F(\cdots, a_{ij}, \cdots)$, $F = \sum_{\underline{n}} c_{\underline{n}} \cdot \prod_{i,j} X_{ij}^{n_{ij}}$, such that

$\sum_{i,j} n_{ij}(j-i) \le d$, for each monomial $\prod_{i,j} X_{ij}^{n_{ij}}$ occurring in F. Then $A_0 = k$, and each A_d is a subspace. Define

$$V_d = \{ v \in V \mid \sigma(v) \in V \otimes A_d \},$$

which is a subspace of $V = \bigcup_{d \ge 0} V_d$. If $V_d = 0$ for some d, consider the image of V_{d+1} under

$$\begin{tikzcd} V \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & V \otimes A \arrow[d, "id \otimes \Delta"] \\ V \otimes A \arrow[r, "\sigma \otimes id"] & V \otimes A \otimes A \end{tikzcd} .$$

Since $\Delta(A_{d+1}) \subset \sum_i A_i \otimes A_{d+1-i}$ (exercise) $\subset A \otimes A_d + A_d \otimes A$, we have

$$(id \otimes \Delta) \circ \sigma (V_{d+1}) \subset V \otimes (A \otimes A_d + A_d \otimes A).$$

$V_d = 0 \Rightarrow \bar{\sigma} : V \to V \otimes A/A_d$ is injective $\Rightarrow$

$$(\bar{\sigma} \otimes id) \circ \bar{\sigma} : V \longrightarrow V \otimes A/A_d \otimes A/A_d$$

is injective, so $(\bar{\sigma} \otimes id) \circ \bar{\sigma}(V_{d+1}) = 0 \Rightarrow V_{d+1} = 0$.

If $V_0 = 0$, then all $V_d = 0$, hence $V = 0$. So if $V \ne 0$, then $V_0 \ne 0$. Let $v \in V_0$, $v \ne 0$, and write

$$\sigma(v) = v' \otimes 1, \text{ some } v' \in V.$$

Then $v = \varepsilon(1) \cdot v' = v'$, i.e., v is a fixed vector. $\square$.

• Cor. 4.5. ~~#~~ Algebraic subgroups and extensions of unip. groups are unip.

Pf: For alg. subgroups, use (4.4). Let $N \triangleleft G$, with N and G/N unipotent, and let (V, ρ) be a f. dim'l nonzero G-rep. Then $V^N \neq 0$, and it is a G-stable subspace on which N acts trivially, so it descends to an action by G/N:

$$G \underset{G/N}{\times} G \overset{p_1}{\underset{p_2}{\rightrightarrows}} G \circlearrowright V^N$$

$p_1^*(\rho)$ and $p_2^*(\rho)$ are the same rep. of $G \times_{G/N} G$ on V^N: given $(g_1, g_2) \in (G \times_{G/N} G)(R) = G(R) \times_{G/N(R)} G(R)$, we have $g_1 = g_2 \cdot n$ for some $n \in N(R')$, so $\rho_R(g_1) = \rho_R(g_2)$ on $(V^N)_R$. Therefore, we have the factorization

$$\begin{tikzcd} V^N \arrow[r, "\sigma^N"] \arrow[d, dashed] & V^N \otimes k[G] \arrow[r, shift left, "id \otimes i_1"] \arrow[r, shift right, "id \otimes i_2"'] & V^N \otimes \left(k[G] \otimes_{k[G/N]} k[G]\right) \\ V^N \otimes k[G/N] \arrow[ur] & & \end{tikzcd}$$

$V^N \otimes k[G/N] = $ (equalizer of $id \otimes i_1$ and $id \otimes i_2$).

As G/N is unip., $\exists\, v \neq 0$ in V^N s.t. $\sigma^N(v) = v \otimes 1$ in $V^N \otimes k[G/N]$, hence $= v \otimes 1$ in $V^N \otimes k[G]$, too. $\square$.

• Cor. 4.6. An alg. group G is unip. $\Leftrightarrow$ $\exists$ a filtration by alg. subgroups

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_N = \mathbb{1}$$

such that $G_i / G_{i+1} \cong \mathbb{G}_a$, $\forall i$.

Pf: ($\Leftarrow$). As $\mathbb{G}_a \simeq \mathbb{U}_2$ is unip., this follows from (4.5).

($\Rightarrow$). Take an embedding $G \hookrightarrow \mathbb{U}_n$ by (4.4). Such a filtration exists for $\mathbb{U}_n$:

$$\mathbb{U}_n \triangleright \underbrace{\begin{pmatrix} 1 & 0 & * & \cdots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ & & & \ddots & 0 \\ & & & & 1 \end{pmatrix}}_{\mathbb{G}_a^{n-1}} \triangleright \underbrace{\begin{pmatrix} 1 & 0 & 0 & * & \cdots & * \\ & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & * \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots & 0 \\ & & & & & 1 \end{pmatrix}}_{\mathbb{G}_a^{n-2}} \triangleright \cdots \triangleright \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 & * \\ & \ddots & 0 & \cdots & 0 \\ & & \ddots & & \vdots \\ & & & \ddots & 0 \\ & & & & 1 \end{pmatrix}}_{\mathbb{G}_a} \triangleright \mathbb{1}.$$

Let
$$\mathbb{U}_n = H_0 \triangleright H_1 \triangleright \cdots \triangleright H_N = \mathbb{1}$$
be such that $H_i/H_{i+1} \cong \mathbb{G}_a$, and take $G_i = G \cap H_i$. Then
$$G_i/G_{i+1} \hookrightarrow H_i/H_{i+1} \quad \text{(allow me to skip the details ...)}$$

Closed subsets of $\mathbb{G}_a$ are either the whole $\mathbb{G}_a$ or a finite set of closed points, and the latter cannot be an alg. subgroup unless it is $\{e\}$ (since char.$(k) = 0$). $\blacksquare$.

• **Cor.** 4.7. <u>Let G be a unip. alg. group and</u> H <u>a diagonalizable alg. group. Then</u> $\mathrm{Hom}_{k\text{-gr.}}(G, H) = 0$. <u>Consequently, if U is a unip. subgroup and D is a diag. subgroup of some alg. group, then</u> $U \cap D = \mathbb{1}$.

Pf. By (1.8 i)) and (4.2), a hom. $\varphi: G \to H$ factors through $H^0 \cong \mathbb{G}_m^n$, so it suffices to consider $\varphi: G \to \mathbb{G}_m$, namely a 1-dim. rep. of G, which must be trivial by def'n.

By (4.5), $U \cap D$ is unip., so the inclusion

$$U \cap D \hookrightarrow D$$

must be trivial. $\square$.

Finally, returning to the comment at the beginning of this section, we give a characterization in terms of its "group elements".

• Prop. 4.8. An alg. group G is unip. $\Leftrightarrow$ all $g \in G(k)$ are unip. (recall: $g_s = 1$).

Pf: ($\Rightarrow$). Clear, by (4.4).

($\Leftarrow$). Take an embedding $G \hookrightarrow GL_V$. If one can show that, w.r.t. some k-basis of V, $G(k)$ is contained in $\mathbb{U}_n(k)$, then $G \hookrightarrow \mathbb{U}_n$ (reason: $G \cap \mathbb{U}_n \hookrightarrow G$ induces $(G \cap \mathbb{U}_n)(k) = G(k)$, so $G \cap \mathbb{U}_n = G$), so that G is unip. by (4.4). To prove $G(k) \hookrightarrow \mathbb{U}_n(k)$, it suffices to prove that $V^{G(k)} \neq 0$, for one can then use induction on $V/V^{G(k)}$ to find such a basis (see the proof of (4.3)).

Let W be a simple $G(k)$-subrep. of V (in part., $W \neq 0$); we will show that $G(k)$ acts on W trivially. Let A be the sub-k-algebra of $\mathrm{End}_k(W)$ generated by the $g|_W$, $g \in G(k)$. Then W is a simple (left) A-module, so A, which acts faithfully on

W, is a simple algebra (see, e.g., Bourbaki, <u>Algèbre</u>, Ch. 8, §5, n°2, prop. 9). The subspace

$$I := \langle g|_W - 1 \;;\; g \in G(k) \rangle \quad \subset A$$

is a two-sided ideal in A $\left(h\cdot(g-1) = (hg-1) - (h-1) \ldots \right)$. Since all eigenvalues of $g|_W$ are 1, we have

$$\mathrm{Tr}\,(g|_W - 1) = 0,$$

thus $\mathrm{Tr}(I) = 0$, but $\mathrm{Tr}(1_A) = \dim(W)$, so $I \neq A$, so $I = 0 \Rightarrow g|_W = 1$, $\forall g \in G(k)$, namely $G(k)$ acts trivially on W. $\square$.

• <u>Rem. 4.9</u>. i). We mention in passing that for char.$(k) = 0$, there is an equiv. of categories

$$\mathrm{Lie} : \left\{ \begin{array}{l} \text{unipotent} \\ k\text{-alg. groups} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{f. dim. nilpotent} \\ \text{Lie algebras over } k \end{array} \right\}.$$

Note that it is not obvious a priori that any nilpotent Lie algebra arises from an alg. group.

ii). If char.$(k) = p$, the def'n (4.1) still applies. (4.6) is modified: G is unip. $\Leftrightarrow \exists$ filtr. s.t. each G_i/G_{i+1} is isom. to an alg. subgroup of $\mathbb{G}_a$, e.g. $(\mathbb{Z}/p\mathbb{Z})_k$, α_{p^r} $(r \geq 1)$ (where $\alpha_{p^r}(R) = \{ x \in R \mid x^{p^r} = 0 \}$). Th. (4.4) remains valid. Prop. (4.8) fails: $\mu_p(k) = \{1\}$, but μ_p is diag., not unip..

§5. Solvable groups.

(Hom. between diag. gr. and unip. gr. are trivial, but Ext. may not be trivial, hence solvable gr.)

We give two definitions of a solvable alg. group, one being rep.-theoretic (under the name of a "trigonalizable group"), one being group-theoretic (using the termination of the derived series, as for abstract groups), and then show that they are equivalent (for connected groups).

• Def. 5.1. An affine k-group G is trigonalizable, if each rep. $(V \neq 0, \rho)$ of G has a G-stable line ($\Leftrightarrow \exists\, v \neq 0$ in V, s.t. $\sigma(v) = v \otimes a$ for some $a \in k[G]$; a is then necessarily group-like, hence corresponds to a $\chi_a \in X(G)$).

• Prop. 5.2. An alg. group G is trigonalizable $\iff \forall$ f. dim. G-rep. (V, ρ), $\exists$ a k-basis of V with respect to which $\rho: G \to GL_n$ factors through $\mathbb{T}_n$ (with $n = \dim(V)$).

Pf: Similar to that of (4.3). $\square$.

• Prop. 5.3. An alg. group G is trigonalizable $\iff$ $\exists$ injective hom. $G \overset{\varphi}{\hookrightarrow} \mathbb{T}_n$, for some n.

Pf: ($\Rightarrow$). It follows from (5.2) and (I, 5.2).

($\Leftarrow$). Let U be the kernel of the composition ψ

$$\psi: G \overset{\varphi}{\hookrightarrow} \mathbb{T}_n \xrightarrow{\text{can.}} \mathbb{D}_n$$

$$\begin{pmatrix} a_{11} & & * \\ & \ddots & \\ & & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}.$$

Then we have exact sequences

$$\begin{array}{ccccccccc} 1 & \to & U & \longrightarrow & G & \longrightarrow & \Psi(G) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbb{U}_n & \longrightarrow & \mathbb{T}_n & \longrightarrow & \mathbb{D}_n & \longrightarrow & 1 \end{array},$$

in which U is unipotent (4.5) and $\Psi(G)$ is diagonalizable (2.4.2). Given a nonzero G-rep. (V, ρ), we have $V^U \neq 0$, on which the G-action factors through $G/U \simeq \Psi(G)$ (see p.34). By (2.7), V^U has a line stable by $\Psi(G)$, hence by G. $\square$.

- Exercise 5.3.1. Alg. subgr. and quot. of a trig. alg. gr. are trig.. //.

Before getting into solvable alg. groups, let us look at comm. alg. groups first.

- Prop. 5.4. A commutative alg. group G is trigonalizable.

Pf: Take a f. dim'l faithful rep. (by (I, 5.2)) of G:

$$G \hookrightarrow GL_V.$$

Then $G(k)$ is a commuting set of linear automorphisms (endom. is fine) of V, which can be simultaneously upper triangularized (a thm. of Frobenius) (an exercise in linear alg.: take a proper eigenspace W of one operator, and apply induction hypothesis to W and V/W ...), $G(k) \subset \mathbb{T}_n(k)$ w.r.t. some basis of V. Then $G \cap \mathbb{T}_n \subset G \Rightarrow G \cap \mathbb{T}_n = G \Rightarrow G \subset \mathbb{T}_n \Rightarrow G$ is trig., by (5.3). $\square$.

• **Cor. 5.5.** Let G be a comm. alg. group. Then $\exists!$ alg. subgroups U, D of G, s.t. U is unip. and D is diag., and that the mult. map

$$U \times_k D \longrightarrow G$$

is an isom. Furthermore, $U(k)$ (resp. $D(k)$) is the set of unip. (resp. semisimple) elements in $G(k)$.

Pf: Take an embedding $G \hookrightarrow \mathbb{T}_n$ (by (5.4, 5.3)), and take $U := G \cap \mathbb{U}_n$, $D := G \cap \mathbb{D}_n$; then U is unip. (4.5) and D is diag. (2.4.2). The mult. map

$$U \times_k D \longrightarrow G$$

is a hom. (since G is comm.) with kernel $U \cap D \subset \mathbb{U}_n \cap \mathbb{D}_n = \mathbb{1}$. It is surjective (by checking k-points) since $\mathbb{T}_n = \mathbb{U}_n \cdot \mathbb{D}_n$.

As elements in $G(k)$ are realized as upper-triangular matrices, it is clear that $U(k)$ (resp. $D(k)$) is the set of unip. (resp. s.s) elements in $G(k)$. Uniqueness follows: if U_1 and U_2 are two such groups, then

$$\left.\begin{array}{r} U_1 \cap U_2 \subset U_1 \\ (U_1 \cap U_2)(k) = U_1(k) \end{array}\right\} \Rightarrow \left.\begin{array}{r} U_1 \cap U_2 = U_1 \Rightarrow U_1 \subset U_2 \\ U_1(k) = U_2(k) \end{array}\right\}$$

$$\Rightarrow U_1 = U_2. \quad \square.$$

Let $\mathcal{D}G$ be G^{der}, the derived subgroup, and $\mathcal{D}^n G := \mathcal{D}(\mathcal{D}^{n-1}G)$.

• <u>Def. 5.6.</u> An alg. group G is <u>solvable</u>, if $\mathcal{D}^n G = \mathbb{1}$ for some $n \geqslant 1$.

• <u>Exercise 5.6.1.</u> An alg. group G is solvable $\Longleftrightarrow$ $\exists$ a filtration

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \mathbb{1}$$

s.t. each G_i/G_{i+1} is commutative. Consequently, alg. subgroups, quotient groups and extensions of solvable groups are solvable. //.

• <u>Examples</u>. Unipotent groups are solvable (4.6), so $\mathbb{T}_n$ is solvable (being an extension of $\mathbb{D}_n$ by $\mathbb{U}_n$), so trigonalizable groups are solvable (5.3). The finite alg. groups $(\Gamma)_k$ is solvable if Γ is, without any doubt.

• <u>Th. 5.7.</u> (Lie-Kolchin). ~~Let V be a f. dim'l faithful rep. of a~~ <u>Let</u> G <u>be a connected solvable alg. group</u>. <u>Then</u> G <u>is trigonalizable</u>.

<u>Pf</u>: Take an embedding $G \overset{\rho}{\hookrightarrow} GL_V$. It suffices to show that $G(k) \subset \mathbb{T}_n(k)$ w.r.t. some basis of V (using tricks $G \cap \mathbb{T}_n$ ... that we have used before, this implies that $G \subset \mathbb{T}_n$). Using induction on $\dim(V)$, it suffices to show that all g in $G(k)$ have a common eigenvector in V. We use induction on the length of the derived series of G. When G is comm., this is done (5.4). By induction hypothesis, for $\mathcal{D}G$, we may

assume that $\exists\, \chi \in X(\mathcal{D}G)$ s.t. the χ-eigenspace in V for the $\mathcal{D}G$-action is nonzero. (Note that $\mathcal{D}G \subset G$, being the increasing union of the images of

$$\prod_{2n} G \longrightarrow G$$
$$(x_1, y_1, \cdots, x_n, y_n) \longmapsto \prod_{i=1}^{n} [x_i ; y_i] ,$$

is again connected, so that induction hypothesis applies.).

The eigenspaces for different characters $\chi_i \in X(\mathcal{D}G)$ are linearly indep., and $\dim(V) < \infty$, so there are only finitely many such characters occurring in V; let

$$\Sigma = \{ \chi \in X(\mathcal{D}G) \mid \chi \subset \rho|_{\mathcal{D}G} \}$$

be this finite set, on which $G(k)$ acts: given $g \in G(k)$, $\chi \in X(\mathcal{D}G)$, let $\chi^g := \chi(g^{-1} \cdot \square \cdot g) \in X(\mathcal{D}G)$ (note that $\mathcal{D}G \lhd G$), and an easy calculation shows that $\rho(g)(V_\chi) = V_{\chi^g}$, so $\chi \in \Sigma \Rightarrow \chi^g \in \Sigma$.

$\forall \chi \in \Sigma$, $\chi = \chi^g \iff g \in Stab_{G(k)}(V_\chi) = Stab_G(V_\chi)(k)$, which is a subgroup of $G(k)$, closed w.r.t. the Zariski top. ~~indeed, if $e_1, \dots, e_r$ is a basis of V_χ, then $g \in Stab_{G(k)}(V_\chi) \iff$ $\rho(g)(e_i) \wedge e_1 \wedge \dots \wedge e_r = 0$ in $\Lambda^{r+1}(V)$, $\forall i = 1, \dots, r$ so that $Stab_{G(k)}(V_\chi)$ is the intersection of the inverse images of "0" via some morphisms $G(k) \to \Lambda^{r+1}(V)$.~~ As $Orb_{G(k)}(\chi)$ is finite, $[G(k) : Stab_{G(k)}(\chi)] < \infty$, $G(k)$ is connected, so $G(k) =$

$\mathrm{Stab}_{G(k)}(\chi)$. Therefore, $G(k)$ stabilizes each V_χ, $\chi \in \Sigma$.

$\forall\, x, y \in G(k)$, $\det(\rho([x,y]); V_\chi) = 1$, so $(\mathcal{D}G)(k)$ acts on each V_χ by operators of det. 1, i.e.

$$\chi_k(g)^{\dim(V_\chi)} = 1, \quad \forall\, g \in (\mathcal{D}G)(k),$$

so $\mathrm{Im}(\chi : \mathcal{D}G \to \mathbb{G}_m) \subset \mu_{\dim(V_\chi)}$ is finite. As $\mathcal{D}G$ is connected, χ must be trivial (so Σ is a singleton), so the action on V_χ by $G(k)$ factors through $G(k)/(\mathcal{D}G)(k)$, which is commutative, therefore, ~~on V_χ, operators in $G(k)$ can be simultaneously upper triangularized $\Rightarrow$~~ $\exists$ a common eigenvector in V_χ. $\square$.

• <u>Rem. 5.7.1.</u> The theorem extends to char. p (with the additional assumption that G be smooth), but does not extend to base fields k that are not alg. closed: $SO_{2,\mathbb{R}}$ is a torus (hence commutative) but is not trigonalizable: the tautological rep.

$$SO_{2,\mathbb{R}} \hookrightarrow GL_{2,\mathbb{R}}$$

is in fact irreducible, hence has no stable line in $\mathbb{R}^2$.

• <u>Cor. 5.8.</u> <u>Let</u> G <u>be a connected solvable alg. group. Then $\exists!$ unipotent normal alg. subgroup</u> $G_u \subset G$ <u>s.t.</u> G/G_u <u>is diagonalizable</u>.

<u>Pf.</u> By (5.7, 5.3), $\exists$ an embedding $G \hookrightarrow \mathbb{T}_n$. Let

G_u be $G \cap \mathbb{U}_n$, as in the proof of (5.3). For uniqueness, note that $G_u(k) = G(k) \cap \mathbb{U}_n(k)$ consists of precisely the unip. elements in $G(k)$, then use the trick in the proof of (5.5). $\square$.

§6. The radical of an alg. group.

Let H and N be alg. subgroups of an alg. group G, and assume one of them, say N, to be normal, so that $H \cdot N$ is still an alg. subgroup. As $H \cdot N/N \simeq H/H \cap N$, we see that

H and N are unip. (resp. solvable) $\Rightarrow$ so is $H \cdot N$,

by applying (4.2, 4.5) (resp. (5.6.1)) to the exact sequence

$$1 \longrightarrow N \longrightarrow H \cdot N \longrightarrow H/H \cap N \longrightarrow 1.$$

For the reason of dimensions, there is a unique maximal connected normal unipotent (resp. max. conn. normal solvable) alg. subgroup in G, namely the composite of all such groups. (The connectedness on unip. groups is indeed a requirement in char. p.).

• Def. 6.1. The unip. radical (resp. radical) of an alg. group G is the max. conn. normal unip. (resp. conn. normal solvable) alg. subgroup of G, denoted by $R_u(G)$ (resp. $R(G)$).

Clearly, $R_u(G) \subset R(G)$, and $R_u(G) = R(G)_u$ in the notation of (5.8). The quotient $R(G)/R_u(G)$, being connected, is a torus.

• Def. 6.2. A connected alg. group G is semisimple (resp. reductive) if $R(G) = \mathbb{1}$ (resp. $R_u(G) = \mathbb{1}$). (For non-alg. closed field k, we demand $R(G_{\bar{k}}) = \mathbb{1}$ (resp. $R_u(G_{\bar{k}}) = \mathbb{1}$).)

• Exercise 6.3. Let G be a conn. alg. group.

i). $G/R(G)$ is semisimple and $G/R_u(G)$ is reductive;

ii). G is semisimple $\Longleftrightarrow$ every conn. normal comm. subgroup is trivial;

iii). G is reductive $\Longleftrightarrow$ every conn. normal comm. subgroup is a torus. //.

• Exercise 6.4. Let G be an alg. group, (V, ρ) a semisimple G-rep., and N a normal alg. subgroup in G. Then $(V, \rho|_N)$ is a semisimple N-rep. //.

• Prop. 6.5. Let G be a conn. alg. group. If all f. dim'l G-rep. are semisimple, then G is reductive.

Pf: Let (V, ρ) be a f. dim'l faithful G-rep. (I, 5.2). By (6.4), $(V, \rho|_{R_u(G)})$ is semisimple. By def'n, the only irred. rep. of a unip. group is the trivial one, so $R_u(G)$ acts faithfully and trivially on $V \Rightarrow R_u(G) = \mathbb{1}$. $\square$.

• <u>Rem. 6.6.</u> An alg. group G is called <u>linearly reductive</u> if $\mathrm{Rep}(G)$ is a semisimple abelian category. So (6.5) says that G is conn. linearly red. $\Rightarrow$ G is red. The converse, under our assumption that $\mathrm{char.}(k) = 0$, is also true and will be proved later in the course. (see (III, 4.3)).

Course notes of « Linear Algebraic Groups ».

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Ch. III. Lie Algebras.

After reviewing some basic notions and facts for Lie algebras, we explain how they may arise from alg. groups, and then use Lie alg. to prove some results for alg. groups, such as G is linearly reductive $\Leftrightarrow$ G° is reductive. As always, we assume $k=\bar{k}$, char. $(k)=0$.

§1. Review.

We give a brief review for the basic theory of Lie algebras. No proof will be given (even though some are important / difficult theorems); they can be found in standard texts (e.g. Bourbaki, Humphreys, ...).

(1.1). A Lie algebra over k is a pair $(\mathfrak{g}, [-,-])$, where $\mathfrak{g}$ is a k-vector space, and $[\ ,\ ]$ is an alternating bilinear ~~form~~ product on $\mathfrak{g}$

$$[\ ,\ ]: \mathfrak{g}\wedge\mathfrak{g} \longrightarrow \mathfrak{g}$$

satisfying the "Jacobi identity"

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0,$$

$\forall\, x, y, z \in \mathfrak{g}$. A <u>homomorphism</u> of Lie algebras is a k-linear map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ s.t. $[\varphi(x), \varphi(y)]' = \varphi([x,y])$. When φ is injective, $\mathfrak{g}$ is also called a (<u>Lie</u>) <u>subalgebra</u> of $\mathfrak{g}'$. An <u>ideal</u> in $\mathfrak{g}$ is a subspace $\mathfrak{a} \subset \mathfrak{g}$ s.t.

$$[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}.$$

The <u>center</u> of $\mathfrak{g}$ is $z(\mathfrak{g}) := \{ x \in \mathfrak{g} \mid [x, \mathfrak{g}] = 0 \}$; it is an ideal.

A Lie alg. $\mathfrak{g}$ is <u>abelian</u> if $\mathfrak{g} = z(\mathfrak{g})$ (i.e. $[\mathfrak{g}, \mathfrak{g}] = 0$); <u>nilpotent</u> if the central series $C^n\mathfrak{g}$ $(n \geq 1)$ terminates (i.e. $C^n\mathfrak{g} = 0$ for some n), where $C^1\mathfrak{g} = \mathfrak{g}$ and $C^n\mathfrak{g} = [\mathfrak{g}, C^{n-1}\mathfrak{g}]$; <u>solvable</u> if the derived series $D^n\mathfrak{g}$ $(n \geq 1)$ terminates, where $D^1\mathfrak{g} = \mathfrak{g}$ and $D^n\mathfrak{g} = [D^{n-1}\mathfrak{g}, D^{n-1}\mathfrak{g}]$. The Lie alg. $\mathfrak{g}$ is nilpotent (resp. solvable) $\Leftrightarrow \exists$ filtration by ideals

$$\mathfrak{g} = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_n = 0$$

s.t. $[\mathfrak{g}, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$ (resp. $[\mathfrak{a}_i, \mathfrak{a}_i] \subset \mathfrak{a}_{i+1}$).

The <u>radical</u> $r(\mathfrak{g})$ is the (unique) maximal solvable ideal in $\mathfrak{g}$ (fact: $\mathfrak{a}_1, \mathfrak{a}_2$ solvable ideals $\Rightarrow$ so is $\mathfrak{a}_1 + \mathfrak{a}_2$). The Lie alg. $\mathfrak{g}$ is <u>semisimple</u> if $\underbrace{r(\mathfrak{g}) = 0}_{\dim_k(\mathfrak{g}) < \infty \text{ and}}$ ($\Leftrightarrow \nexists$

nontrivial abelian ideal in $\mathfrak{g}$); <u>simple</u> if $\mathfrak{g}$ is not abelian and $\nexists$ ideal $\mathfrak{a} \subset \mathfrak{g}$ other than 0 and $\mathfrak{g}$. As a consequence of Cartan's criterion (to be mentioned later), a semisimple Lie alg. $\mathfrak{g}$ is the direct product (as Lie alg.) of its nonzero minimal ideals (there are only finitely many)

$$\mathfrak{g} = \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_n,$$

and each $\mathfrak{a}_i$ is a simple Lie algebra; any ideal in $\mathfrak{g}$ takes the form $\prod_{i\in I} \mathfrak{a}_i$ for some $I \subset [1, \cdots, n]$. Conversely, a finite product of (f. dim'l) simple Lie alg. is semisimple.

Subalgebras and quotients (by ideals) of a nilpotent (resp. solvable) Lie alg. are nilpotent (resp. solvable). If $\mathfrak{a} \subset \mathfrak{g}$ is a solvable ideal s.t. $\mathfrak{g}/\mathfrak{a}$ is solvable, then $\mathfrak{g}$ is solvable. Ideals and quotients of a semisimple Lie alg. are semisimple (being a finite product of simple algebras). If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (for it is so for $\mathfrak{g}$ simple).

From now on, we assume that all Lie alg. $\mathfrak{g}$ and all k-vectors spaces V are finite-dim'l.

<u>(1.2)</u>. Let V be a k-vector space. Define $\mathfrak{gl}_V$ to be the Lie alg. $\mathrm{End}_k(V)$ with $[\varphi, \psi] = \varphi \circ \psi - \psi \circ \varphi$. After

choosing a basis of V, it is also **denoted by** $\mathfrak{gl}_n$. A hom. $\rho: \mathfrak{g} \to \mathfrak{gl}_V$ of Lie alg. is called a (<u>linear</u>) <u>**representation**</u> of $\mathfrak{g}$; define $V^{\mathfrak{g}} := \{v \in V \mid \rho(\mathfrak{g}) \cdot v = 0\}$.

The subspaces

$$\mathfrak{b}_n := \begin{pmatrix} * & * & \cdots & * \\ & \ddots & \ddots & * \\ & & \ddots & \vdots \\ & & & * \end{pmatrix}, \quad \mathfrak{n}_n = \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix}$$

of $\mathfrak{gl}_n$ are Lie subalgebras. More intrinsically, let F be a full flag in V:

$$V = F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_n = 0 \quad \text{(subspaces)},$$

and define

$$\mathfrak{b}(F) := \{\varphi \in \mathfrak{gl}_V \mid \varphi(F_i) \subset F_i, \ \forall i\}$$

$$\mathfrak{n}(F) := \{\varphi \in \mathfrak{gl}_V \mid \varphi(F_i) \subset F_{i+1}, \ \forall i\}.$$

$\mathfrak{b}(F)$ is solvable and $\mathfrak{n}(F)$ is nilpotent.

Let (V, ρ) be a rep. of $\mathfrak{g}$, and $W \subset V$ a subspace. The <u>stabilizer</u> and <u>centralizer</u> of W in $\mathfrak{g}$ are

$$\mathrm{stab}_{\mathfrak{g}}(W) := \{x \in \mathfrak{g} \mid \rho(x)(W) \subset W\}$$

$$\mathrm{cent}_{\mathfrak{g}}(W) := \{x \in \mathfrak{g} \mid \rho(x)(W) = 0\}$$

are Lie subalgebras of $\mathfrak{g}$. For $v \in V$, one also has its

<u>isotropy</u> <u>subalgebra</u> $\mathrm{cent}_{\mathfrak{g}}(v) := \mathrm{cent}_{\mathfrak{g}}(\langle v \rangle)$.

Write $\mathrm{ad}(x)(y) := [x, y]$. Then

For a subalg. $\mathfrak{h} \subset \mathfrak{g}$, define its <u>normalizer</u> and <u>centralizer</u> to be $n_{\mathfrak{g}}(\mathfrak{h}) := \mathrm{stab}_{\mathrm{ad}}(\mathfrak{h})$, $c_{\mathfrak{g}}(\mathfrak{h}) := \mathrm{cent}_{\mathrm{ad}}(\mathfrak{h})$.

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}_{\mathfrak{g}}$$

is a hom. (by the Jacobi identity), called the <u>adjoint</u> <u>rep.</u> of $\mathfrak{g}$. We have $\mathfrak{z}(\mathfrak{g}) = \ker(\mathrm{ad})$. $\forall x \in \mathfrak{g}$, $\mathrm{ad}(x)$ is a <u>derivation</u> on $\mathfrak{g}$, i.e. a k-linear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ s.t.

$$\varphi([y, z]) = [\varphi(y), z] + [y, \varphi(z)].$$

All derivations on $\mathfrak{g}$ form a Lie subalgebra $\mathrm{Der}(\mathfrak{g})$ of $\mathfrak{gl}_{\mathfrak{g}}$. When $\mathfrak{g}$ is **semisimple**, ad gives an isom.

$$\mathrm{ad} : \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}(\mathfrak{g}),$$

namely, "all derivations **are** inner".

The symmetric bilinear form on $\mathfrak{g}$

$$K_{\mathfrak{g}}(x, y) := \mathrm{Tr}\,(\mathrm{ad}(x) \cdot \mathrm{ad}(y))$$

is called the <u>Killing form</u> of $\mathfrak{g}$. It is an <u>invariant</u> form, i.e. a form $K(x, y)$ s.t.

$$K([x, y], z) + K(y, [x, z]) = 0.$$

Consequently, if $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, so is

$$\mathfrak{a}^{\perp} := \{ x \in \mathfrak{g} \mid K_{\mathfrak{g}}(x, \mathfrak{a}) = 0 \}.$$

(1.3). Some theorems:

(1.3.1). (Ado). $\forall$ Lie alg. $\mathfrak{g}$ (assumed f. dim'l), $\exists$ (f. dim'l) faithful rep. (V, ρ) of $\mathfrak{g}$, s.t. $\forall$ nilp. ideal $\mathfrak{n} \subset \mathfrak{g}$, $\rho(\mathfrak{n})$ consists of nilp. operators.

(1.3.2). (Engel). Let $\rho: \mathfrak{g} \to \mathfrak{gl}_V$ be a rep. of $\mathfrak{g}$, s.t. $\rho(\mathfrak{g})$ consists of nilpotent operators, then $\exists v \in V$, $v \neq 0$, s.t. $\rho(\mathfrak{g}) \cdot v = 0$; consequently, $\exists$ a full flag F in V s.t. $\rho(\mathfrak{g}) \subset \mathfrak{n}(F)$.

A Lie alg. $\mathfrak{g}$ is nilp. $\Leftrightarrow$ $\mathrm{ad}(x)$ is nilp., $\forall x \in \mathfrak{g}$. (This is the Lie-alg. analogue of (II, 4.8).)

(1.3.3). (Lie). Let $\rho: \mathfrak{g} \to \mathfrak{gl}_V$ be a rep. of $\mathfrak{g}$. If $\mathfrak{g}$ is solvable, then $\exists$ a full flag F in V s.t. $\rho(\mathfrak{g}) \subset \mathfrak{b}(F)$. (This is the Lie-alg. analogue of (II, 5.7).)

By taking ρ to be faithful, we see that $\mathfrak{g}$ solvable $\Rightarrow [\mathfrak{g}, \mathfrak{g}]$ nilpotent, because

$$\rho([\mathfrak{g}, \mathfrak{g}]) \subset [\mathfrak{b}(F), \mathfrak{b}(F)] \subset \mathfrak{n}(F).$$

In contrast with (II, 5.7), this works only for char. $(k) = 0$. (But fine for char$(k) = p$ as long as $p > \dim(V)$.)

(1.3.4). (Cartan). Let $\mathfrak{g} \subset \mathfrak{gl}_V$ be a subalgebra.

If $\mathrm{Tr}(x\circ y, V) = 0$, $\forall x, y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable (converse is clearly false : think of $\mathfrak{b}_n$). Consequently, if $\mathrm{Tr}(x\circ y, V) = 0$, $\forall x, y \in D^n\mathfrak{g}$, then $D^n\mathfrak{g}$, and hence $\mathfrak{g}$, is solvable. Conversely, if $\mathfrak{g}$ is solvable, then $\mathrm{Tr}(x\circ y, V) = 0$, $\forall x \in \mathfrak{g}$, $y \in [\mathfrak{g}, \mathfrak{g}]$.

If $\kappa_{\mathfrak{g}}(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$, then $\mathfrak{g}$ is solvable.

(1.3.5). (Cartan-Killing). A Lie alg. $\mathfrak{g}$ is semisimple $\Longleftrightarrow$ the Killing form $\kappa_{\mathfrak{g}}$ is nondegenerate. That "$\mathfrak{g}$ semisimple $\Rightarrow \mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$" is a consequence.

(1.3.6). (Weyl). **A Lie alg. $\mathfrak{g}$ is semisimple $\Longleftrightarrow$ each f. dim'l rep. of $\mathfrak{g}$ is semisimple (namely, the abelian category $\mathrm{Rep}(\mathfrak{g})$ is semisimple).**

(1.3.7). (additive Jordan decomposition). Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\forall x \in \mathfrak{g}$, $\exists!\ x_s, x_n \in \mathfrak{g}$ s.t. $x = x_s + x_n$, that $[x_s, x_n] = 0$, and that $\forall$ rep. (V, ρ) of $\mathfrak{g}$, $\rho(x_s)$ is a semisimple operator and $\rho(x_n)$ is a nilpotent operator. This is a consequence of (1.3.6). The semisimplicity assumption is necessary : $\exists$ Lie subalg. $\mathfrak{g} \subset \mathfrak{gl}_V$, $x \in \mathfrak{g}$, but $x_s, x_n \notin \mathfrak{g}$.

<u>(1.4)</u>. <u>About root systems</u>. Let V be a f. dim'l v. s./ a field F of char. 0, $\alpha \in V$, $\alpha \neq 0$. A <u>symmetry</u> with α is an F-linear auto. s_α of V s.t. $s_\alpha(\alpha) = -\alpha$ and $\ker(id_V - s_\alpha)$ has codim. 1 in V. Such symm. with α are in 1-1 correspondence with $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$:

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle . \alpha , \quad \ker(id_V - s_\alpha) = \ker(\alpha^\vee).$$

A subset $R \subset V$ is called a (reduced) <u>root system</u> in V if

- $0 \notin R$ is a finite set spanning V;
- $\forall \alpha \in R$, $\exists(!)$ symm. s_α with α s.t. $s_\alpha(R) \subset R$;
- $\forall \alpha, \beta \in R$, $\exists n \in \mathbb{Z}$ s.t. $s_\alpha(\beta) = \beta + n\alpha$;
- $\alpha, c\alpha \in R$, $c \in F \Rightarrow c = \pm 1$.

Elements in R are called <u>roots</u>, and $\dim(V)$ is called the <u>rank</u> of R. Every root system (V, R) can be written in the form

$$(V, R) = (\bigoplus V_i , \coprod R_i)$$

for some root systems (V_i, R_i) that are no longer decomposable.

The linear relations among the roots are all defined over $\mathbb{Q}$: if we put $V_0 := \mathbb{Q}\langle R \rangle \subset V$, then R is a root system in V_0, and $F \otimes_{\mathbb{Q}} V_0 \xrightarrow{\sim} V$. Hence one may as well assume that $F = \mathbb{R}$ and talk about inner products etc.

The <u>Weyl group</u> W of (V, R) is the subgroup of $GL(V)$ generated by s_α $(\alpha \in R)$; it acts simply transitively on the <u>Weyl chambers</u> $\pi_0(V \setminus \bigcup_{\alpha \in R} \ker(\mathrm{id}_V - s_\alpha))$. By averaging, one sees that $\exists$ W-invariant inner product $(-,-)$ on $V/\mathbb{R}$. The ratio, $\forall \alpha \in R$,

$$2 \cdot \frac{(-, \alpha)}{(\alpha, \alpha)},$$

is indep. of the choice of $(-,-)$, and equals $\langle -, \alpha^\vee \rangle$.

$\forall \alpha \in R$, let $\alpha^\vee \in V^\vee$ be the linear form corresp. to the unique s_α s.t. $s_\alpha(R) \subset R$; so $\langle \alpha, \alpha^\vee \rangle = 2$ and $\langle R, \alpha^\vee \rangle \subset \mathbb{Z}$. Then $R^\vee := \{\alpha^\vee \mid \alpha \in R\}$ is a root system in $V^\vee$, the <u>dual root system</u>.

For a root system (V, R), its <u>root lattice</u> $Q(R)$ is the subgroup $\mathbb{Z}\langle R \rangle$ of V generated by the roots, and its <u>weight lattice</u> $P(R)$ is the dual lattice of $Q(R^\vee)$:

$$P(R) := \{x \in V \mid \langle x, \alpha^\vee \rangle \in \mathbb{Z}, \forall \alpha \in R\}.$$

We have $Q(R) \subset P(R)$, and $[P(R) : Q(R)] < \infty$ (since they both span V). **Define** the <u>fundamental group</u> of the root system to be $\pi_1(R) := P(R)/Q(R)$.

One may also introduce a *base* $S \subset R$, the *Cartan matrix* $(\langle \alpha, \beta^\vee \rangle)_{\alpha, \beta \in S}$, and *Dynkin diagram* of the root system (V, R). The connected Dynkin diagrams correspond to indecomp. root systems, and are classified into the (well-known) "A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2" types. Let us not get into these notions.

(1.5). The root system associated to a semisimple Lie algebra.

Let $\mathfrak{g}$ be a s.s. Lie alg. over k. A *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a nilp. subalg. $\mathfrak{h} \subset \mathfrak{g}$ that is *self-centralizing* (or, *self-normalizing*): $x \in \mathfrak{g}$, $[x, \mathfrak{h}] = 0 \Rightarrow x \in \mathfrak{h}$. (In this case, it is also the max. *toral subalg.* of $\mathfrak{g}$, hence abelian.) The operators $\mathrm{ad}_{\mathfrak{g}}(x)$, $x \in \mathfrak{h}$, form a commuting family of s.s. operators on $\mathfrak{g}$, which can then be diagonalized simultaneously.

Let $R \subset \mathfrak{h}^\vee$ be the set of $\alpha \neq 0$ s.t.

$$\mathfrak{g}_\alpha := \{ x \in \mathfrak{g} \mid [y, x] = \alpha(y) \cdot x, \ \forall y \in \mathfrak{h} \}$$

is nonzero. The eigenspace $\mathfrak{g}_0$ of $0 \in \mathfrak{h}^\vee$ is by def'n $\mathfrak{h}$ itself. We have a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha .$$

The subset R is a root system in $\mathfrak{h}^\vee$ over k, called the <u>root system associated to the pair</u> $(\mathfrak{g}, \mathfrak{h})$. Any two such pairs $(\mathfrak{g}, \mathfrak{h}_1)$ and $(\mathfrak{g}, \mathfrak{h}_2)$, with $\mathfrak{h}_1, \mathfrak{h}_2$ two Cartan subalgebras of $\mathfrak{g}$, are isomorphic, so the root system R is well-def'd up to isom.

A decomposition of s.s. Lie algebras with Cartan subalg.

$$(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{g}_1, \mathfrak{h}_1) \oplus (\mathfrak{g}_2, \mathfrak{h}_2)$$

leads to a decomp. of the associated root system, hence $\mathfrak{g}$ is simple $\Leftrightarrow$ R is indecomposable ($\Leftrightarrow$ Dynkin is connected).

Every root system over k arises from a unique (up to isom.) s.s. Lie algebra over k.

§2. Relations with alg. groups.

$k[\varepsilon_1] := k[T]/(T^2)$, the <u>ring of dual numbers</u>.

$\forall$ k-alg. R, $R[\varepsilon_1] := R \otimes_k k[\varepsilon_1] = R[T]/(T^2)$.

We have a (the unique) k-alg. hom. $\pi : k[\varepsilon_1] \to k$.
$$a + b\varepsilon_1 \mapsto a$$

- <u>Def. 2.1</u>. Let G be an alg. group over k. Define $\mathrm{Lie}(G)$

to be the group

$$\mathrm{Lie}(G) := \ker\left(G(k[\varepsilon_1]) \xrightarrow{G(\pi)} G(k)\right).$$

Let $m = \ker(\varepsilon : k[G] \to k)$. Then we have

- <u>Lem. 2.2.</u> <u>There is a natural isom. of groups</u>

$$\mathrm{Lie}(G) \simeq (m/m^2)^\vee = \mathrm{Hom}_k(m/m^2, k),$$

<u>from which</u> $\mathrm{Lie}(G)$ <u>gets the structure of a</u> k-<u>vector space</u>.

Pf: Clear from the diagram

$$\begin{tikzcd}
 & m/m^2 \arrow[r, dashed, "\bar{t}"] & k\cdot\varepsilon_1 \arrow[d] \\
m \arrow[ur] \arrow[urr, "t"'] \arrow[d] & & k[\varepsilon_1] \arrow[d, "\pi"] \\
k[G] \arrow[urr, "t"] \arrow[d, "\varepsilon"] & & k \\
k \arrow[urr, equal] & &
\end{tikzcd}$$

;

[Hart.], II, Ex. 2.8 merely gives this as a bijection.

$k\oplus m \longrightarrow k[\varepsilon_1]$
$a = \varepsilon(a) + (a-\varepsilon(a)) \mapsto \varepsilon(a) + t\cdot(a-\varepsilon(a))\cdot\varepsilon_1$
$b = \cdots \mapsto \varepsilon(b) + t(b-\varepsilon(b))\cdot\varepsilon_1$,
use $(a-\varepsilon(a))(b-\varepsilon(b)) \in m^2$ to show that it preserves mult.

only need to check that group structures correspond: since

$$\begin{array}{ccccccc}
k[G] \xrightarrow{\Delta} & k[G]\otimes k[G] & \xrightarrow{t_1\otimes t_2} & k[\varepsilon_1]\otimes k[\varepsilon_1] & \xrightarrow{m} & k[\varepsilon_1] \\
 & m\otimes m & \longmapsto & & & 0
\end{array},$$

this follows from that for $x\in m$, we have

$$\Delta x \equiv x\otimes 1 + 1\otimes x \pmod{m\otimes m},$$

a fact that we have proved in (I, 3.5.1). □

So $\mathrm{Lie}(G)$ is the Zariski tangent space $T_e(G)$.

- <u>Cor. 2.3.</u> <u>We have</u> $\dim_k \mathrm{Lie}(G) = \dim(G)$.

Pf: This is because G is smooth (I, 3.5). □

For a (comm.) k-algebra A, we have the A-module $\Omega_{A/k}$ of <u>Kähler differentials</u>, s.t. $\forall$ A-module M,

$$\mathrm{Hom}_A(\Omega_{A/k}, M) \simeq \mathrm{Der}_k(A, M).$$

Now for $A = k[G]$ and $M = k$ (viewed as an A-alg. via $\varepsilon: A \to k$), $\mathrm{Hom}_A(\Omega_{k[G]/k}, k)$ is identified with the tangent space $(m/m^2)^\vee$ (see [Hart.], II, 8.17), so we get an isom.

$$\mathcal{L}ie(G) \simeq \mathrm{Der}_k(k[G], k),$$

given by $(t: k[G] \to k[\varepsilon_1]) \mapsto (a \mapsto \underset{\text{(with "}\varepsilon_1\text{" dropped)}}{t(a - \varepsilon(a))} : k[G] \to k)$,

~~and~~ (one verifies immediately that it is a derivation:

$$t(ab - \varepsilon(ab)) = \underset{\substack{\uparrow \\ \text{this is how } A \text{ acts} \\ \text{on } k}}{\varepsilon(a)} \cdot t(b - \varepsilon(b)) + \varepsilon(b) \cdot t(a - \varepsilon(a)).$$), with

inverse $\left(\partial_e \in \mathrm{Der}_k(k[G], k)\right) \mapsto \left(a \mapsto \varepsilon(a) + \partial_e(a - \varepsilon(a)) \cdot \varepsilon_1 : k[G] \to k[\varepsilon_1]\right)$.

As $\varepsilon: A \to k$ is by def'n A-linear, we have

$$\partial \mapsto \varepsilon \circ \partial : \mathrm{Der}_k(A, A) \longrightarrow \mathrm{Der}_k(A, k)$$

taking a "vector field" ∂ on G to its "restriction ∂_e at the identity $e \in G$". Using the group structure on G, one can define a splitting, taking ∂_e to the unique "left-invariant" vector field $\{\underset{\substack{\uparrow \\ \text{derivative of left translation by } g}}{d(l_g)_e}(\partial_e)\}_{g \in G}$. Formally, given a k-derivation

$\partial_e : A \to k$, define $\partial : A \to A$ to be

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{id \otimes \partial_e} A \otimes k \simeq A,$$

which is a k-derivation (Δ pres. mult. and $id \otimes \partial_e$ satisfies the Leibniz rule), and $\varepsilon \circ \partial = \partial_e$. So $\partial_e \mapsto \partial$ identifies $\mathrm{Der}_k(A, k)$ with a subspace of $\mathrm{Der}_k(A, A)$, the <u>left-invariant derivations</u> $A \to A$.

- <u>Lem. 2.4</u>. <u>This subspace is a Lie subalgebra of</u> $\mathrm{Der}_k(A, A) \subset \mathfrak{gl}_A$.

<u>Pf</u>: Let $\partial = (id_A \otimes \partial_e) \circ \Delta$ and $\partial' = (id_A \otimes \partial'_e) \circ \Delta$ be two left inv. derivations. Then

$$[\partial, \partial'] = \left[(id_A \otimes \partial_e) \circ \Delta \circ (id_A \otimes \partial'_e) - (id_A \otimes \partial'_e) \circ \Delta \circ (id_A \otimes \partial_e)\right] \circ \Delta$$

$$= \left[\partial \circ (id_A \otimes \partial'_e) - \partial' \circ (id_A \otimes \partial_e)\right] \circ \Delta,$$

and

$$[\partial, \partial']_e = \varepsilon \circ [\partial, \partial'] = \left[\partial_e \circ (id_A \otimes \partial'_e) - \partial'_e \circ (id_A \otimes \partial_e)\right] \circ \Delta$$

$$= (\partial_e \otimes \partial'_e - \partial'_e \otimes \partial_e) \circ \Delta.$$

To show $[\partial, \partial'] = (id_A \otimes [\partial, \partial']_e) \circ \Delta$, it suffices to show half of it: $\partial \circ (id_A \otimes \partial'_e) \circ \Delta = \left[id_A \otimes \left((\partial_e \otimes \partial'_e) \circ \Delta\right)\right] \circ \Delta$, in which the right-hand side equals $(\partial \otimes \partial'_e) \circ \Delta$:

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A\otimes A \arrow[d, "id_A\otimes\Delta"] & \\
A\otimes A \arrow[r, "\Delta\otimes id_A"] \arrow[rr, bend right, "\underbrace{((id_A\otimes\partial_e)\circ\Delta)}_{\partial}\otimes\partial'_e"'] & A\otimes A\otimes A \arrow[r, "id_A\otimes\partial_e\otimes\partial'_e"] & A\otimes k\otimes k
\end{tikzcd}.$$

Clearly, $\partial\circ(id_A\otimes\partial'_e) = \partial\otimes\partial'_e$:

$$\begin{tikzcd}
A\otimes A \arrow[r, "id_A\otimes\partial'_e"] \arrow[rr, bend right, "\partial\otimes\partial'_e"'] & A\otimes k\simeq A \arrow[r, "\partial"] & A
\end{tikzcd}. \quad \square.$$

In this way, we put a Lie alg. structure on $\mathcal{L}ie(G)\simeq \mathrm{Der}_k(A,k)\hookrightarrow \mathrm{Der}_k(A,A)$; it is called the <u>Lie algebra</u> of G.

- <u>Lem. 2.5.</u> (Functoriality). <u>Let</u> $f: G\to G'$ <u>be a hom. of alg. group. Then the induced linear map</u>

$$d(f)_e: \mathcal{L}ie(G)\longrightarrow \mathcal{L}ie(G')$$

<u>is a **hom.** of Lie algebras.</u>

<u>Pf</u>:

$$\begin{tikzcd}
A' \arrow[r, "f^\#"] \arrow[rr, bend left, "\partial'_e"] \arrow[d, "\Delta'"] \arrow[dd, bend right=60, "\partial'"'] & A \arrow[r, "\partial_e"] \arrow[d, "\Delta"] \arrow[dd, bend left=60, "\partial"] & k \\
A'\otimes A' \arrow[r, "f^\#\otimes f^\#"] \arrow[d, "id_{A'}\otimes\partial'_e"] & A\otimes A \arrow[d, "id_A\otimes\partial_e"] & \\
A'\otimes k \arrow[r, "f^\#\otimes id_k"'] & A\otimes k &
\end{tikzcd}$$

$d(f)_e : \partial_e \mapsto \partial'_e$

so $\partial\circ f^\# = f^\#\circ\partial'$. If ∂_1, ∂_2 are both left inv. on A, giving rise to ∂'_1, ∂'_2 on A', then

$$\begin{tikzcd}
A' \arrow[r, "\partial'_1"] \arrow[d, "f^\#"'] & A' \arrow[r, "\partial'_2"] \arrow[d, "f^\#"] & A' \arrow[r, "\varepsilon'"] \arrow[d, "f^\#"] & k \\
A \arrow[r, "\partial_1"] & A \arrow[r, "\partial_2"] & A \arrow[ur, "\varepsilon"'] &
\end{tikzcd},$$

i.e. $(\partial_2 \circ \partial_1) \circ f^{\#} = f^{\#} \circ (\partial_2' \circ \partial_1')$, so $[\partial_1', \partial_2']_e = d(f)_e([\partial_1, \partial_2]_e)$. $\square$

Therefore, we have a functor

$$G \mapsto \mathrm{Lie}(G) : \{\text{alg. groups}/k\} \longrightarrow \{\text{Lie alg.}/k\}.$$

A Lie alg. $\mathfrak{g}$ is <u>algebraic</u> if $\mathfrak{g} \cong \mathrm{Lie}(G)$ for some G. All nilpotent Lie alg. are algebraic (as we mentioned in (II, 4.9)), and all semisimple Lie alg. are algebraic (as we will prove later). There are non-algebraic solvable Lie alg., e.g. $\mathfrak{g} = \bigoplus_{i=1}^{5} k \cdot x_i$ with $\mathfrak{z}(\mathfrak{g}) = k \cdot x_5$, and $[x_1, x_2] = x_5$, $[x_1, x_3] = x_3$, $[x_2, x_4] = x_4$, $[x_1, x_4] = [x_2, x_3] = [x_3, x_4] = 0$ (this example is taken from Bourbaki, Groupes et algèbres de Lie, I, §5, Ex. 6). This shows the difference between Lie groups and alg. groups.

(Ado + Lie $\Rightarrow$ any $\mathbb{R}$-Lie alg. (f. dim.) is the Lie alg. of a Lie group.).

• <u>Examples 2.6.</u>

<u>(2.6.1)</u>. $G = GL_n$. Then $\ker(GL_n(k[\varepsilon_1]) \longrightarrow GL_n(k))$

$= \{ I_n + M \cdot \varepsilon_1 \, ; \, M \in M_n(k) \} \simeq M_n(k)$, because

$$\det(I_n + M \cdot \varepsilon_1) \equiv 1 \mod (\varepsilon_1) \Rightarrow \det \in k[\varepsilon_1]^{\times}.$$

$\forall A \in M_n(k)$, $I_n + A \cdot \varepsilon_1$ is a k-alg. hom. (again denoted A) $k[G] \to k[\varepsilon_1]$ given by $T_{ij} \mapsto \delta_{ij} + a_{ij} \cdot \varepsilon_1$, which gives a

k-derivation

$$\partial^A_{I_n} : k[G] \longrightarrow k$$
$$f \longmapsto A(f - f(I_n)) \quad (\text{without } \varepsilon_1).$$

The corresponding left-inv. derivation $\partial^A = (id_{k[G]} \otimes \partial^A_{I_n}) \circ \Delta$, satisfying the Leibniz rule, is determined by its values on the generators T_{ij}. We have

$$\partial^A(T_{ij}) = (id \otimes \partial^A_{I_n})\left(\sum_{\ell=1}^n T_{i\ell} \otimes T_{\ell j}\right) = \sum_{\ell=1}^n T_{i\ell} \otimes A(T_{\ell j} - \delta_{\ell j})$$

$$= \sum_{\ell=1}^n T_{i\ell} \cdot a_{\ell j}.$$

Given another $B = (b_{ij}) \in M_n(k)$, we have

$$\partial^B \circ \partial^A(T_{ij}) = \sum_{\ell=1}^n a_{\ell j} \cdot \partial^B(T_{i\ell}) = \sum_{\ell=1}^n a_{\ell j} \sum_{m=1}^n b_{m\ell} \cdot T_{im},$$

and so

$$[\partial^A, \partial^B](T_{ij}) = \partial^A \circ \partial^B(T_{ij}) - \partial^B \circ \partial^A(T_{ij}) = \sum_{\ell, m=1}^n T_{im} \cdot$$

$$(a_{m\ell} \cdot b_{\ell j} - b_{m\ell} \cdot a_{\ell j}) = \sum_{m=1}^n T_{im} \cdot \underbrace{\sum_{\ell=1}^n (a_{m\ell} \cdot b_{\ell j} - b_{m\ell} \cdot a_{\ell j})}_{(m,j)\text{-entry of } AB-BA}$$

$$= \partial^{AB-BA}(T_{ij}),$$

hence $[\partial^A, \partial^B] = \partial^{AB-BA}$ (since they are both derivations). This shows that $\mathrm{Lie}(GL_n) \simeq \mathfrak{gl}_n$ as Lie algebras.

<u>(2.6.2).</u> $G = SL_n$. We have $\mathrm{Lie}(SL_n) = \mathrm{Lie}(GL_n) \cap SL_n(k[\varepsilon_1])$. We have $\det(I_n + M \cdot \varepsilon_1) = 1 + tr(M) \cdot \varepsilon_1$, so $\mathfrak{sl}_n := \mathrm{Lie}(SL_n) = \{M \in M_n(k) \mid tr(M) = 0\}$, with $[A, B] = AB - BA$ by (2.5).

(2.6.3). $G = O_n$, or $O_n^\circ (= SO_n)$. Again, $\mathrm{Lie}(O_n) = \{M \in M_n(k) \mid (I_n + M^T\cdot\varepsilon_1)(I_n + M\cdot\varepsilon_1) = I_n\} = \{M \mid M^T = -M\} =: \mathfrak{so}_n$.

(2.6.4). Similarly,

$$\mathrm{Lie}(\mathbb{U}_n) = I_n + \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix}\cdot\varepsilon_1 \simeq \begin{pmatrix} 0 & & * \\ & \ddots & \\ & & 0 \end{pmatrix} = \mathfrak{n}_n, \quad \text{(nilpotent)}$$

$$\mathrm{Lie}(\mathbb{D}_n) = I_n + \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}\cdot\varepsilon_1 \simeq \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} =: \mathfrak{d}_n, \quad \text{(abelian)}$$

$$\mathrm{Lie}(\mathbb{T}_n) = I_n + \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}\cdot\varepsilon_1 \simeq \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix} = \mathfrak{b}_n, \quad \text{(solvable)}$$

all of which are Lie subalg. of $\mathfrak{gl}_n$.

$\mathfrak{d}_n \cap \mathfrak{sl}_n$ is a Cartan subalgebra of $\mathfrak{sl}_n$, and $\mathfrak{b}_n \cap \mathfrak{sl}_n$ is a <u>Borel subalgebra</u> of $\mathfrak{sl}_n$.

(2.6.5). $G = D(M)$, M a f.g. abelian group. We have by (II, 2.1)

$$\begin{aligned}\mathrm{Lie}(G) &= \ker(\mathrm{Hom}(M, k[\varepsilon_1]^\times) \longrightarrow \mathrm{Hom}(M, k^\times)) \\ &\simeq \mathrm{Hom}(M, 1 + k\varepsilon_1) = \mathrm{Hom}(M, k) \\ &= \mathrm{Hom}(X(G), k) = \mathrm{Hom}(X(G), \mathbb{Z}) \otimes k,\end{aligned}$$

an abelian Lie algebra.

(2.7). Let G be an alg. group, $\mathfrak{g} = \mathrm{Lie}(G)$. Define a linear rep. of G on $\mathfrak{g}$ as follows: $\forall g \in G(R)$, $x \mapsto gxg^{-1} : G_R \to G_R$ is an auto. of the R-group scheme G_R, inducing an auto. of the R-Hopf algebra $k[G]_R$, sending m_R onto m_R, and therefore induces an R-linear auto. of $(m/m^2)^\vee \otimes R \simeq \mathrm{Hom}_R(m_R/m_R^2, R)$. This assignment preserves multiplication

$$\mathrm{Ad}_R : G(R) \longrightarrow GL(\mathfrak{g}_R),$$

and is functorial in R (exercise), hence we have defined a rep.

$$\mathrm{Ad} : G \longrightarrow GL_{\mathfrak{g}},$$

called the <u>adjoint representation</u> of G.

Here is another way defining Ad. $\forall$ k-alg. R, one can show (in the same way as (2.2)) that

$$\mathcal{L}ie(G_R) := \ker\left(G(R[\varepsilon_1]) \xrightarrow{G(\pi_R)} G(R)\right) \simeq \mathrm{Hom}_k(m/m^2, R) \simeq (m/m^2)^\vee \otimes R,$$

which provides the group $\mathcal{L}ie(G_R)$ with an R-module structure.

Given $g \in G(R)$, $x \in \mathcal{L}ie(G_R)$, consider $g \cdot x \cdot g^{-1} \in G(R[\varepsilon_1])$. As $G(R) \to G(R[\varepsilon_1])$ gives a splitting of $G(\pi_R)$, we have

$$G(\pi_R)\left(g \cdot x \cdot g^{-1}\right) = g \cdot G(\pi_R)(x) \cdot g^{-1} = 1,$$

so that $g \cdot x \cdot g^{-1} \in \mathcal{L}ie(G_R)$.

- <u>Exercise</u>: Show that $x \mapsto g \cdot x \cdot g^{-1}$ on $\mathcal{L}ie(G_R)$ corresponds, via $\mathcal{L}ie(G_R) \simeq (m/m^2)^\vee \otimes R$, to the action $\mathrm{Ad}_R(g)$. //

- <u>Prop. 2.7.1.</u> <u>The derivative of Ad at the identity is</u> ad.

<u>Pf</u>: Take an embedding $G \hookrightarrow GL_n$, which gives $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$. Then G acts on $\mathfrak{gl}_n$ by $\mathrm{Ad}_{GL_n}|_G$ and $\mathfrak{g}$ is a G-stable subspace, on which G acts by Ad_G. If one can prove the proposition for GL_n, then it also holds for G:

$$\mathfrak{g} \hookrightarrow \mathfrak{gl}_n \xrightarrow{\ \mathrm{ad}_{\mathfrak{gl}_n}\ } \mathrm{End}(\mathfrak{gl}_n)$$
$$x \longmapsto (y \mapsto xy - yx)$$

in particular, for $y \in \mathfrak{g}$, $d(\mathrm{Ad}_G)_e(x)(y) = xy - yx$, and this is the Lie bracket in $\mathfrak{g}$, as $\mathfrak{g} \subset \mathfrak{gl}_n$.

So now let $G = GL_n$.

$$\begin{tikzcd}
A \arrow[d, mapsto] & \mathrm{Lie}(G) \arrow[r] \arrow[d, hook] & \mathrm{End}(\mathfrak{g}) \arrow[d, hook] & M \arrow[d, mapsto] \\
I_n + \varepsilon_1 \cdot A & G(k[\varepsilon_1]) \arrow[r, "\mathrm{Ad}_{k[\varepsilon_1]}"] \arrow[d] & GL(\mathfrak{g} \otimes k[\varepsilon_1]) \arrow[d] & \mathrm{id} + \varepsilon_1 \cdot M \\
& G(k) \arrow[r, "\mathrm{Ad}_k"] & GL(\mathfrak{g}) &
\end{tikzcd}$$

$I_n + \varepsilon_1 A$ is sent to the operator on $\mathfrak{g} \otimes k[\varepsilon_1]$

$$X + \varepsilon_1 Y \mapsto (I_n + \varepsilon_1 A)(X + \varepsilon_1 Y)(I_n - \varepsilon_1 A) = X + \varepsilon_1 Y + \varepsilon_1 \cdot \overbrace{\mathrm{ad}(A)(X)}^{AX - XA}$$
$$= (\mathrm{id} + \varepsilon_1 \cdot \mathrm{ad}(A))(X + \varepsilon_1 Y),$$

so $A \in \mathrm{Lie}(G) = M_n(k)$ is sent to $\mathrm{ad}(A) \in \mathrm{End}(\mathfrak{g})$. $\square$.

- Rem. 2.7.2. For a k-Lie algebra $\mathfrak{g}$, define a subgroup functor $\underline{\mathrm{Aut}}_{\mathfrak{g}}$ of $GL_{\mathfrak{g}}$ as follows: $\forall$ k-alg. R, let

$$\underline{\mathrm{Aut}}_{\mathfrak{g}}(R) = \{\varphi \in GL(\mathfrak{g}_R) \mid \varphi \text{ is an } R\text{-Lie alg. hom.}\}.$$

It is clear (if not, work out this exercise, e.g. by taking a basis $\{e_1, \cdots, e_n\}$ of $\mathfrak{g}$) that this functor is representable by an alg. subgroup $\mathrm{Aut}_{\mathfrak{g}} \subset GL_{\mathfrak{g}}$, and one can show that

$$\mathrm{Lie}(\mathrm{Aut}_{\mathfrak{g}}) = \mathrm{Der}(\mathfrak{g}).$$

Now if $\mathfrak{g} = \mathrm{Lie}(G)$, because $x \mapsto g \cdot x \cdot g^{-1}$ is a hom.

of R-group schemes, the induced auto. $Ad_R(g)$ on $\mathfrak{g}_R$ is in $\underline{Aut}_{\mathfrak{g}}(R)$, therefore Ad factors through $\underline{Aut}_{\mathfrak{g}}$:

$$\begin{tikzcd} G \arrow[rr, "Ad"] \arrow[dr, dashed] & & GL_{\mathfrak{g}} \\ & \underline{Aut}_{\mathfrak{g}} \arrow[ur, hook] & \end{tikzcd},$$

which is consistent with the factorization

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "ad"] \arrow[dr, dashed] & & \mathfrak{gl}_{\mathfrak{g}} \\ & \mathrm{Der}(\mathfrak{g}) \arrow[ur, hook] & \end{tikzcd}$$

(see p. 5).

- <u>Excercises 2.8.</u> Let $G, H \ldots$ be alg. groups and $\mathfrak{g}, \mathfrak{h} \ldots$ be their Lie algebras.

i). G is finite $\Longleftrightarrow \mathfrak{g} = 0$;

ii). let $G \xrightarrow{\varphi} H \xleftarrow{\psi} G'$ be hom., then

$$\mathrm{Lie}(G \times_H G') = \mathfrak{g} \times_{\mathfrak{h}} \mathfrak{g}' ;$$

iii). let $1 \to N \to G \to H$ be an exact seq. of alg. groups, then the seq. of vector spaces

$$0 \to \mathrm{Lie}(N) \to \mathrm{Lie}(G) \to \mathrm{Lie}(H)$$

is exact ; (hint : $N = G \times_H \mathbb{1}$.)

iv). let H_i $(i \in I)$ be a family of alg. subgroups of G, then

$$\mathrm{Lie}\Big(\bigcap_{i\in I} H_i\Big) = \bigcap_{i \in I} \mathfrak{h}_i \text{ in } \mathfrak{g}. \quad //.$$

• <u>Cor. 2.8.1.</u> <u>Let</u> $1 \to N \to G \to H \to 1$ <u>be an exact sequence of alg. groups. Then</u>

$$0 \to \mathrm{Lie}(N) \to \mathrm{Lie}(G) \to \mathrm{Lie}(H) \to 0$$

<u>is exact.</u>

<u>Pf</u>: By (I, 4.5), $G \to H$ is flat, so by [Hart.], III, 9.5, we have $\dim(G) = \dim(N) + \dim(H)$, and by (2.3) and (2.8, iii), we have $\mathrm{Lie}(G) \twoheadrightarrow \mathrm{Lie}(H)$. □.

• <u>Rem. 2.8.2.</u> By (2.8.1), $G \twoheadrightarrow G' \Rightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}'$. Conversely, given a hom. $G \to G'$ s.t. $\mathfrak{g} \twoheadrightarrow \mathfrak{g}'$, we have $G^\circ \twoheadrightarrow G'^\circ$.

To see this, we need the simple fact that, if $H \subset G = G^\circ$ and $\mathfrak{h} = \mathfrak{g}$, then $H = G$ (by (2.3) and [Hart.], I, Ex. 1.10d).

Now consider

$$\begin{array}{l} G^\circ \twoheadrightarrow \mathrm{Im}(G^\circ) \hookrightarrow G'^\circ \\ \mathfrak{g} \twoheadrightarrow \mathrm{Lie}(\mathrm{Im}(G^\circ)) \hookrightarrow \mathfrak{g}' \quad \Rightarrow \mathrm{Lie}(\mathrm{Im}(G^\circ)) = \mathfrak{g}' \end{array}$$

$\Rightarrow \mathrm{Im}(G^\circ) = G'^\circ$.

Clearly $G \hookrightarrow G' \Rightarrow \mathfrak{g} \hookrightarrow \mathfrak{g}'$; the converse is false even if they are connected groups (e.g. $SL_n \to PSL_n$).

Define a hom. $f: G \to G'$ to be an <u>isogeny</u> if f is (ex. $\Leftrightarrow$ f is surj. and $\ker(f)$ is finite.)

finite and **surjective**. By (2.8, ii) and (2.8.1), we see that $f: G \to G'$ is an isogeny $\Rightarrow$ $f|_{G^\circ}: G^\circ \to G'^\circ$ is an isogeny $\Leftrightarrow$ $\mathfrak{g} \xrightarrow[df]{\sim} \mathfrak{g}'$. ((II, 2.9) step 2 shows that any isogeny between conn. alg. groups is central.)

- <u>Cor. 2.8.3.</u> <u>The functor</u>

$$G \mapsto \mathrm{Lie}(G) : \{\text{connected } \underline{\text{Alg.}}\ \text{Gr.}\} \longrightarrow \{\underline{\text{Lie Alg.}}\}$$

<u>is faithful</u> (<u>but not full</u>).

Pf.: Given two hom. $\varphi, \psi : G \rightrightarrows H$ such that

$$\mathrm{Lie}(\varphi) = \mathrm{Lie}(\psi) : \mathfrak{g} \to \mathfrak{h},$$

consider

$$\begin{tikzcd} G \underset{\varphi,H,\psi}{\times} G \arrow[rr] \arrow[d] \arrow[dr, phantom, "\subset"] & & G \arrow[dd, "\varphi"] \\ & G \times G \arrow[ur] \arrow[dl] & \\ G \arrow[rr, "\psi"] & & H \end{tikzcd} \quad \overset{\mathrm{Lie}}{\rightsquigarrow} \quad \begin{tikzcd} \mathrm{Lie}(G \underset{H}{\times} G) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\subset"] & & \mathfrak{g} \arrow[dd, "\mathrm{Lie}(\varphi)"] \\ & \mathfrak{g} \times \mathfrak{g} \arrow[ur] \arrow[dl] & \\ \mathfrak{g} \arrow[rr, "\mathrm{Lie}(\psi)"] & & \mathfrak{h}. \end{tikzcd}$$

By (2.8, ii), $\mathrm{Lie}(G \underset{H}{\times} G)$ contains diagonal $\Delta\mathfrak{g} \subset \mathfrak{g} \times \mathfrak{g}$, so by considering $(G \underset{H}{\times} G) \cap \Delta G$, we see that $\Delta G \subset G \underset{\varphi,H,\psi}{\times} G$, i.e. $\varphi = \psi$.

As for non-fullness, ~~connectivity is~~ consider

$$\begin{array}{c} \mathrm{Hom}(PSL_n, SL_n) \hookrightarrow \mathrm{Hom}(sl_n, sl_n), \\ \| \\ \{1\} \end{array}$$

or

$$\begin{array}{c} \mathrm{Hom}(\mathbb{G}_a, \mathbb{G}_m) \hookrightarrow \mathrm{Hom}(k, k) \\ (\mathrm{II}, 4.7) \rightarrow \| \\ \{1\} \end{array}$$

in which id_k, when $k = \mathbb{C}$, corresponds to the hom.

$$exp : \mathbb{C} \longrightarrow \mathbb{C}^\times$$

of complex Lie groups. □.

§3. Structure of semisimple groups.

• Def. 3.1. An alg. group G is *simple* if it is connected, non-commutative, and ~~any~~ the only normal alg. subgroups are $\mathbb{1}$ and G itself; *almost-simple* if it is conn., non-comm., and any proper normal alg. subgroup is finite. An alg. group G is the *almost-direct product* of its alg. subgroups $G_1, \cdots, G_n$, if the mult. map $\prod_{i=1}^{n} G_i \longrightarrow G$ is an isogeny. Two alg. groups G and G' are *isogenous* if there exists a "roof" of isogenies

$$\begin{tikzcd} & H \arrow[dl] \arrow[dr] & \\ G & & G' \end{tikzcd}.$$

• <u>Lem. 3.2.</u> <u>A connected alg. group G is almost-simple if and only if $Z(G)$ is finite and $G^{ad} = G/Z(G)$ is simple.</u>

<u>Pf</u>: ($\Longrightarrow$). As $Z(G)$ is a proper normal subgroup in G, it must be finite. Let $\overline{N}$ be a proper normal alg. subgroup in G^{ad}, and let N be its inverse image in G, so that N is proper normal in G, hence N is finite. As G is connected, any action of G on N must be trivial (see step 2 of the proof of (II, 2.9)), a fortiori, the conjugation action of G on N is trivial, hence $N \subset Z(G) \Rightarrow N = Z(G) \Rightarrow \overline{N} = \mathbb{1}$, so G^{ad} is simple.

($\Longleftarrow$). As $G^{ad} \neq \mathbb{1}$, G is not commutative. Let N be a proper normal alg. subgroup in G, and consider the image $N \cdot Z$ of the mult. map (where $Z = Z(G)$)

$$N \times Z \longrightarrow G.$$

It is again an alg. subgroup $\left((n_1 z_1)\cdot(n_2 z_2) = (n_1 n_2)(z_1 z_2)\right)$, and is normal in G $\left(g(nz)g^{-1} = (gng^{-1})\cdot z\right)$, so it corresponds to a normal alg. subgroup in G^{ad}, which is either $\mathbb{1}$ or G^{ad}. As G is connected and N is proper closed, we have

$$\dim(G^{ad}) = \dim(G) > \dim(N) = \dim(N \times Z) \geqslant \dim(N\cdot Z) \geqslant \dim(N\cdot Z/Z),$$

so $N \cdot Z = Z$, hence $N \subset N\cdot Z = Z$ is finite. $\square$

• <u>Exercise</u>. $PSL_n(k)$ is a simple group (in the ordinary sense). The same proof also shows that PSL_n $(n \geqslant 2)$ is a simple alg. gr. //.

We will prove in this section that a conn. alg. group G is semisimple $\Leftrightarrow$ G is isogenous to a finite product of simple alg. groups $\Leftrightarrow$ $\mathfrak{g} = \mathcal{L}ie(G)$ is a semisimple Lie alg., both justifying the name of a "semisimple alg. group".

Recall $\mathcal{S}tab_G(W)$, $\mathcal{C}ent_G(W)$ (for a G-rep. (V,ρ) and a subspace $W \subset V$), $N_G(H)$, $C_G(H)$ (for an alg. subgroup $H \subset G$), $\mathrm{stab}_{\mathfrak{g}}(W)$, $\mathrm{cent}_{\mathfrak{g}}(W)$, $n_{\mathfrak{g}}(\mathfrak{h})$, $c_{\mathfrak{g}}(\mathfrak{h})$; see (I, 4.7) and (1.2) on p. 4–5.

For a rational rep. $G \xrightarrow{\rho} GL_V$ of G, we get a rep. $\mathcal{L}ie(\rho) = d(\rho)_e : \mathfrak{g} \to \mathfrak{gl}_V$ of $\mathfrak{g} = \mathcal{L}ie(G)$.

• Prop. 3.3. We have

$$\mathcal{L}ie(\mathcal{S}tab_G(W)) = \mathrm{stab}_{\mathfrak{g}}(W) \quad \text{and}$$
$$\mathcal{L}ie(\mathcal{C}ent_G(W)) = \mathrm{cent}_{\mathfrak{g}}(W) \quad \text{in } \mathfrak{g},\ \forall\, W \subset V.$$

Pf: The diagrams

$$\begin{array}{ccc} \mathcal{S}tab_G(W) & \longrightarrow & \mathcal{S}tab_{GL_V}(W) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\rho} & GL_V \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{stab}_{\mathfrak{g}}(W) & \longrightarrow & \mathrm{stab}_{\mathfrak{gl}_V}(W) \\ \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{d\rho} & \mathfrak{gl}_V \end{array}$$

are clearly Cartesian, so by (2.8, ii), we reduce to prove the case when $G = GL_V$ and $\rho = \mathrm{id}_G$ (the tautological rep.).

For $x \in \mathrm{End}(V)$,

$\mathrm{id}_{V_{k[\varepsilon_1]}} + x\cdot\varepsilon_1 \in \mathrm{Lie}(\mathrm{Stab}_{GL_V}(W)) \iff \mathrm{id}_{V_{k[\varepsilon_1]}} + x\cdot\varepsilon_1 \in \mathrm{Stab}_{GL_V}(W)(k[\varepsilon_1])$

$\iff (\mathrm{id}_{V_{k[\varepsilon_1]}} + x\cdot\varepsilon_1)(W\oplus W\cdot\varepsilon_1) = W\oplus W\cdot\varepsilon_1 \iff xW\subset W \iff$

$x\in \mathrm{stab}_{\mathfrak{gl}_V}(W)$,

and

$\mathrm{id}_{V_{k[\varepsilon_1]}} + x\cdot\varepsilon_1 \in \mathrm{Lie}(\mathrm{Cent}_{GL_V}(W)) \iff \mathrm{id}_{V_{k[\varepsilon_1]}} + x\cdot\varepsilon_1$ fixes every vector in $W\oplus W\cdot\varepsilon_1 \iff x\cdot\varepsilon_1(W\oplus W\cdot\varepsilon_1)=0 \iff xW=0$

$\iff x\in \mathrm{cent}_{\mathfrak{gl}_V}(W)$. $\square$.

- Cor. 3.3.1. Let (V,ρ) be a linear rep. of G, and let $W\subset V$ be a subspace. Then
  - i). W is a G-subrep. $\Rightarrow$ W is a $\mathfrak{g}$-submodule;
  - ii). $V^G \subset V^{\mathfrak{g}}$.

If G is connected, then
  - i bis). W is a G-subrep. $\iff$ W is a $\mathfrak{g}$-submodule;
  - ii bis). $V^G = V^{\mathfrak{g}}$.

Pf: Immediate from (3.3) and the "simple fact" stated in (2.8.2). $\square$.

- Cor. 3.3.2. If G is connected, then the functor $(V,\rho)\mapsto (V, d\rho) : \mathrm{Rep}(G)\longrightarrow \mathrm{Rep}(\mathfrak{g})$ is fully faithful.

Pf: This follows from (3.3.1, ii bis): if V and U are G-reps., then

$$\mathrm{Hom}_G(V, U) = (V^\vee \otimes U)^G, \quad \mathrm{Hom}_{\mathfrak{g}}(V, U) = (V^\vee \otimes U)^{\mathfrak{g}}. \quad \square.$$

• <u>Cor. 3.3.3.</u> <u>Let</u> $H \subset G$ <u>be an alg. subgroup with Lie algebra</u> $\mathfrak{h} \subset \mathfrak{g}$. <u>If</u> H <u>is normal in</u> G, <u>then</u> $\mathfrak{h}$ <u>is an ideal in</u> $\mathfrak{g}$. <u>If</u> H <u>and</u> G <u>are connected</u>, <u>and</u> $\mathfrak{h}$ <u>is an ideal</u>, <u>then</u> H <u>is normal</u>.

Pf: If $H \triangleleft G$, then H is G-stable under the conjugation action of G on G, so $\mathfrak{h}$ is G-stable under the adjoint rep. Ad of G on $\mathfrak{g}$. By (3.3.1, i), $\mathfrak{h}$ is $\mathfrak{g}$-stable under the adjoint rep. ad of $\mathfrak{g}$ on $\mathfrak{g}$, i.e. $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

H is normal in G if and only if we have the factorization

$$\begin{tikzcd} & H \arrow[dr, hook] & \\ G \times_k H \arrow[ur, dashed] \arrow[rr, "f"] & & G \\ g, h \arrow[rr, mapsto] & & g h g^{-1} \end{tikzcd}$$

on R-points. It is an exercise (I, p.25) that ~~As H is reduced, f factors through $H \Leftrightarrow$ the underlying set of $\mathrm{Im}(f)$ is contained in $|H|$, for which~~ it suffices to check the k-points. $\forall g \in G(k)$, $\mathrm{Lie}(g \cdot H \cdot g^{-1}) = \mathrm{Ad}(g) \cdot \mathfrak{h}$, so if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ (i.e. $\mathfrak{g}$-stable under ad), then $\mathfrak{h}$ is G-stable under Ad, so that $\mathrm{Lie}(g \cdot H \cdot g^{-1}) = \mathfrak{h} = \mathrm{Lie}(H)$.

Then

$$\begin{tikzcd}[column sep=small] & H & & & & \mathfrak{h} \arrow[d, phantom, "\parallel"] \\ (gHg^{-1}) \cap H \arrow[ur, hook] \arrow[dr, hook] & & \overset{\text{Lie}}{\rightsquigarrow} & \mathfrak{h} \arrow[urr, phantom, "="] \arrow[drr, phantom, "\parallel"] & & \\ & g \cdot H \cdot g^{-1} & & & & \mathfrak{h} \end{tikzcd}$$

$$\xRightarrow{H \text{ conn.}} \quad H = (gHg^{-1}) \cap H = gHg^{-1} \Rightarrow H \triangleleft G. \quad \square.$$

- **Cor. 3.3.4.** *Let* $H \subset G$ *be an alg. subgroup with Lie alg.* $\mathfrak{h} \subset \mathfrak{g}$. *Then*

$$\mathrm{Lie}(N_G(H)) \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \quad \text{and} \quad \mathrm{Lie}(\cancel{\mathrm{Cent}_G(H)}\ C_G(H)) \subset \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h}).$$

If H *is connected, then both inclusions are equalities.*

Pf: $H \triangleleft N_G(H) \xRightarrow{(3.3.3)} \mathfrak{h}$ is an ideal in $\mathrm{Lie}(N_G(H))$, i.e.

$$[\mathrm{Lie}(N_G(H)), \mathfrak{h}] \subset \mathfrak{h},$$

hence $\mathrm{Lie}(N_G(H)) \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$.

The conjugation action of $C_G(H)$ on H is trivial, so is the adjoint action of ~~Lie~~ $C_G(H)$ (and hence $\mathrm{Lie}(C_G(H))$, by (3.3.1, ii)) on $\mathfrak{h}$, thus

$$[\mathrm{Lie}(C_G(H)), \mathfrak{h}] = 0,$$

i.e. $\mathrm{Lie}(C_G(H)) \subset \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$.

Now assume that H is connected. Let $H' := \mathrm{Stab}_G(\mathfrak{h})^{\circ}$, whose Lie alg. is $\mathfrak{h}' = \mathrm{stab}_{\mathfrak{g}}(\mathfrak{h}) =: \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ by (3.3). We claim that $H' \subset N_G(H)$, so that $\mathfrak{h}' \subset \mathrm{Lie}(N_G(H))$. It suffices to check the k-points:

$$\forall\, g \in H'(k) \overset{?}{\Longrightarrow} g \cdot H \cdot g^{-1} = H.$$

As $\mathfrak{h}$ is $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$-stable (under the adjoint action), it is also H'-stable by (3.3.1, i bis), so

$$\mathrm{Lie}(H) = \mathfrak{h} = \mathrm{Ad}(g)(\mathfrak{h}) = \mathrm{Lie}(g \cdot H \cdot g^{-1}),$$

and by considering $H \cap (g \cdot H \cdot g^{-1})$ we see that

$$H = H \cap (g \cdot H \cdot g^{-1}) = g \cdot H \cdot g^{-1}.$$

This proves that $\mathrm{Lie}(N_G(H)) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$.

Similarly, now let $H' := \mathrm{Cent}_G(\mathfrak{h})^{\circ}$, whose Lie alg. is $\mathfrak{h}' = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$ by (3.3). We claim that $H' \subset C_G(H)$, so that $\mathfrak{h}' \subset \mathrm{Lie}(C_G(H))$. It suffices to check the k-points:

$$\forall\, g \in H'(k) \overset{?}{\Longrightarrow} x \mapsto g \cdot x \cdot g^{-1} : H \to H \text{ is } \mathrm{id}_H.$$

By (2.8.3), it suffices to show that its derivative, which is $\mathrm{Ad}(g) : \mathfrak{h} \to \mathfrak{h}$, is $\mathrm{id}_{\mathfrak{h}}$, and this is clear since $H' \subset \mathrm{Cent}_G(\mathfrak{h})$. This proves that $\mathrm{Lie}(C_G(H)) = \mathfrak{c}_{\mathfrak{g}}(\mathfrak{h})$. $\square$.

- Exercise 3.3.5. Let G be a connected alg. group. Then $\mathrm{Lie}(Z(G)) = \mathfrak{z}(\mathfrak{g})$, and G is abelian if and only if $\mathfrak{g}$ is. //.

• Th. 3.4. Let G be a connected alg. group. Then G is a semi-simple group if and only if $\mathfrak{g}$ is a semi-simple Lie algebra.

Pf. ($\Rightarrow$). Let $\mathfrak{h} \subset \mathfrak{g}$ be an abelian ideal, and let $H' := C_G(\mathfrak{h})^\circ$ (w.r.t. the Ad rep. of G on $\mathfrak{g}$). Its Lie alg. is, by (3.3)

$$\mathfrak{h}' = c_{\mathfrak{g}}(\mathfrak{h}) \supset \mathfrak{h} \quad \text{(as } \mathfrak{h} \text{ is abelian)}.$$

By the Jacobi identity, $\mathfrak{h}'$ is an ideal, so by (3.3.3), H' is normal in G. Then $Z(H')^\circ$ is a conn. normal abelian alg. subgroup in G, which must be trivial, and so also is its Lie algebra

$$0 = \mathrm{Lie}(Z(H')^\circ) \overset{(3.3.5)}{=\!=} \mathfrak{z}(\mathfrak{h}') \supset \mathfrak{h} \Rightarrow \mathfrak{h} = 0.$$

($\Leftarrow$). Let $H \trianglelefteq G$ be a conn. normal abelian alg. subgroup of G. Then by (3.3.3, 3.3.5), its Lie alg. $\mathfrak{h} \subset \mathfrak{g}$ is an abelian ideal, which must be trivial, so $H = \mathbb{1}$. $\square$

• Exercise 3.4.1. For a conn. alg. group G, we have $\mathrm{Lie}(R(G)) = \mathfrak{r}(\mathfrak{g})$ and $\ker(\mathrm{Ad}: G \to GL_{\mathfrak{g}}) = Z(G)$.
Also, G is almost simple $\Leftrightarrow$ $\mathfrak{g}$ is simple. //

• Th. 3.5. Let G be a connected alg. group. Then G is semisimple if and only if G is isogenous to a finite product of simple alg. groups. More precisely, if G is semisimple, then it has only finitely many almost-simple normal alg. subgroups, and G is the almost-direct product of them.

Pf.: ($\Leftarrow$) follows from (3.4, 3.4.1, 2.8 ii).

($\Rightarrow$). By (3.4), $\mathfrak{g}$ is semisimple, so (see p. 3)

$$\mathfrak{g} = \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_n ,$$

where the $\mathfrak{a}_i$'s are simple ideals of $\mathfrak{g}$. Let

$$G_i = \mathrm{Cent}_G(\mathfrak{a}_1 \times \cdots \underset{\text{absent}}{\hat{\mathfrak{a}}_i} \cdots \times \mathfrak{a}_n)^\circ \subset G .$$

By (3.3), $\mathrm{Lie}(G_i) = c_{\mathfrak{g}}(\mathfrak{a}_1 \times \cdots \hat{\mathfrak{a}}_i \cdots \times \mathfrak{a}_n) = \mathfrak{a}_i$, so by (3.3.3, 3.4.1), G_i is an almost-simple normal alg. subgroup in G, and is the only one with Lie algebra $\mathfrak{a}_i$ (if there is another one, take their intersection). As the $\mathfrak{a}_i$'s are the only simple ideals in $\mathfrak{g}$, the G_i's are the only almost-simple normal alg. subgroups in G. The mult. map (which is a hom. since $\mathfrak{a}_i \subset c_{\mathfrak{g}}(\mathfrak{a}_j) = \mathrm{Lie}(C_G(G_j)) \Rightarrow G_i \subset C_G(G_j)$, i.e. the G_i's commute with each oth

$$G_1 \times \cdots \times G_n \longrightarrow G$$

induces the isom. on Lie algebras

$$\mathfrak{a}_1 \times \cdots \times \mathfrak{a}_n \longrightarrow \mathfrak{g} ,$$

so it is an isogeny (see p. 23). By (3.2), the diagram

$$\begin{tikzcd} & \prod G_i \arrow[dl] \arrow[dr] & \\ G & & \prod G_i^{ad} \end{tikzcd}$$

gives a roof of isogenies between G and a finite product of simple alg. groups. $\square$.

• Cor. 3.5.1. Let G be a semisimple alg. group, with almost-simple normal alg. subgroups $G_1, \cdots, G_n$, and let H be a conn. normal alg. subgroup of G, $I = \{ i \in [1,n] \mid G_i \subset H \}$. Then H is also semisimple, $H = \prod_{i \in I}$ (in G) of G_i (i.e. image of $\prod_{i\in I} G_i$ under $\prod_{\text{all } i} G_i \to G$), and $C_G(H)^0 = \prod_{i \notin I}$ (in G) of G_i.

Quotients ($\neq 1$) of G are also semisimple.

Pf.: $\mathfrak{h} \lhd \mathfrak{g}$, so $\mathfrak{h}$ = product of some $\mathfrak{a}_i$'s, those with $i \in I$ (by looking at $G_i \cap H$). Then $\prod_{i\in I} G_i \longrightarrow H$ induces an isom. on Lie alg., hence is an isogeny. So H is semisimple, and is the product (not direct) of these G_i's. By (3.3.4),

$$\mathrm{Lie}\ C_G(H)^\circ = c_{\mathfrak{g}}(\mathfrak{h}) = \prod_{i \notin I} \mathfrak{a}_i ,$$

so $C_G(H)^\circ = \prod_{i\notin I}$ (in G) of G_i (again by considering the Lie alg. of their intersection). Note that this may not be the full $C_G(H)$: $Z(H)$, which is finite, also centralizes H.

The Lie alg. of a quotient group of G is a quotient algebra of $\mathfrak{g}$ (by (2.8.1)), hence is a product of some $\mathfrak{a}_i$'s, which is then semisimple; then apply (3.4) (quot. of G are certainly conn.). $\square$.

- Example. $SL_n \times SL_n / \{(I,I), (-I,-I)\}$ is a s.s. group, but not a direct product of the G_i's (each being isom. to SL_n).

(derived subgroup)

- Cor. 3.5.2. If G is semisimple, then $\mathcal{D}G = G$.

Pf: This is true if G is almost-simple, as $\mathcal{D}G$ is connected and normal. So this is true for a finite product $\prod G_i$ of almost-simple groups. Note that if $H \twoheadrightarrow G$, then $\mathcal{D}H \twoheadrightarrow \mathcal{D}G$. $\square$.

Let G be an alg. group. Define $\mathrm{Inn}(G)$ to be the group of inner automorphisms, i.e. those of the form

$$x \mapsto g x g^{-1} \text{ (on } R\text{-points), for some } g \in G(k).$$

We have $G^{ad}(k) \simeq \mathrm{Inn}(G) \subset \mathrm{Aut}(G)$. (In fact, $\mathrm{Aut}(G)$ also "has" the structure of an alg. group.) The group $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$ $(\psi(g\cdot\psi^{-1}(x)\cdot g^{-1}) = \psi(g)\cdot x\cdot \psi(g)^{-1})$, and $\mathrm{Aut}(G)/\mathrm{Inn}(G) =: \mathrm{Out}(G)$ is the group of outer automorphisms.

• <u>Prop. 3.6.</u> <u>If</u> G <u>is semisimple</u>, <u>then</u> $\mathrm{Out}(G)$ <u>is a</u> <u>finite</u> <u>group</u>.

<u>Pf</u>: By (2.8.3), $\mathrm{Aut}(G) \hookrightarrow \mathrm{Aut}(\mathfrak{g})$, so it suffices to show that $[\mathrm{Aut}(\mathfrak{g}) : \mathrm{Ad}(G(k))] < \infty$.

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & \mathcal{A}ut_{\mathfrak{g}}^{\circ} \subset \mathcal{A}ut_{\mathfrak{g}} \\ & & \big\downarrow \wr \ \mathcal{L}ie \\ \mathfrak{g} & \xrightarrow[\sim]{\mathrm{ad}} & \mathcal{D}er(\mathfrak{g}) \end{array} \qquad \text{(see (2.7.2))}.$$

(see p.5)

so $\mathrm{Ad} : G \twoheadrightarrow \mathcal{A}ut_{\mathfrak{g}}^{\circ}$, and therefore

$$[\mathrm{Aut}(\mathfrak{g}) : \mathrm{Ad}(G(k))] = \# \pi_0(\mathcal{A}ut_{\mathfrak{g}})(k) < \infty. \qquad \square.$$

§4. <u>Reductive groups are linearly reductive</u>.

A rep. (V, ρ) of G (resp. $\mathfrak{g}$) is <u>semisimple</u> if $\forall$ subrep. $W \subset V$, $\exists$ a subrep. $U \subset V$ with $V = W \oplus U$.

This is equivalent to that V is a direct sum of simple (i.e. no nontrivial sub-) rep..

• <u>Prop. 4.1.</u> <u>If</u> G <u>is semisimple</u>, <u>then</u> $\mathrm{Rep}(G)$ <u>is semisimple</u> (<u>i.e. all</u> <u>f.dim'l rep. of</u> G <u>are semisimple</u>).

<u>Pf</u>: By (3.4), $\mathfrak{g}$ is semisimple, so by Weyl's theorem

(1.3.6), Rep($\mathfrak{g}$) is semisimple. Given a G-rep. V and a sub-G-rep. $W \subset V$, $\exists$ sub-$\mathfrak{g}$-rep. $U \subset V$ with $V = W \oplus U$. By (3.3.1, i bis), U is also G-stable. $\square$.

• Prop. 4.2. Let G be a reductive group. Then

i). $Z(G)^\circ = R(G)$ is a torus;

ii). $R(G) \cap \mathcal{D}G$ is finite;

iii). $G = R(G) \cdot \mathcal{D}G$;

iv). $Z(\mathcal{D}G) = Z(G) \cap \mathcal{D}G$;

v). $\mathcal{D}G$ is semisimple.

Pf: i). The connected center $Z(G)^\circ$ is clearly a characteristic subgroup of G (i.e. one that is stable under any automorphism of G), hence is a connected normal commutative alg. ~~gp~~ subgroup, so $Z(G)^\circ \subset R(G)$. Since $R(G)$ is a torus, and G is connected, the conjugation action of G on $R(G)$ is trivial by rigidity (II, 2.9), i.e. $R(G) \subset Z(G)$. Therefore, $Z(G)^\circ = R(G)$.

ii). It suffices to show that $\left|(R(G) \cap \mathcal{D}G)(k)\right| < \infty$. Take $\rho: G \hookrightarrow GL_V$, and decompose V into eigenspaces w.r.t. the $Z(G)^\circ$-action (II, 2.7): $V = \bigoplus_{\chi \in X(Z(G)^\circ)} V_\chi$. So

$$\rho(Z(G)^\circ)(k) = \left\{ \begin{pmatrix} \chi_1(t)\cdot I_{n_1} & & 0 \\ & \ddots & \\ 0 & & \chi_\ell(t)\cdot I_{n_\ell} \end{pmatrix} ;\ t \in Z(G)^\circ(k) \right\},$$

and hence

$$C_{GL_V}\left(\rho(Z(G)^\circ)\right)(k) = C_{GL(V)}\left(\rho(Z(G)^\circ)(k)\right) \xlongequal[\text{exercise}]{\text{linear alg.}}$$

$$\left\{ \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_\ell \end{pmatrix} \;\middle|\; A_i \in GL_{n_i}(k) \right\}.$$

As $Z(G)^\circ$ clearly commutes with G, we have

$$\rho(G) \subset C_{GL_V}\left(\rho(Z(G)^\circ)\right),$$

so

$$\rho(\mathcal{D}G) \subset \begin{pmatrix} SL_{n_1} & & 0 \\ & \ddots & \\ 0 & & SL_{n_\ell} \end{pmatrix} \subset GL_V.$$

Therefore,

$$\rho\left(Z(G)^\circ \cap \mathcal{D}G\right) \subset \begin{pmatrix} \mu_{n_1} & & 0 \\ & \ddots & \\ 0 & & \mu_{n_\ell} \end{pmatrix},$$

hence is finite.

iii). We have $Z(G)^\circ \cdot \mathcal{D}G \triangleleft G$. The quotient group $G/Z(G)^\circ \cdot \mathcal{D}G$, being a quotient of $G/\mathcal{D}G$, is abelian, and being a quotient of the semisimple group $G/R(G)$, is semisimple or trivial (3.5.1), hence it is trivial.

iv). Clearly $Z(\mathcal{D}G) \subset C_G(\mathcal{D}G) \cap \underbrace{C_G(Z(G))}_{= G} = C_G(Z(G)\cdot\mathcal{D}G)$, which, by iii), is equal to $Z(G)$.

v). The canonical morphism

$$\mathcal{D}G \longrightarrow G/R(G) \simeq \mathcal{D}G/_{R(G)\cap \mathcal{D}G}$$

is an isogeny by ii), and $G/R(G)$ is semisimple. As $\mathcal{D}G$ is connected, it is also semisimple. $\square$

• <u>Rmk. 4.2.1.</u> Any reductive group G can be realized as a (central) extension of a semisimple group by a torus:

$$1 \longrightarrow R(G) \longrightarrow G \longrightarrow G/R(G) \longrightarrow 1,$$

and as an extension of a torus by a semisimple group:

$$1 \longrightarrow \mathcal{D}G \longrightarrow G \longrightarrow G^{ab} \longrightarrow 1.$$

(Here by a "semisimple group" we allow the trivial group $\mathbb{1}$.)

For example,

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\lambda \cdot I_n} GL_n \longrightarrow PGL_n \longrightarrow 1,$$

$$1 \longrightarrow SL_n \longrightarrow GL_n \xrightarrow{\det} \mathbb{G}_m \longrightarrow 1.$$

Define the <u>group of symplectic similitudes</u> GSp_ψ, associated to a non-degenerate alternating bilinear form ψ on V, to be

$$GSp_\psi(R) := \left\{ \alpha \in GL(V_R) \;\middle|\; \begin{array}{l} \psi_R(\alpha x, \alpha y) = c(\alpha)\cdot \psi_R(x,y), \\ \forall x, y \in V_R, \text{ for some } c(\alpha) \in R^\times \end{array} \right\};$$

it is reductive, and the sequence

$$1 \longrightarrow Sp_\psi \longrightarrow GSp_\psi \xrightarrow{\alpha \mapsto c(\alpha)} \mathbb{G}_m \longrightarrow 1$$

is exact.

• **Th. 4.3.** Let $G \neq 1$ be an alg. group. TFAE:

i). G° is reductive;

ii). $\mathrm{Rep}(G^\circ)$ is semisimple;

iii). $\mathrm{Rep}(G)$ is semisimple;

iv). $\exists$ a faithful f. dim'l semisimple G-rep.

Pf: i) $\Rightarrow$ ii) (namely, the title of this section). Let (V, ρ) be a f. dim'l G°-rep., and decompose V w.r.t. the action of the torus $R(G^\circ)$:

$$V = \bigoplus_{\chi \in X(R(G^\circ))} V_\chi .$$

As $R(G^\circ) = Z(G^\circ)^\circ$ commutes with $\mathcal{D}(G^\circ)$, each V_χ is also $\mathcal{D}(G^\circ)$-stable (exercise, if not clear). By (4.1, 4.2, v), V_χ is a semisimple $\mathcal{D}(G^\circ)$-rep., and each simple factor is clearly G°-stable and is G°-simple (each G°-stable subspace is also $\mathcal{D}(G^\circ)$-stable). Therefore, V is a s.s. G°-rep.

ii) $\Rightarrow$ iii). Let V be a f. dim'l G-rep., $W \subset V$ a G-stable subspace. Then $\exists$ a G°-equiv. section

$$\begin{tikzcd} W \arrow[r, hook] & V \arrow[l, two heads, bend right, "p"'] \end{tikzcd} \qquad \begin{tikzcd} w & w \arrow[l, bend right, "p"'] \\ w \arrow[r, mapsto] & w \end{tikzcd} .$$

Let r be the rank of $\pi_0(G)$, and let $g_1, \cdots, g_r \in G(k)$

be a set of representatives of $\pi_0(G)(k)$. Define

$$\tilde{p} : V \longrightarrow W$$
$$v \longmapsto \frac{1}{r} \sum_{i=1}^{r} g_i^{-1} \cdot p(g_i \cdot v) \,,$$

(makes sense, since $\mathrm{char}(k) = 0$.)

which is a section. To verify that it is G-equiv. (i.e. $\forall$ k-alg. R, $\tilde{p}_R$ is $G(R)$-equiv.), namely to verify that $\tilde{p} \in \mathrm{Hom}(V, W)$ is fixed by G (i.e. $\sigma(\tilde{p}) = \tilde{p} \otimes 1$), it suffices to check all k-pts. of G (again, because

$$\bigcap_{g \in G(k)} \ker(\mathrm{id}_V \otimes g : V \otimes k[G] \to V \otimes k) = 0),$$

or equivalently, all $g \in G^\circ(k)$ and the g_i's, which we leave as an exercise.

<u>iii) $\Rightarrow$ iv)</u>. Clear, by (I, 5.2).

<u>iv) $\Rightarrow$ i)</u>. Let (V, ρ) be a f.dim'l faithful s.s. G-rep.; then by (II, 6.4), $(V, \rho|_{G^\circ})$ is a faithful s.s. G°-rep.. Then the proof goes verbatim as that of (II, 6.5), by analyzing the image of $R_u(G^\circ)$. $\square$

• <u>Exercise 4.3.1</u>. Let Γ be a group, and let V, W be semisimple rep. of Γ over k. Then $V \otimes W$ is also a semisimple Γ-rep.. (<u>Hint</u>: Take the Zariski closure of the image of Γ in $GL(V) \times GL(W)$.). //. (A theorem of Chevalley).

Course notes of « Linear Algebraic Groups »

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Ch. IV. Classification of semisimple groups.

§1. Review of the classical approach.

Given a conn. alg. group G, one can consider its maximal tori. They are all conjugate to one another: $\forall$ max. tori $T, T' \subset G$, $\exists\, g \in G(k)$ s.t. $g \cdot T \cdot g^{-1} = T'$. To prove this, one can either introduce Borel subgroups in G and use (essentially) the "Borel fixed points theorem", or, at least for G semisimple (the case we are interested in here), reduce to Cartan subalgebras in $\mathfrak{g}$ and use similar results for them.

Now assume that G is semisimple, $T \subset G$ a max. torus, with Lie alg. $\mathfrak{h} \subset \mathfrak{g}$. Then $\mathrm{Ad}|_T$ gives an action of T on $\mathfrak{g}$: $\mathfrak{g} = \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha$. Let $R = R(G,T)$ be the set of non-trivial characters $\alpha \in X(T)$ s.t. $\mathfrak{g}_\alpha \neq 0$. One can view $X(T)$ as a lattice in $\mathfrak{h}^\vee = \mathrm{Hom}_k(\mathfrak{h}, k)$, via

$$(\alpha : T \to \mathbb{G}_m) \longmapsto (d\alpha : \mathfrak{h} \to k) \quad (\text{see } (\mathrm{III}, 2.8.3))$$

Then R is a reduced root system in $\mathfrak{h}^\vee$, or ~~better~~ equivalently, in the $\mathbb{Q}$-span of R in $\mathfrak{h}^\vee$:

$$X(T) \otimes \mathbb{Q} \subset X(T) \otimes k = \mathfrak{h}^\vee, \qquad (\mathrm{II}, 2.6.5)$$

with Weyl group $W = (N_G(T)/T)(k)$, operating on $\mathfrak{h}$ (which induces an action on $\mathfrak{h}^\vee$) by the adjoint rep..

Moreover, we have

$$\underset{\text{root lattice}}{Q(R)} \subset X(T) \subset \underset{\text{weight lattice}}{P(R)} \quad \text{in } \mathfrak{h}^\vee$$

By a <u>diagram</u> we mean a pair $((V, R), X)$ where R is a (reduced, as always) root system in a $\mathbb{Q}$-vector space V, and X is a $\mathbb{Z}$-lattice in V **between** $Q(R)$ and $P(R)$:

$$Q(R) \subset X \subset P(R).$$

We will prove in this chapter that $G \mapsto ((X(T)_{\mathbb{Q}}, R(G,T)), X(T))$ gives a one-to-one **correspondence** between the isom. classes of semisimple alg. groups over k and the isom. classes of "diagrams". In particular, the base field k is irrelevant.

We sketch the classical approach of Killing, Cartan, Weyl, for $k = \mathbb{C}$. Firstly, a few words about the relation between alg. groups (over $\mathbb{R}$ or $\mathbb{C}$) and Lie groups. We have a functor

$$G \mapsto G(\mathbb{R}) \ : \{\mathbb{R}\text{-Alg. Groups}\} \longrightarrow \{\text{Lie groups}\},$$

faithful on connected alg. groups (think of $\mu_{n,\mathbb{R}}$, with n odd), but far from being full (think of $\exp : \mathbb{R} \to \mathbb{R}^{\times}$) or essentially surjective: the universal covering

$$\widetilde{SL_2(\mathbb{R})} \longrightarrow SL_2(\mathbb{R}) \ , \ \text{a } \mathbb{Z}\text{-covering},$$

provides one example, and the existence of non-algebraic Lie algebras (III, p.16) shows that infinite covering is not the only reason for the existence of non-algebraic Lie groups. Nevertheless, with the simple global topological property of compactness, "analytic functions can be replaced by polynomials." To be precise, given a compact Lie group K, by the theorem of Peter-Weyl, $\exists$ a faithful $\mathbb{R}$-rep. of finite dim.

$$K \hookrightarrow GL_n(\mathbb{R}),$$

and by the theorem of Stone-Weierstrass, the matrix coefficients $\{a_{ij}(x)\}_{i,j}$ generate the $\mathbb{R}$-algebra $\mathcal{T}(K)$ of real-valued representative functions, so that $G := \mathrm{Spec}(\mathcal{T}(K))$ is of finite type. Then G is an $\mathbb{R}$-alg. group, and one verifies that

$$x \mapsto ev_x : K \longrightarrow \mathrm{Hom}_{\mathbb{R}\text{-alg.}}(\mathcal{T}(K), \mathbb{R}) = G(\mathbb{R})$$

is an isom. of Lie groups.

One can further pass to the complexification

$$G_{\mathbb{C}} = \operatorname{Spec}(\mathcal{T}(K)\otimes\mathbb{C}).$$

K is connected $\Longleftrightarrow$ $G_{\mathbb{C}}$ is connected (w.r.t. the Zariski top.) $\Longleftrightarrow$ $G_{\mathbb{C}}(\mathbb{C})$ is connected (w.r.t. the analytic top.), K is a maximal compact subgroup in $G_{\mathbb{C}}(\mathbb{C})$, and is homotopic to $G_{\mathbb{C}}(\mathbb{C})$. $\mathrm{Lie}(K)$ is the unique compact real form of $\mathrm{Lie}(G_{\mathbb{C}})$ (an $\mathbb{R}$-Lie alg. is <u>compact</u> if its Killing form is negative-definite). When K is connected, the $\mathbb{C}$-alg. group $G_{\mathbb{C}}$ is reductive, and every reductive $\mathbb{C}$-alg. group arises this way (take $G_{\mathbb{C}} \hookrightarrow GL_{n,\mathbb{C}}$, and consider $K := G_{\mathbb{C}}(\mathbb{C}) \cap U(n)$); moreover, K is semisimple $\Longleftrightarrow$ $G_{\mathbb{C}}$ is semisimple. Finally, $\forall$ $\mathbb{C}$-alg. group G', we have

$$\mathrm{Hom}_{\text{Lie gr.}}(K, G'(\mathbb{C})) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}\text{-alg. gr.}}(G_{\mathbb{C}}, G');$$

in particular, by taking $G' = GL_n$, we have an equiv. of their rep. categories

$$\mathrm{Rep}_{\mathbb{C}}(K) \xrightarrow{\sim} \mathrm{Rep}(G_{\mathbb{C}}).$$

One has the following classifications:

$$\{\text{root systems}\} \xleftrightarrow{\text{Killing, Cartan}} \{\text{semisimple Lie algebras}/\mathbb{C}\}$$

$\forall$ semisimple real Lie algebra $\mathfrak{g}$, with root system R,
($\Leftrightarrow \mathfrak{g}_{\mathbb{C}}$ is s.s.)

$$\left\{\begin{matrix}\text{lattices between} \\ Q(R) \text{ and } P(R)\end{matrix}\right\} \xleftrightarrow{\text{Weyl, Cartan}} \left\{\begin{matrix}\text{conn. compact semisimple} \\ \text{Lie groups } K \text{ with } \mathrm{Lie}(K) \simeq \mathfrak{g}\end{matrix}\right\}$$

Together with the relation between conn. compact s.s. Lie groups and s.s. $\mathbb{C}$-alg. groups described above, these lead to a classification of s.s. $\mathbb{C}$-alg. groups. In particular, any s.s. Lie alg. $/\mathbb{C}$ is algebraic; note that to prove this purely algebraic fact, the approach above involves ~~analysis~~ transcendental method in an essential way. Like Weyl's theorem (III, 1.3.6), this classification, as well as the algebricity of s.s. Lie alg., can be proved purely algebraically. One such proof is given by Chevalley, who associates to each diagram $((V, R), X)$ a group scheme G over $\mathbb{Z}$, the <u>Chevalley group</u>, such that $\forall$ alg. closed field k, G_k is a s.s. k-alg. group corresponding to $((V, R), X)$. Another approach, due to Milne, is more categorical and less algebro-geometric, and we will follows this one in the notes.

§2. Tannakian duality.

We want to find more structures of $\mathrm{Rep}(G)$, for some affine k-group G. First, some notions from category theory.

A <u>symmetric monoidal</u> (or <u>tensor</u>) <u>category</u> consists of $(\mathscr{C}, \otimes, \mathbb{1}, \gamma, \tau, \theta)$ where

— $\mathscr{C}$ is a category,

— $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ is a functor,

— $\mathbb{1}$ is an object in $\mathscr{C}$,

— $\gamma_{X,Y,Z} : (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z)$ is a natural isomorphism, $X, Y, Z \in \mathrm{obj}(\mathscr{C})$,

— $\tau_{X,Y} : X\otimes Y \xrightarrow{\sim} Y\otimes X$ is a natural isom., $X, Y \in \mathrm{obj}(\mathscr{C})$,

— $\theta_X : X \xrightarrow{\sim} \mathbb{1}\otimes X$ is a natural isom., $X \in \mathrm{obj}(\mathscr{C})$,

such that

$$\begin{tikzcd}
((X\otimes Y)\otimes Z)\otimes W \arrow[r, "\gamma_{X,Y,Z}\otimes id"] \arrow[d, "\gamma_{X\otimes Y,Z,W}"'] & (X\otimes(Y\otimes Z))\otimes W \arrow[d, "\gamma_{X,Y\otimes Z,W}"] \\
 \arrow[r, phantom, "\circlearrowright"] & X\otimes((Y\otimes Z)\otimes W) \arrow[d, "id\otimes\gamma_{Y,Z,W}"] \\
(X\otimes Y)\otimes(Z\otimes W) \arrow[r, "\gamma_{X,Y,Z\otimes W}"] & X\otimes(Y\otimes(Z\otimes W))
\end{tikzcd},$$

$$\begin{tikzcd}
(X\otimes Y)\otimes Z \arrow[r, "\gamma_{X,Y,Z}"] \arrow[d, "\tau_{X,Y}\otimes id"'] & X\otimes(Y\otimes Z) \arrow[r, "\tau_{X,Y\otimes Z}"] \arrow[d, phantom, "\circlearrowright"] & (Y\otimes Z)\otimes X \arrow[d, "\gamma_{Y,Z,X}"] \\
(Y\otimes X)\otimes Z \arrow[r, "\gamma_{Y,X,Z}"] & Y\otimes(X\otimes Z) \arrow[r, "id\otimes\tau_{X,Z}"] & Y\otimes(Z\otimes X)
\end{tikzcd},$$

and $\tau_{Y,X}\circ\tau_{X,Y} = id_{X\otimes Y}$, $\tau_{\mathbb{1},\mathbb{1}} = id_{\mathbb{1}\otimes\mathbb{1}}$.

(Equivalently, the neutral object $(\mathbb{1}, \theta)$ can also be given by $(\mathbb{1}, u: \mathbb{1} \xrightarrow{\sim} \mathbb{1}\otimes\mathbb{1})$ such that $X \mapsto \mathbb{1}\otimes X : \mathcal{C}\to\mathcal{C}$ is an equiv., ~~the relation with the previous one is that~~

~~$\mathbb{1}\otimes X \xrightarrow{u\otimes id} (\mathbb{1}\otimes\mathbb{1})\otimes X \xrightarrow{\gamma_{\mathbb{1},\mathbb{1},X}} \mathbb{1}\otimes(\mathbb{1}\otimes X)$~~

~~is $\theta_{\mathbb{1}\otimes X}$, and any object in $\mathcal{C}$ is isom to some $\mathbb{1}\otimes X$~~

and that

$$\begin{tikzcd}
\mathbb{1}\otimes\mathbb{1} \arrow[r, "u\otimes id"] \arrow[dr, "id\otimes u"'] & (\mathbb{1}\otimes\mathbb{1})\otimes\mathbb{1} \arrow[d, "\gamma_{\mathbb{1},\mathbb{1},\mathbb{1}}"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathbb{1}\otimes(\mathbb{1}\otimes\mathbb{1})
\end{tikzcd};$$

the relation between them is $u = \theta_{\mathbb{1}}$, and

$$\mathbb{1}\otimes X \xrightarrow{u\otimes id} (\mathbb{1}\otimes\mathbb{1})\otimes X \xrightarrow{\gamma_{\mathbb{1},\mathbb{1},X}} \mathbb{1}\otimes(\mathbb{1}\otimes X)$$

is $\theta_{\mathbb{1}\otimes X}$, and any object in $\mathcal{C}$ is isom. to some $\mathbb{1}\otimes X$.)

A <u>tensor functor</u> from $(\mathcal{C}, \otimes, \mathbb{1}, \gamma, \tau, \theta)$ to $(\mathcal{C}', \otimes', \ldots)$ is (F, α, β) where $F: \mathcal{C}\to\mathcal{C}'$ is a functor, $\alpha_{X,Y} : F(X)\otimes' F(Y) \xrightarrow{\sim} F(X\otimes Y)$ is a natural isom. (for $X, Y \in \mathrm{obj}(\mathscr{C})$), and $\beta : \mathbb{1}' \xrightarrow{\sim} F(\mathbb{1})$ is an isom.,

such that

$$\begin{tikzcd}
(F(X)\otimes' F(Y))\otimes' F(Z) \arrow[rr, "\gamma'"] \arrow[d, "\alpha_{X,Y}\otimes id"'] & & F(X)\otimes'(F(Y)\otimes' F(Z)) \arrow[d, "id\otimes\alpha_{Y,Z}"] \\
F(X\otimes Y)\otimes' F(Z) \arrow[d, "\alpha_{X\otimes Y,Z}"'] \arrow[rr, phantom, "\circlearrowright"] & & F(X)\otimes' F(Y\otimes Z) \arrow[d, "\alpha_{X,Y\otimes Z}"] \\
F((X\otimes Y)\otimes Z) \arrow[rr, "F(\gamma)"] & & F(X\otimes(Y\otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
F(X)\otimes' F(Y) \arrow[r, "\alpha_{X,Y}"] \arrow[d, "\tau'_{F(X),F(Y)}"'] \arrow[dr, phantom, "\circlearrowright"] & F(X\otimes Y) \arrow[d, "F(\tau_{X,Y})"] \\
F(Y)\otimes' F(X) \arrow[r, "\alpha_{Y,X}"'] & F(Y\otimes X)
\end{tikzcd}$$

$$\begin{tikzcd}
F(X) \arrow[r, "F(\theta_X)"] \arrow[d, "\theta'_{F(X)}"'] \arrow[dr, phantom, "\circlearrowright"] & F(\mathbb{1}\otimes X) \\
\mathbb{1}'\otimes F(X) \arrow[r, "\beta\otimes id"'] & F(\mathbb{1})\otimes' F(X) \arrow[u, "\alpha_{\mathbb{1},X}"']
\end{tikzcd} ,$$

for all $X, Y, Z \in \mathrm{obj}(\mathscr{C})$.

Given tensor functors

$$\begin{tikzcd}
(\mathscr{C}, \otimes, \cdots) \arrow[r, bend left, "{(F,\alpha,\beta)}"] \arrow[r, bend right, "{(F',\alpha',\beta')}"'] & (\mathscr{C}', \otimes', \cdots)
\end{tikzcd} ,$$

a natural *transformation* $\varphi : F \Rightarrow F'$ is called a <u>*morphism* of *tensor functors*</u>, if $\forall\, X, Y \in \mathrm{obj}(\mathscr{C})$, the diagrams commute:

$$\begin{tikzcd} F(X)\otimes' F(Y) \arrow[r, "\alpha_{X,Y}"] \arrow[d, "\varphi_X\otimes'\varphi_Y"'] & F(X\otimes Y) \arrow[d, "\varphi_{X\otimes Y}"] \\ F'(X)\otimes' F'(Y) \arrow[r, "\alpha'_{X,Y}"] & F'(X\otimes Y) \end{tikzcd} \qquad \begin{tikzcd} & F(\mathbb{1}) \arrow[d, "\varphi_{\mathbb{1}}"] \\ \mathbb{1}' \arrow[ur, "\beta"] \arrow[r, "\beta'"'] & F'(\mathbb{1}) \end{tikzcd}.$$

We denote by $\mathrm{Hom}^{\otimes}((F,\alpha,\beta),(F',\alpha',\beta'))$ the set of all such nat. transf.; also $\mathrm{End}^{\otimes}(F,\alpha,\beta)$ ... .

A tensor category $(\mathcal{C},\otimes,\ldots)$ is called $\otimes$-<u>closed</u>, if $\forall X\in \mathrm{obj}(\mathcal{C})$, the functor

$$\square\otimes X : \mathcal{C}\longrightarrow\mathcal{C}$$

has a right adjoint functor, **denoted** $\underline{\mathrm{Hom}}(X,-)$. So $\forall\, T, Y\in \mathrm{obj}(\mathcal{C})$, there is a nat. ~~iso~~ bijection

$$\mathrm{Hom}(T\otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}(T, \underline{\mathrm{Hom}}(X,Y)).$$

By taking $X=\mathbb{1}$, we have

$$\mathrm{Hom}(T,Y)\xrightarrow{\sim}\mathrm{Hom}(T\otimes\mathbb{1},Y)\xrightarrow{\sim}\mathrm{Hom}(T,\underline{\mathrm{Hom}}(\mathbb{1},Y)),$$

so there is a nat. isom., by Yoneda's lemma:

$$Y\xrightarrow{\sim}\underline{\mathrm{Hom}}(\mathbb{1},Y).$$

Define the <u>dual object</u> $X^\vee$ of X to be $\underline{\mathrm{Hom}}(X,\mathbb{1})$.

- <u>Exercise 2.1.</u> There are nat. isom.

$$\underline{\mathrm{Hom}}(X,\underline{\mathrm{Hom}}(Y,Z))\xrightarrow{\sim}\underline{\mathrm{Hom}}(X\otimes Y, Z)$$

and

$$\mathrm{Hom}(\mathbb{1}, \underline{\mathrm{Hom}}(X,Y)) \xrightarrow{\sim} \mathrm{Hom}(X,Y).$$

We call $\underline{\mathrm{Hom}}$ the <u>internal hom</u>. //.

The identity $id_{X^\vee} : X^\vee \to \underline{\mathrm{Hom}}(X, \mathbb{1})$ gives a morp.

$$ev_X : X^\vee \otimes X \longrightarrow \mathbb{1},$$

the <u>evaluation morphism</u>. The morphism

$$X \otimes X^\vee \xrightarrow{\tau_{X,X^\vee}} X^\vee \otimes X \xrightarrow{ev_X} \mathbb{1}$$

gives the **<u>biduality morphism</u>**

$$i_X : X \longrightarrow X^{\vee\vee}.$$

The morp. (we ignore the issue of γ...)

$$\underline{\mathrm{Hom}}(X_1,Y_1) \otimes \underline{\mathrm{Hom}}(X_2,Y_2) \otimes X_1 \otimes X_2 \xrightarrow[id \otimes \tau \otimes id]{\sim}$$

$$\underline{\mathrm{Hom}}(X_1,Y_1) \otimes X_1 \otimes \underline{\mathrm{Hom}}(X_2,Y_2) \otimes X_2 \xrightarrow[ev_1 \otimes ev_2]{} Y_1 \otimes Y_2$$

induces

$$\underline{\mathrm{Hom}}(X_1,Y_1) \otimes \underline{\mathrm{Hom}}(X_2,Y_2) \xrightarrow{\boxtimes} \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2).$$

A tensor category $(\mathcal{C}, \otimes, \cdots)$ is said to **be <u>rigid</u>**, if it is $\otimes$-closed and a right adjoint $\underline{\mathrm{Hom}}(X,-)$ is specified for each X, and i_X and $\boxtimes$ are isom., $\forall X, X_i, Y_i \in \mathrm{obj}(\mathcal{C})$. Note that as special cases of $\boxtimes$, we then have nat. isom.

$$(X \otimes Y)^\vee \simeq X^\vee \otimes Y^\vee$$

and

$$X^\vee \otimes Y \simeq \underline{\mathrm{Hom}}(X, Y).$$

We also have a nat. morp. $\delta_X : \mathbb{1} \to X^\vee \otimes X$, given by

$$\begin{array}{ccccc} \mathrm{Hom}(X,X) & \simeq & \mathrm{Hom}(\mathbb{1}, \underline{\mathrm{Hom}}(X,X)) & \simeq & \mathrm{Hom}(\mathbb{1}, X^\vee \otimes X) \\ \mathrm{id}_X & & \longleftrightarrow & & \delta_X \end{array}$$

> **Rem**: For $\mathcal{C} = \mathrm{Vec}_k$, $\delta_X : k \to X^\vee \otimes X$ sends 1 to $\sum e_i^* \otimes e_i$, where $\{e_i\}$ is a basis of X, e_i^* = dual basis; this sum is indep. of the choice of $\{e_i\}$, hence δ_X is intrinsically defined.

One verifies that the composite

$$X \xrightarrow{\theta_X} \mathbb{1} \otimes X \xrightarrow{\tau_{\mathbb{1},X}} X \otimes \mathbb{1} \xrightarrow{\mathrm{id} \otimes \delta_X}$$
$$X \otimes X^\vee \otimes X \xrightarrow{\tau_{X,X^\vee} \otimes \mathrm{id}} X^\vee \otimes X \otimes X \xrightarrow{ev_X \otimes \mathrm{id}}$$
$$\mathbb{1} \otimes X \xrightarrow{\theta_X^{-1}} X$$

is id_X, and similarly, the composite

$$X^\vee \xrightarrow{\theta_{X^\vee}} \mathbb{1} \otimes X^\vee \xrightarrow{\tau_{\mathbb{1},X^\vee}} X^\vee \otimes \mathbb{1} \xrightarrow{\mathrm{id} \otimes \delta_X} X^\vee \otimes X^\vee \otimes X \xrightarrow{\mathrm{id} \otimes \tau_{X^\vee,X}}$$
$$X^\vee \otimes X \otimes X^\vee \xrightarrow{ev_X \otimes \mathrm{id}} \mathbb{1} \otimes X^\vee \xrightarrow{\theta_{X^\vee}^{-1}} X^\vee$$

(exercise)

is $\mathrm{id}_{X^\vee}$. Conversely, these two properties characterize the dual object $X^\vee$, ~~when $(\mathcal{C}, \otimes)$ is rigid~~ so one can also define a dual object of X to be $(Y, ev: Y \otimes X \to \mathbb{1}, \delta: \mathbb{1} \to Y \otimes X)$ such that the two composites above are identities, then define $(\mathcal{C}, \otimes, \ldots)$ to be rigid if every object X has a dual object (unique up to unique isom.), and put $\underline{\mathrm{Hom}}(X,Y)$ to be $X^\vee \otimes Y$. Choosing a dual object $X^\vee$ for X amounts to choosing a right adjoint $\overset{H}{\vee}$ for $\Box\otimes X$, namely

$$\Box\otimes X \dashv X^\vee\otimes\Box\ , \quad X^\vee = H(\mathbb{1}).$$

Given a morphism $f : X\to Y$ in the tensor cat. $(\mathcal{C},\otimes,\cdots)$, we have

$$\begin{array}{ccc} \mathrm{Hom}(T\otimes Y,\mathbb{1}) & \longrightarrow & \mathrm{Hom}(T\otimes X,\mathbb{1}) \\ \wr\!\wr & & \wr\!\wr \\ \mathrm{Hom}(T,Y^\vee) & & \mathrm{Hom}(T,X^\vee) \end{array},$$

functorial in T. By Yoneda (i.e. by putting $T=Y^\vee$), we have the <u>dual morphism</u>

$$f^\vee : Y^\vee \to X^\vee.$$

Then

$$\begin{aligned} \mathcal{D} : \mathcal{C}^{op} &\longrightarrow \mathcal{C} \\ X &\longmapsto X^\vee \\ f &\longmapsto f^\vee \end{aligned}$$

becomes a functor, which is an equivalence (and equals its own quasi-inverse) if $\mathcal{C}$ is rigid (exercise); in particular,

$$\mathrm{Hom}(X,Y) \xrightarrow{\ \sim\ } \mathrm{Hom}(Y^\vee,X^\vee).$$

• <u>Prop. 2.2.</u> <u>Let</u> $(\mathcal{C},\otimes,\cdots)$ <u>and</u> $(\mathcal{C}',\otimes',\cdots)$ <u>be rigid tensor categories</u>, <u>and let</u> (F,α,β) <u>and</u> (F',α',β') <u>be tensor functors</u>. <u>Then every morphism</u> $\varphi : F\Rightarrow F'$ <u>of tensor</u>

functors is an isom.. In particular, $\mathrm{End}^{\otimes}(F) = \mathrm{Aut}^{\otimes}(F)$.

Pf: First we prove that $F(X^\vee) \simeq F(X)^\vee$. Precisely, consider

$$F(X^\vee) \otimes' F(X) \xrightarrow{\alpha_{X^\vee, X}} F(X^\vee \otimes X) \xrightarrow{F(ev_X)} F(\mathbb{1}) \xrightarrow{\beta^{-1}} \mathbb{1}'$$

and

$$\mathbb{1}' \xrightarrow{\beta} F(\mathbb{1}) \xrightarrow{F(\delta_X)} F(X^\vee \otimes X^\bullet) \xrightarrow{\alpha^{-1}_{X^\vee, X^\bullet}} F(X^\vee) \otimes' F(X^\bullet).$$

We claim that these two morp. ~~due~~ induce an isom. between $F(X^\vee)$ and $F(X)^\vee$, as the remark on p. 11. This verification (that $F(X) \to \cdots \to F(X)$ is $id_{F(X)}$, and ditto for $F(X^\vee)$) is routine and tedious, which is then omitted here.

The diagram

$$\begin{tikzcd}
F(X^\vee) \otimes' F(X) \arrow[r, "\alpha"] \arrow[d, "\varphi_{X^\vee} \otimes \varphi_X"'] & F(X^\vee \otimes X) \arrow[r, "F(ev_X)"] \arrow[d, "\varphi_{X^\vee \otimes X}"] & F(\mathbb{1}) \arrow[r, "\beta^{-1}"] \arrow[d, "\varphi_{\mathbb{1}}"] & \mathbb{1}' \arrow[d, equal] \\
F'(X^\vee) \otimes' F'(X) \arrow[r, "\alpha'"] & F'(X^\vee \otimes X) \arrow[r, "F'(ev_X)"] & F'(\mathbb{1}) \arrow[r, "\beta'^{-1}"] & \mathbb{1}'
\end{tikzcd}$$

is commutative, inducing a comm. diagram

$$\begin{tikzcd}
F(X^\vee) \arrow[r, "a", "\sim"'] \arrow[d, "\varphi_{X^\vee}"'] & \underline{Hom}(F(X), \mathbb{1}') = F(X)^\vee \\
F'(X^\vee) \arrow[r, "b", "\sim"'] & \underline{Hom}(F'(X), \mathbb{1}') = F'(X)^\vee \arrow[u, "\varphi_X^\vee"']
\end{tikzcd},$$

hence $\varphi_X^\vee \circ b \circ \varphi_{X^\vee} = a$ and $\varphi_{X^\vee} \circ a^{-1} \circ \varphi_X^\vee = b^{-1}$, i.e. $a^{-1} \circ \varphi_X^\vee \circ b$ is the inverse morp. of $\varphi_{X^\vee}$. Since every object in $\mathcal{C}$ is a dual object ($i_X : X \xrightarrow{\sim} X^{\vee\vee}$), we deduce that φ_X is bijective, $\forall X \in obj(\mathcal{C})$. $\square$.

• <u>Example</u>. The category $Proj_R$ of f.g. projective R-modules, endowed with $\otimes_R$, is a rigid **tensor** category.

A k-<u>linear category</u> $\mathcal{C}$ is a category $\mathcal{C}$ together with a f. dim'l k-vector space structure on each Hom set, such that **the composition** law is k-bilinear; **in other** words, it is a cat. $\mathcal{C}$ together with a lifting

$$\begin{tikzcd} & & Vec_k = \{\text{f. dim'l v.s.}/k\} \arrow[d, "\text{forgetful functor}"] \\ \mathcal{C}^{op} \times \mathcal{C} \arrow[urr, dashed] \arrow[rr, "Hom"'] & & \underline{Sets} \end{tikzcd}.$$

A k-<u>linear tensor category</u> is a tensor category $(\mathcal{C}, \otimes, \cdots)$ which is **endowed** with a k-linear structure **such that** $\otimes$ is k-bilinear (i.e. $h \otimes (f_1 + f_2) = \cdots$).

• <u>Def. 2.3</u>. A <u>neutral Tannakian category over</u> k is an abelian k-linear rigid tensor category $\mathcal{C}$, **such that** there exists a faithful exact **k-linear** tensor functor

$$\omega : \mathcal{C} \longrightarrow Vec_k ,$$

called a <u>fiber functor</u>. A <u>neutralized Tannakian category</u> is such a pair $(\mathcal{C}, \omega)$.

• <u>Examples 2.3.1.</u>

— Let G be an affine k-group. Then $\mathrm{Rep}(G)$, the cat. of f. dim'l G-rep. (V, ρ), with

$$\omega^G : \mathrm{Rep}(G) \longrightarrow \mathcal{V}ec_k$$
$$(V,\rho) \longmapsto V \quad ,$$

is a neutralized Tann. cat..

— Let Γ be an abstract group. Then $\mathrm{Rep}_k(\Gamma)$, the cat. of f. dim'l k-linear rep. of Γ, with $\omega(V,\rho) = V$, is a neutr. Tann. cat..

— Let X be a connected top. space, and let $\mathcal{L}oc_k(X)$ be the cat. of k-linear local systems on X (i.e. locally constant sheaves $\mathcal{F}$ of f. dim'l k-vector spaces on X). Then $\mathcal{L}oc_k(X)$ is an abelian k-linear rigid tensor cat., and for each point $x \in X$, taking the fiber at x defines a fiber functor

$$\omega_x : \mathcal{L}oc_k(X) \longrightarrow \mathcal{V}ec_k$$
$$\mathcal{F} \longmapsto \mathcal{F}_x \, ,$$

hence the name "fiber functor". We have an equivalence of abelian k-linear tensor cat. compatible with their fiber functors :

$$\begin{tikzcd} \mathcal{L}oc_k(X) \arrow[rr, "\sim"] \arrow[dr, "\omega_x"'] & & \mathrm{Rep}_k(\pi_1(X,x)) \arrow[dl, "\omega"] \\ & \mathcal{V}ec_k & \end{tikzcd}$$

— Let K be a top. group (e.g. a Galois group with profinite topology, or a Lie group), and let $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{R}}(K)$ be the cat. of f. dim'l continuous $\mathbb{R}$-linear rep. of K. Then it is a neut. Tann. cat. over $\mathbb{R}$. One can replace $\mathbb{R}$ with other fields with norms, e.g. $\mathbb{C}$, $\mathbb{Q}_p$.

— Let $\mathbb{Z}$-<u>Gr. Vec</u>$_k$ be the cat. of f. dim'l $\mathbb{Z}$-graded k-vector spaces, with degree preserving k-linear maps:

$$V, \quad (V_n)_{n\in\mathbb{Z}}, \quad V = \bigoplus_n V_n .$$

Define $(V\otimes W)_n := \bigoplus_i (V_i \otimes W_{n-i})$, $\mathbb{1} := k$ in degree 0, and $\underline{\mathrm{Hom}}((V,(V_n)),(W,(W_n))) := \mathrm{Hom}_k(V,W)$ with degree n part the k-linear maps $f: V\to W$ s.t. $f(V_i)\subset W_{i+n}, \forall i$. With the fiber functor $(V,(V_n)) \mapsto V$, $\mathbb{Z}$-<u>Gr. Vec</u>$_k$ is a neut. Tann. cat..

One can replace $\mathbb{Z}$ with any commutative group M, and consider M-graded vector spaces of f. dim. M-<u>Gr. Vec</u>$_k$.

We need M to be comm. in order for $\otimes$ to be symm.:

$$(V \otimes W)_m := \bigoplus_{\substack{m_1, m_2 \in M \\ m_1 m_2 = m}} (V_{m_1} \otimes W_{m_2}).$$

—— Let $MHS_{\mathbb{R}}$ be the cat. of $\mathbb{R}$- <u>mixed Hodge structures</u> $(V, W_\bullet, F^\bullet)$, i.e. a f. dim'l $\mathbb{R}$-vector space V, with an increasing filtration $W_\bullet$ on V and a decreasing filtration $F^\bullet$ on $V_{\mathbb{C}}$, s.t. $\forall\, n, p \in \mathbb{Z}$, we have

$$Gr_n^W(V_{\mathbb{C}}) = F^p Gr_n^W(V_{\mathbb{C}}) \oplus \overline{F^{n+1-p} Gr_n^W(V_{\mathbb{C}})}.$$

Define for $(V, W_\bullet, F^\bullet)$ and $(U, W_\bullet, F^\bullet)$,

$$W_n(V \otimes U) = \sum_i W_i(V) \otimes W_{n-i}(U), \quad \text{on } V \otimes U,$$

$$F^p(V_{\mathbb{C}} \otimes_{\mathbb{C}} U_{\mathbb{C}}) = \sum_i F^i(V_{\mathbb{C}}) \otimes_{\mathbb{C}} F^{p-i}(U_{\mathbb{C}}), \quad \text{on } (V \otimes U)_{\mathbb{C}},$$

$\underline{Hom}(V, U) := Hom_{\mathbb{R}}(V, U)$ with

$$W_n \underline{Hom} := \{ f : V \to U \mid f(W_i(V)) \subset W_{n+i}(U), \forall i \}$$

and

$$F^p \underline{Hom}_{\mathbb{C}} := \{ f : V_{\mathbb{C}} \to U_{\mathbb{C}} \mid f(F^i(V_{\mathbb{C}})) \subset F^{p+i}(U_{\mathbb{C}}), \forall i \},$$

and $\mathbb{1} := \mathbb{R}(0)$. Then equipped with $(V, W_\bullet, F^\bullet) \mapsto V$, $MHS_{\mathbb{R}}$ becomes a neut. Tann. cat. over $\mathbb{R}$. One can also replace $\mathbb{R}$ with $\mathbb{Q}$, obtaining a cat. $MHS_{\mathbb{Q}}$ that is much more complicated than $MHS_{\mathbb{R}}$.

• Th. 2.4. Let $\mathcal{C}$ be a k-linear abelian category, and let $\omega : \mathcal{C} \to \mathcal{V}ec_k$ be a faithful k-linear exact functor. Then there is a coassociative k-coalgebra B with coidentity, and a k-linear equivalence of $\mathcal{C}$ and the category $\mathrm{Comod}(B)^f$ of right B-comodules over k of finite dimension, making a (2-) commutative diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "\sim"] \arrow[dr, "\omega"'] & & \mathrm{Comod}(B)^f \arrow[dl, "\text{forgetful functor}"] \\
& \mathcal{V}ec_k &
\end{tikzcd}
$$

Pf: Some basic observations:

~~— $\omega(0) = 0$ (since the zero object in an abelian category is characterized to be an object whose identity morp. is zero);~~

~~— given a morp. f in $\mathcal{C}$,~~

— for $X \in \mathrm{obj}(\mathcal{C})$, we have $X = 0 \Leftrightarrow \omega(X) = 0$ (since the zero object in an abelian category is characterized to be an object whose identity morp. is zero);

— given a morp. f in $\mathcal{C}$, we have that f is injective (resp. surjective) if and only if $\omega(f)$ is so;

— given a subobject $Y \subset X$ in $\mathcal{C}$, we see from the above that $Y \subsetneq X \Leftrightarrow \omega(Y) \subsetneq \omega(X)$; consequently, each object in $\mathcal{C}$ has finite length.

The theorem ~~means~~ asks that we shall find some algebra (or coalgebra) that acts (or coacts) on the vector spaces $\omega(X)$, functorially in X. For an "object" T, the most natural algebra that operates on it is $\mathrm{End}(T)$, so this leads us to consider $\mathrm{End}(\omega)$; but this algebra may not be finite dim'l, a property which is needed to endow a coalgebra structure on its dual space (in fact, ω is only "pro-representable" in general, by a theorem of Grothendieck), so we turn to "$\mathrm{End}(\omega|_{\langle X\rangle})$", for various objects X in $\mathcal{C}$.

An object $X \in \mathrm{obj}(\mathcal{C})$ is called <u>cyclic</u> (not standard) if $\exists\, x \in \omega(X)$ such that $\forall$ subobject $Y \subset X$, $x \in \omega(Y)$ only if $Y = X$. Let $\langle X\rangle$ be the full subcategory of $\mathcal{C}$ consisting of objects isomorphic to subquotients of X^n, $n \in \mathbb{N}$.

• <u>Lem. 2.4.1.</u> $\dim_k \omega(Y)$, <u>for</u> Y <u>a cyclic object in</u> $\langle X\rangle$ (<u>with</u> X <u>a fixed object of</u> $\mathcal{C}$), <u>is bounded</u>.

<u>Pf</u>: We may assume that $Y \subset X^m$ for some m. In fact, let $Y = Y_1/Y_2$ be generated by $y \in \omega(Y)$, with $Y_1 \subset X^m$ for some m, and let $y_1 \in \omega(Y_1)$ be a lifting of y. Let Y' be the subobject of Y_1 generated by y_1 (i.e. intersection of all ...,

this is a finite intersection, since Y_1 has finite length).
Then Y' is cyclic, and $Y' \hookrightarrow Y_1 \to Y$ is surjective, so that $\dim_k \omega(Y') \geq \dim_k \omega(Y)$, and it suffices to bound $\dim_k \omega(Y')$.

Let $n = \dim_k \omega(X)$. If Y can be embedded into X^m for some $m > n$, write $y \in \omega(Y) \subset \omega(X^m) = \omega(X)^m$ **as**

$$y = (y_1, \cdots, y_m), \quad y_i \in \omega(X).$$

As $\dim_k \omega(X) = n < m$, these y_i's **are k-linearly dependent**: $\sum a_i y_i = 0$ for some $a_i \in k$, not all zero.
Since End(X) is a k-~~vector space~~ algebra (associative), the a_i's define a morp.

$$(a_1, \cdots, a_m): X^m \to X,$$

whose kernel X', which is isom. to X^{m-1} **(exercise)**, **satisfies** that $y \in \omega(X')$. Then $Y \subset X'$ (by considering $Y \cap X'$ in X^m). Repeating the process, we see that Y is embeddable into X^n, and therefore

$$\dim_k \omega(Y) \leq \dim_k \omega(X^n) = n^2. \quad \square.$$

Now let $P_X \in \mathrm{obj}\langle X \rangle$ be a **cyclic object** with **maximal** $\dim_k \omega(P_X)$, generated by $p \in \omega(P_X)$.

- <u>Lem. 2.4.2</u>. <u>The pair</u> (P_X, p) <u>represents</u> $\omega|_{\langle X \rangle}$.

Pf. The element p defines a natural transformation (by Yoneda)

$$\mathrm{Hom}(P_X,-) =: h'_{P_X} \Longrightarrow \omega|_{\langle X\rangle} :$$

$$h'_{P_X}(Y) \longrightarrow \omega(Y)$$

$$(f: P_X \to Y) \longmapsto \omega(f)(p).$$

$\forall\, y \in \omega(Y)$, to construct a morp. $P_X \to Y$, we first construct its "graph": let Z be the subobject of $P_X \times Y$ generated by $(p, y) \in \omega(P_X) \times \omega(Y) \simeq \omega(P_X \times Y)$. Then the projection $Z \to P_X$ is surjective, hence

$$\dim_k \omega(Z) \geqslant \dim_k \omega(P_X);$$

but since Z is also cyclic, we must have an equality, by the choice of P_X, forcing $Z \to P_X$ to be an isom.. Then $P_X \xleftarrow{\simeq} Z \to Y$ is a morp. sending p to y after ω is applied, proving the surjectivity. If f_1 and f_2 are two morp. $P_X \to Y$ with $\omega(f_i)(p) = y$, $i=1,2$, then $p \in \boxed{\omega(\ker(f_1 - f_2))} \boxed{=} \boxed{\ker(\omega(f_1) - \omega(f_2))}$, so $\ker(f_1 - f_2) = P_X$, i.e. $f_1 = f_2$. $\square$.

Let A_X be the associative k-algebra $\mathrm{End}(P_X)^{op}$, or equivalently by Yoneda, $A_X = \mathrm{End}(\underset{\text{on } \langle X\rangle}{\underset{\downarrow}{h'_{P_X}}}) = \mathrm{End}(\omega|_{\langle X\rangle})$.

Each $\omega(Y) = \mathrm{Hom}(P_X, Y)$ (for $Y \in \mathrm{obj}\langle X\rangle$) has a right action by $\mathrm{End}(P_X)$:

$$f, a \longmapsto f\circ a, \quad f \in \mathrm{Hom}(P_X, Y),\ a \in \mathrm{End}(P_X),$$

so A_X acts on $\omega(Y)$ from the left. This defines a k-linear **functor** (clearly exact, **as** ω is)

$$\langle X\rangle \longrightarrow A_X\text{-}\mathrm{Mod}^f$$

where $A_X\text{-mod}^f$ is the cat. of **left** A_X-**modules** over k of finite dimension.

- Lem. 2.4.3. This functor is an equivalence.

Pf: Essential surjectivity: For $V \in \mathrm{obj}.(A_X\text{-}\mathrm{Mod}^f)$, it is of finite presentation

$$A_X^m \xrightarrow{\varphi_M} A_X^n \xrightarrow{\pi} V \to 0$$

$\boxed{(a_1, \cdots, a_m) \overset{\varphi_M}{\longmapsto} (a_1,\cdots,a_m)\cdot M \text{ is left } A_X\text{-linear.}}$

(note that $\dim_k(A_X) < \infty$, so $\dim_k \ker(\pi) < \infty$, **hence** can be **mapped onto** by some A_X^m), **where** φ_M is **defined** by a **matrix** $M \in M_{\cancel{n\times m}\, m\times n}(A_X)$. As

$$\mathrm{Hom}(P_X^m, P_X^n) = M_{\cancel{n\times m}\, m\times n}(\mathrm{End}(P_X)),$$

this matrix M **also gives rise** to a morp. $\psi_M : P_X^m \to P_X^n$; let $Y = \mathrm{coker}(\psi_M)$. Applying ω, we see that

$$\cong \cancel{V \otimes_{A_X} P} \ P_X \otimes_{A_X} V$$

$\omega(P_X) = h'_{P_X}(P_X) = A_X$ (as left A_X-modules) and $\omega(\psi_M) = \varphi_M$, so by exactness of ω, we have

$$V \simeq \omega(Y) \quad \text{as } A_X\text{-modules}.$$

<u>Full faithfulness</u>: It is clearly faithful, as ω is.

To see $\quad \mathrm{Hom}_{\langle X\rangle}(Y, Z) \longrightarrow \mathrm{Hom}_{A_X}(\omega(Y), \omega(Z))$

is surjective, let $f : \omega(Y) \to \omega(Z)$ be a left A_X-linear map. Then it fits in a comm. diagram

$$\begin{tikzcd}
A_X^m \arrow[r, "\varphi_{M_1}"] \arrow[d, "\varphi_{M_4}"] & A_X^n \arrow[r] \arrow[d, "\varphi_{M_2}"] & \omega(Y) \arrow[r] \arrow[d, "f"] & 0 \\
A_X^s \arrow[r, "\varphi_{M_3}"] & A_X^r \arrow[r] & \omega(Z) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. So

$$\begin{tikzcd}
P_X^m \arrow[r, "\psi_{M_1}"] \arrow[d, "\psi_{M_4}"] & P_X^n \arrow[r] \arrow[d, "\psi_{M_2}"] & \mathrm{coker}(\psi_{M_1}) \arrow[r] \arrow[d, dashed, "\exists! \tilde{f}"] & 0 \\
P_X^s \arrow[r, "\psi_{M_3}"] & P_X^r \arrow[r] & \mathrm{coker}(\psi_{M_3}) \arrow[r] & 0,
\end{tikzcd}$$

~~then we the~~ and $\omega(\tilde{f}) = f$. It remains to show that Y is canonically isom. to $P_X \otimes_{A_X} \omega(Y)$. In fact, $V \mapsto P_X \otimes_{A_X} V$ is left adjoint to $Y \mapsto \mathrm{Hom}(P_X, Y)$:

$$P_X \otimes_{A_X} V \to Y \text{ in } \mathcal{C} \xrightarrow{\ \omega\ } V \to \mathrm{Hom}(P_X, Y) \text{ in } A_X\text{-Mod}^f$$

$$\begin{tikzcd} P_X^m \arrow[r] \arrow[drr, "0"'] & P_X^n \arrow[r] \arrow[dr, "(f_1,\dots,f_n)"] & P_X \otimes_{A_X} V \arrow[r] \arrow[d, dashed] & 0 \\ & & Y & \end{tikzcd} \longleftarrow V \to \mathrm{Hom}(P_X, Y)$$

$$\underbrace{v_1,\dots,v_n}_{A_X\text{-generators}} \mapsto f_i : P_X \to Y$$

$$A_X^m \to A_X^n \to V$$
$$e_i \mapsto v_i \quad .$$

In particular, $id : V = \mathrm{Hom}(P_X, Y) \to \mathrm{Hom}(P_X, Y)$ corresponds to a morp. $P_X \otimes_{A_X} V \to Y$, which is an isom. after applying ω, hence it is an isom. (by looking at its kernel and cokernel). $\square$.

• Exercise 2.4.4. Let A be a finite dim'l assoc. k-algebra, with $A^\vee$ its dual vector space with the induced coasso. coalgebra structure; let V be a ~~finite dim'l~~ k-vector space. Consider the adjunction

$$\mathrm{Hom}_k(A \otimes V, V) \longleftrightarrow \mathrm{Hom}_k(V, V \otimes A^\vee)$$
$$\varphi \longleftrightarrow \psi \text{ if } A \otimes V \xrightarrow[\simeq]{\tau} V \otimes A \xrightarrow{\psi \otimes id} V \otimes A^\vee \otimes A$$
$$\xrightarrow{id \otimes ev} V \otimes k \simeq V \text{ is } \varphi.$$

Show that φ is a left A-module structure on V if and only if ψ is a right $A^\vee$-comodule structure on V. //.

It follows that

$$\begin{tikzcd} \langle X \rangle \arrow[rr, "\sim"] \arrow[dr, "\omega|\langle X \rangle"'] & & \mathrm{Comod}(A_X^\vee)^f \arrow[dl] \\ & \mathrm{Vec}_k & \end{tikzcd}$$

For $X, Y \in \mathrm{Obj}(\mathcal{C})$, define $X \le Y$ if $X \in \langle Y \rangle$.

This defines a partial ordering on the isom. classes of objects in $\mathcal{C}$. For $X \leq Y$, we have a restriction hom.

$$\begin{array}{ccc} \mathrm{End}(\omega|\langle Y\rangle) & \longrightarrow & \mathrm{End}(\omega|\langle X\rangle) \\ \| & & \| \\ A_Y & & A_X \end{array},$$

inducing a hom. of coalgebras $A_X^\vee \to A_Y^\vee$. Define

$$B := \underset{[X] = \text{isom. class}}{\mathrm{colim}} \{A_X^\vee\};$$

it has a natural k-coalgebra structure. $\forall X \in \mathrm{obj}(\mathcal{C})$, every right $A_X^\vee$-comodule V becomes a right B-comodule naturally

$$V \longrightarrow V \otimes A_X^\vee \longrightarrow V \otimes B;$$

conversely, every right B-comodule V of f. dim. comes from an $A_X^\vee$-comodule: take basis $v_1, \cdots, v_n$ of V, and write $\sigma(v_j) = \sum_i v_i \otimes b_{ij}$, $b_{ij} \in B$. Let X be "large" enough (note that $X, Y \leq X \oplus Y$) so that all $b_{ij} \in A_X^\vee$; then σ factors through $V \otimes A_X^\vee$. We have that

$$\mathcal{C} = \underset{[X]}{2\text{-}\mathrm{colim}} \langle X \rangle, \quad \mathrm{Comod}(B)^f = \underset{[X]}{2\text{-}\mathrm{colim}}\, \mathrm{Comod}(A_X^\vee)^f,$$

hence

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\sim"] \arrow[dr, "\omega"'] & & \mathrm{Comod}(B)^f \arrow[dl] \\ & \mathrm{Vec}_k & \end{tikzcd} \qquad \square.$$

• <u>Rem. 2.5.</u> Let A be a f. dim'l assoc. k-algebra, with $\omega : A\text{-Mod}^f \to Vec_k$ the forgetful functor. Each $a \in A$ defines by homothety an endomorphism of ω. Conversely, $\forall\ \alpha \in \mathrm{End}(\omega)$, consider A as a left A-module and let $a = \alpha_A(1)$. Then $\forall\ V \in A\text{-Mod}^f$, the action map

$$(a, v) \mapsto a.v : A \otimes_k \omega(V) \longrightarrow V$$

is A-linear, hence

$$\begin{tikzcd}
A \otimes \omega(V) \arrow[r] \arrow[d, "\alpha_{A\otimes\omega(V)} = \alpha_A \otimes \mathrm{id}_{\omega(V)}"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\alpha_V"] \\
A \otimes \omega(V) \arrow[r] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
1\otimes v \arrow[r, mapsto] \arrow[d, mapsto] & v \arrow[d, mapsto] \\
a\otimes v \arrow[r, mapsto] & \alpha_V(v) = av
\end{tikzcd},$$

so α_V is the homothety a on V. This shows that

$$A \xrightarrow{\sim} \mathrm{End}(\omega).$$

Let B be a coassoc. k-coalgebra with coidentity. The regular comodule B is the directed union of its f. dim'l subcomodules V_i (I, 2.6), and each V_i is contained in a f. dim'l sub-coalgebra B_i (choose a basis for V_i, and let B_i be the k-vector space generated by the coef. "b_{ij}"). Let $\omega : \mathrm{Comod}(B)^f \to Vec_k$ be the forgetful functor. As $\mathrm{Comod}(B)^f = 2\text{-}\operatorname{colim}_i \{\mathrm{Comod}(B_i)^f\} = 2\text{-}\operatorname{colim}_i \{B_i^\vee\text{-Mod}^f\}$, and A_X-Mod^f, for $X \in \mathrm{Comod}(B)^f$, is cofinal with $B_i^\vee$-Mod^f, and $A_X \simeq \mathrm{End}(\omega|_{A_X\text{-}\mathrm{Mod}^f}) = \mathrm{End}(\omega|_{\langle X\rangle})$, we see that

$$B = \operatorname*{colim}_{[X]} \{\mathrm{End}(\omega|_{\langle X\rangle})^\vee\}.$$

In particular, the coalgebra B in (2.4) is uniquely determined by $(\mathcal{C}, \omega)$ up to unique isom.. This observation also leads to the following functoriality.

• Prop. 2.5.1. Let B and B' be coassoc. k-coalgebras with coidentity, and let $\varphi : \mathrm{Comod}(B)^f \to \mathrm{Comod}(B')^f$ be a (k-linear) functor such that

$$\begin{tikzcd} \mathrm{Comod}(B)^f \arrow[rr, "\varphi"] \arrow[dr, "\omega"'] & \circlearrowleft & \mathrm{Comod}(B')^f \arrow[dl, "\omega'"] \\ & \mathrm{Vec}_k & \end{tikzcd}.$$

Then there is a unique hom. $f : B \to B'$ of coalgebras inducing φ.

Pf: $\forall X \in \mathrm{Comod}(B)^f$, we have

$$\begin{tikzcd} \langle X\rangle \arrow[rr, "\varphi|_{\langle X\rangle}"] \arrow[dr, "\omega|_{\langle X\rangle}"'] & & \langle \varphi(X)\rangle \arrow[dl, "\omega'|_{\langle\varphi(X)\rangle}"] \\ & \mathrm{Vec}_k & \end{tikzcd},$$

so we have

$$\mathrm{End}(\omega'|_{\langle\varphi(X)\rangle}) \longrightarrow \mathrm{End}(\omega|_{\langle X\rangle}) \quad \rightsquigarrow$$

$$B' = \operatorname*{colim}_{[X']} \mathrm{End}(\omega'|_{\langle X'\rangle})^\vee \longleftarrow A^\vee_{\varphi(X)} \longleftarrow A^\vee_X \quad \rightsquigarrow$$

$$B \longrightarrow B'. \qquad \square.$$

- <u>Lem. 2.6.</u> <u>Let</u> $\mathcal{C} = \mathrm{Comod}(B)^f \times \mathrm{Comod}(B')^f$, <u>with</u>

$$\omega: \mathcal{C} \longrightarrow \mathcal{V}ec_k$$

<u>defined by</u> $\omega(V, V') = V \otimes V'$. <u>Then the associated coalgebra in</u> (2.4) <u>is</u> $B \otimes B'$.

<u>Pf</u>: Let ω^B and $\omega^{B'}$ be forgetful functors on $\mathrm{Comod}(B)^f$ and $\mathrm{Comod}(B')^f$. For f. dim'l k-vector spaces V, W,

$$f, g \longmapsto f \otimes g : \mathrm{End}_k(V) \otimes \mathrm{End}_k(W) \longrightarrow \mathrm{End}_k(V \otimes W)$$

is an isom. (by showing injectivity and comparing dim.). For $V \in \mathrm{Comod}(B)^f$, $V' \in \mathrm{Comod}(B')^f$, we have

$$\mathrm{End}(\omega^B|_{\langle V\rangle}) \otimes \mathrm{End}(\omega^{B'}|_{\langle V'\rangle}) \xrightarrow{\sim} \mathrm{End}(\omega|_{\langle (V,V')\rangle}),$$

because $\langle (V, V')\rangle = \langle V\rangle \times \langle V'\rangle$:

$$\begin{array}{l} \quad\quad\quad\quad\ \hookrightarrow (V, V')^n = (V^n, V'^n) \\ (Y, Y') \\ \quad\quad\quad\quad\ \twoheadrightarrow (Y/Z, Y'/Z'), \end{array}$$

and both sides give, for each $X \in \langle V\rangle$, $X' \in \langle V'\rangle$, a k-linear map $X \otimes X' \to X \otimes X'$, functorial in X and X'. So

$$A_{(V,V')} = A_V \otimes A_{V'},$$

$$A^\vee_{(V,V')} = A^\vee_V \otimes A^\vee_{V'},$$

and taking colimit commutes with tensor product. $\square$.

- Cor. 2.6.1. Every k-bilinear functor

$$\begin{tikzcd}
\mathrm{Comod}(B)^f \times \mathrm{Comod}(B)^f \arrow[rr] \arrow[dr, "\omega^B \otimes \omega^B"'] & \circlearrowleft & \mathrm{Comod}(B)^f \arrow[dl, "\omega^B"] \\
& Vec_k &
\end{tikzcd}$$

is induced by a unique k-coalgebra hom. $B \otimes B \xrightarrow{m} B$. $\square$.

- Lem. 2.7. Let $\mathcal{C} = \mathrm{Comod}(B)^f$, and let

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[rr, "\otimes"] \arrow[dr, "\omega^B \otimes \omega^B"'] & \circlearrowleft & \mathcal{C} \arrow[dl, "\omega^B"] \\
& Vec_k &
\end{tikzcd}$$

be a k-bilinear functor, with $m: B \otimes B \to B$ the assoc. hom. of coalgebras. Then $\otimes \circ (\otimes \times id_{\mathcal{C}})$ and $\otimes \circ (id_{\mathcal{C}} \times \otimes)$ in $\mathrm{Func}(\mathcal{C} \times \mathcal{C} \times \mathcal{C}, \mathcal{C})$ are naturally isomorphic if and only if $m \circ (m \otimes id_B) = m \circ (id_B \otimes m) : B \otimes B \otimes B \to B$.

Pf: $\otimes \circ (\otimes \times id_{\mathcal{C}})$ and $\otimes \circ (id_{\mathcal{C}} \times \otimes)$ are induced by $m \circ (m \otimes id_B)$

and $m\circ(id_B\otimes m)$, resp., so the result follows from the uniqueness in (2.5.1). $\square$

Similarly, if τ denotes transposition (on $\mathcal{C}\times\mathcal{C}$ or $B\otimes B$), then $\otimes \simeq \otimes\circ\tau$ in $\mathrm{Fun}(\mathcal{C}\times\mathcal{C},\mathcal{C})$ if and only if $m = m\circ\tau : B\otimes B \longrightarrow B$.

Note that $\mathrm{Comod}(k)^f = \mathcal{V}ec_k$:
$$\begin{tikzcd} V \arrow[r, "\sigma"] \arrow[dr, "id_V"'] & V\otimes k \arrow[d, "id\otimes\varepsilon"] \\ & V \end{tikzcd}$$
$\Rightarrow \sigma = id_V$. So a k-linear functor

$$\mathcal{U} : \mathrm{Comod}(k)^f \longrightarrow \mathcal{C} \quad (= \text{any } k\text{-linear cat.})$$

is determined by $\mathbb{1} := \mathcal{U}(k)$. For $\mathcal{C} = \mathrm{Comod}(B)^f$, the composite

$m : B\otimes B \to B$

$$\begin{array}{ccccccc} \mathcal{C} & \longrightarrow & \mathcal{C}\times\mathrm{Comod}(k)^f & \xrightarrow{id_{\mathcal{C}}\times\mathcal{U}} & \mathcal{C}\times\mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\ V & \longmapsto & (V, k) & \longmapsto & (V, \mathbb{1}) & \longmapsto & V\otimes\mathbb{1} \end{array}$$

is isom. to $id_{\mathcal{C}}$ if and only if

$$\begin{tikzcd} B \arrow[r] \arrow[rrr, bend right, "id"'] & B\otimes k \arrow[r, "id\otimes e"] & B\otimes B \arrow[r, "m"] & B \end{tikzcd}$$

commutes, where $e : k \to B$ is the coalg. hom. corresponding to $\mathcal{U}$.

Finally, we are prepared to prove the main result of this section.

(the other way)

• <u>Th. 2.8.</u> (Tannakian duality). <u>Let</u> $(\mathcal{C}, \omega)$ <u>be a neutralized Tannakian category over</u> k. <u>Then there is an affine</u> k-<u>group scheme</u> G <u>and</u> ~~an~~ <u>a</u> k-<u>linear equivalence of neutr. Tann. categories</u>

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\sim"', "F"'] \arrow[dr, "\omega"'] & & \mathrm{Rep}(G) \arrow[dl, "\omega^G"] \\ & \mathcal{V}ec_k & \end{tikzcd}.$$

<u>Pf</u>: By (2.4), there is a coalgebra (B, Δ, ε) over k and a k-linear equiv.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\sim"', "F"'] \arrow[dr, "\omega"'] & & \mathrm{Comod}(B)^f \arrow[dl, "\omega^B"] \\ & \mathcal{V}ec_k & \end{tikzcd}.$$

The tensor structure on $\mathcal{C}$ ~~transforms~~ leads to a tensor structure on $\mathrm{Comod}(B)^f$:

$$\begin{tikzcd} \mathrm{Comod}(B)^f \times \mathrm{Comod}(B)^f \arrow[r, "\otimes"] \arrow[d, "\omega^B \times \omega^B"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Comod}(B)^f \arrow[d, "\omega^B"] \\ \mathcal{V}ec_k \times \mathcal{V}ec_k \arrow[r, "\otimes"] & \mathcal{V}ec_k \end{tikzcd},$$

(ω is a tensor functor $\Rightarrow$ 2-comm. of)

giving rise to a multiplication $m: B \otimes B \to B$. The associativity, commutativity constraints and the identity object $\mathbb{1}$ in $\mathcal{C}$ implies that m is associative,

commutative with identity $e : k \to B$ (given by the k-linear functor $\mathcal{U} : \mathrm{Comod}(k)^f \to \mathcal{C} \xrightarrow{\sim} \mathrm{Comod}(B)^f$),
$$k \longmapsto \mathbb{1}$$
by (2.7) and the comments after it. Note that m and e are both hom. of coalgebras.

• <u>Exercise 2.8.1</u>. Given a coasso. coalgebra with coidentity (B, Δ, ε), and an assoc. algebra structure with identity on the same underlying vector space (B, m, e), show that Δ and ε are algebra hom. if and only if m and e are coalgebra hom., both being equiv. to the commutativity of

$$\begin{tikzcd}
B\otimes B \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & B \arrow[r, "\Delta"] & B\otimes B \\
B\otimes B\otimes B\otimes B \arrow[rr, "id\otimes\tau\otimes id"'] & & B\otimes B\otimes B\otimes B \arrow[u, "m\otimes m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, "\sim"] \arrow[d, "e"'] & k\otimes k \arrow[d, "e\otimes e"] \\
B \arrow[r, "\Delta"] & B\otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
B\otimes B \arrow[r, "m"] \arrow[d, "\varepsilon\otimes\varepsilon"'] & B \arrow[d, "\varepsilon"] \\
k\otimes k \arrow[r, "\sim"] & k
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \arrow[dr, "\varepsilon"] & \\
k \arrow[ur, "e"] \arrow[rr, "id"'] & & k
\end{tikzcd} . \quad //.$$

Therefore, $G := \mathrm{Spec}(B)$, endowed with Δ and ε, becomes a monoid scheme over k (i.e. a representative functor from k-Alg. to {monoids}), and $\mathrm{Rep}(G) \simeq \mathrm{Comod}(B)^f$ becomes a k-linear abelian tensor category, and the equivalence F is (by definition) a tensor functor. It remains to show that G is a group scheme, which will follow from that $\mathcal{C}$, and hence $\mathrm{Rep}(G)$, is rigid.

Back to the situation in (2.5) for a moment:

$$A\text{-Mod}^f \xrightarrow{\omega^A} Vec_k, \quad A = \text{f. dim'l assoc. } k\text{-algebra.}$$

One can prove verbatim that $\forall\, R \in k\text{-Alg.}$ (commutative), the natural map, "homothety":

$$A \underset{k}{\otimes} R \longrightarrow End(\omega_R^A),$$

where ω_R^A is the composite $\phi_R \circ \omega^A$:

$$\begin{array}{ccccc} A\text{-Mod}^f & \xrightarrow{\omega^A} & Vec_k & \xrightarrow{\phi_R} & Proj_R \\ (V,\sigma) & \longmapsto & V & \longmapsto & V_R \end{array},$$

is an isom. of R-algebras.

Now consider

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\omega"] & Vec_k \arrow[r, "\phi_R"] & Proj_R \\ \langle X \rangle \arrow[u, hook] \arrow[r, phantom, "\underset{(2.4.3)}{\simeq}"] & A_X\text{-Mod}^f \arrow[u, "\omega^{A_X}"] \arrow[ur, "\omega_R^{A_X}"'] & \end{tikzcd} \qquad (\omega_R := \phi_R \circ \omega).$$

We have

$$End(\omega_R) \overset{\mathcal{C} \overset{?}{=} \mathrm{colim}\langle X\rangle}{=} \varprojlim_{[X]} End(\omega_R|_{\langle X\rangle}) = \varprojlim_{[X]} End(\omega_R^{A_X})$$

$$= \varprojlim_{[X]} (A_X \underset{k}{\otimes} R) = \varprojlim_{[X]} Hom_k(A_X^\vee, R) = Hom_k(\mathrm{colim}\, A_X^\vee, R) = Hom_k(B, R).$$

One verifies that the endomorphism of ω_R corresponding to the k-linear map $f: B \to R$ associates to $V \in \mathcal{C} \simeq Comod(B)^f$

the R-endomorphism of V_R obtained from

$$V \xrightarrow{\sigma\ \text{(coaction)}} V \otimes B \xrightarrow{id \otimes f} V \otimes R$$

by extending scalars.

<u>Claim</u>: the nat. endomorphism of ω_R corresp. to $B \xrightarrow{f} R$ is a morp. of tensor functors (cf. p. 8) if and only if f is a hom. of k-algebras.

$\forall\ (V, \sigma), (W, \pi) \in \mathrm{Comod}(B)^f$, consider the diagrams

$$\begin{tikzcd}
V \otimes W \arrow[d, "\sigma \otimes \pi"'] \arrow[dr, "((id_V \otimes f)\circ\sigma)\otimes((id_W\otimes f)\circ\pi)"] \arrow[dr, phantom, shift right=4, "\circlearrowright"] & \\
V \otimes W \otimes B \otimes B \arrow[r, "id_{V\otimes W} \otimes f \otimes f"'] \arrow[d, "id_{V\otimes W} \otimes m"'] & V \otimes W \otimes R \otimes R \arrow[d, "id_{V\otimes W} \otimes m_R"] \\
V \otimes W \otimes B \arrow[r, "id_{V\otimes W} \otimes f"'] & V \otimes W \otimes R
\end{tikzcd}$$

$$\begin{tikzcd}
 & k \otimes R = \omega_R^B(k) \arrow[d, "e \otimes id_R"] \\
 & B \otimes R \arrow[d, "f \otimes id_R"] \\
R \arrow[uur, "\sim"] \arrow[dr, "\sim"'] & R \otimes R \arrow[d, "m_R"] \\
 & k \otimes R = \omega_R^B(k)
\end{tikzcd}$$

$$\begin{tikzcd}
1_R \arrow[r, mapsto] \arrow[dddd, mapsto] & 1_k \otimes 1_R \arrow[d, mapsto] \\
 & 1_B \otimes 1_R \arrow[d, mapsto] \\
 & f(1_B) \otimes 1_R \arrow[d, mapsto] \\
 & f(1_B) \\
1_R \arrow[ur, phantom, "\overset{?}{=}"] &
\end{tikzcd}$$

The second diagram commutes if and only if $f(1_B) = 1_R$ (being R-linear, it suffices to trace 1_R); the first one commutes if f preserves multiplication. Conversely, suppose it commutes, $\forall (V, \sigma)$ and (W, π). One can take $(V, \sigma) = (W, \sigma)$ to be any f. dim'l sub-comodule of $(B, \text{reg.})$ (or the regular comodule B itself, after using (I, 2.6) and the naturality to extend the

R-endomorphisms to all B-comodules), and then apply $\varepsilon|_V \otimes \varepsilon|_W$ to the "$V \otimes W$"-parts in the bottom square to obtain

$$\begin{tikzcd} V \otimes V \arrow[d, hook] & \\ B \otimes B \arrow[r, "f \otimes f"] \arrow[d, "m"'] & R \otimes R \arrow[d, "m_R"] \\ B \arrow[r, "f"] & R \end{tikzcd},$$

hence the result follows by taking V to be larger and larger. The claim is then proved.

Therefore,

$$G(R) = \mathrm{Hom}_{k\text{-alg}}(B, R) = \mathrm{End}^{\otimes}(\omega_R) \overset{(2.2)}{=\!=\!=} \mathrm{Aut}^{\otimes}(\omega_R)$$

is a group, as $\mathcal{C}$ and Proj_R are rigid. $\square$

• Rem. 2.8.2. There is another way, appearantly different from (though essentially the same as) the one in the proof, to prove that the monoid object G is a group object: one can construct the co-inverse S on B categorically. Sketch: for $(\mathcal{C}, \omega)$ in (2.4), corresponding to a coalgebra B, $\mathcal{C}^{op}$ is again k-linear abelian, and

$$\omega' : \mathcal{C}^{op} \xrightarrow{\omega^{op}} \mathcal{V}ec_k^{\ op} \xrightarrow{\sim} \mathcal{V}ec_k$$
$$V \longmapsto V^{\vee}$$

is again faithful k-linear exact, and the pair $(\mathcal{C}^{op}, \omega')$ corresponds to B^{op}, the opposite coalgebra. Then apply the functoriality (2.5.1) upon the duality functor (cf. p.12)

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[rr, "D"] \arrow[dr, "\omega'"'] & & \mathcal{C} \arrow[dl, "\omega"] \\ & \mathcal{V}ec_k & \end{tikzcd} \quad (\Leftarrow \omega(X^\vee) \simeq \omega(X)^\vee)$$

to obtain a coalg. hom. $S: B^{op} \to B$. The vector space B^{op} is also endowed with a mult. structure m', coming from

$$\begin{tikzcd} \mathcal{C}^{op} \times \mathcal{C}^{op} \arrow[rr, "\otimes^{op}"] \arrow[dr, "\omega' \otimes \omega'"'] & & \mathcal{C}^{op} \arrow[dl, "\omega'"] \\ & \mathcal{V}ec_k & \end{tikzcd} \quad (\Leftarrow V^\vee \otimes W^\vee \simeq (V \otimes W)^\vee),$$

w.r.t. which S is a hom. of k-algebras. One may also identify B^{op} with B by identifying $\alpha \in \mathrm{End}(\omega'|\langle X \rangle)$ with $\alpha^\vee \in \mathrm{End}(\omega|\langle X \rangle)$; this identification preserves the k-algebra structures. That S is indeed the co-inverse for Δ is left as an exercise.

- <u>Rem. 2.8.3</u>. As we saw in (II, 3.7),
$$G \simeq \underline{Aut}^{\otimes}(\omega^G) \simeq \underline{Aut}^{\otimes}(\omega);$$
in particular, the affine group G associated to $(\mathcal{C}, \otimes)$ in (2.8) is unique up to unique isom.. Sometimes, people denote G by $\pi_1(\mathcal{C}, \otimes)$, and call it the <u>fundamental group</u> (or <u>Tannaka dual</u>) of the neut. Tann. cat. $(\mathcal{C}, \otimes)$. We have the following

functoriality of "π_1".

- Prop. 2.8.4. Any hom. $f : G \to G'$ of affine k-groups induces a (faithful) k-linear exact tensor functor

$$\begin{tikzcd} \mathrm{Rep}(G') \arrow[rr, "\omega^f"] \arrow[dr, "\omega^{G'}"'] & \circlearrowleft & \mathrm{Rep}(G) \arrow[dl, "\omega^G"] \\ & \mathcal{V}ec_k & \end{tikzcd}$$

Conversely, any tensor functor $F : \mathrm{Rep}(G') \to \mathrm{Rep}(G)$ such that $\omega^G \circ F \cong \omega^{G'}$ (which is then automatically faithful, k-linear and exact) is induced by a unique hom. $f : G \to G'$. Therefore, for neut. Tann. categories $(\mathcal{C}, \omega)$ and $(\mathcal{C}', \omega')$ over k, we have

$$\mathrm{Hom}(\pi_1(\mathcal{C},\omega), \pi_1(\mathcal{C}',\omega')) \xrightarrow{\sim} \mathrm{Hom}^{\otimes}((\mathcal{C}',\omega'), (\mathcal{C},\omega)).$$

Pf: Given F, we shall construct, for each $R \in k\text{-}\mathcal{A}lg.$, a nat. hom.

$$f_R : \underset{\substack{\| \\ \{\varphi_\rho \in GL(\omega^G(V)_R) \mid (V,\rho) \in \mathrm{Rep}(G)\} \\ \text{satisfying a), b), c) in (II, 3.4)}}}{\mathcal{A}ut^{\otimes}(\omega^G_R)} \overset{=G(R)}{\longrightarrow} \underset{\substack{\| \\ \{\ldots\ldots\}}}{\mathcal{A}ut^{\otimes}(\omega^{G'}_R)} = G'(R).$$

This is clear: $\forall (W, \pi) \in \mathrm{Rep}(G')$, we associate the operator $\varphi_{F(\pi)}$ on $\omega^G(F(W))_R = \omega^{G'}(W)_R$. The uniqueness of f ~~is not difficult, and we omit the proof of it~~. follows from that in (2.5.1). $\square$.

• <u>Examples 2.3.1 revisited</u>.

—— Let $G(\Gamma)_k$ be the Tannaka dual of $Rep_k(\Gamma)$; it is called the k-<u>algebraic hull</u> of Γ. I guess there might be some "independent of k"-type of properties, as least when Γ is finitely presented (e.g. $\Gamma = \pi_1$ of an analytic variety $/\mathbb{R}$) and k is "large enough", e.g. such that all irred. Γ-rep. over k are absolutely irred.. Then what about $\pi_0(G(\Gamma)_k)(k)$, $\dim G(\Gamma)_k$? What about (the unip. / diag. part of) the radical of $G(\Gamma)_k$? What about the $\mathbb{Q}$-root system associated to the s.s. group $G(\Gamma)_k^\circ$/its radical ? Can they be read off from the abstract group Γ itself ?

—— For a top. group K, let G be the Tannaka dual of $Rep_{\mathbb{R}}^{cont}(K)$; it is called the <u>real algebraic envelope</u> of K. There is a canonical hom. of top. groups

$$K \longrightarrow G(\mathbb{R}),$$

which is an isom. when K is a compact top. group; see [Gèbres, §5.2].

—— Clearly, the Tannaka dual of $\mathbb{Z}$-<u>Gr-Vec_k</u> is $\mathbb{G}_{m,k}$: to give a $\mathbb{G}_m$-action on V is the same as giving a $\mathbb{Z}$-grading $V = \bigoplus_n V_n$ of V, with $\mathbb{G}_m$ acting on V_n by $t \mapsto t^n$. Similarly, for a comm. group M, the Tannaka dual of M-$\underline{\text{Gr. Vec}}_k$ is $D(M)$.

—— Let $\mathrm{MHS}_{\mathbb{R}}^{\text{split}}$ be the full subcategory of $\mathrm{MHS}_{\mathbb{R}}$ consisting of those $(V, W_\bullet, F^\bullet)$ such that the weight filtr. $W_\bullet$ splits: $V \simeq \bigoplus_n \mathrm{Gr}_n^W(V)$; it is a sub-Tann. cat. of $\mathrm{MHS}_{\mathbb{R}}$, and its Tannaka dual is the $\underline{\text{Deligne torus}}$ $\mathbb{S}/\mathbb{R}$,

$$\mathbb{S}(R) = (R \otimes_{\mathbb{R}} \mathbb{C})^\times, \quad \forall R \in \mathbb{R}\text{-Alg}.$$

$$\simeq \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL_2(R) \right\}.$$

The Tannaka dual of $\mathrm{MHS}_{\mathbb{R}}$ is an extension of $\mathbb{S}$ by a unipotent affine group over $\mathbb{R}$; see [Motives 1, pp. 512–514].

§3. $\underline{\text{Properties of}}$ G $\underline{\text{and of}}$ $\mathrm{Rep}(G)$.

As G can be recovered from $(\mathrm{Rep}(G), \otimes, \omega^G)$, it is possible, at least in principle, to express every group-theoretic property of G in terms of the triple. We give a few examples.

• $\underline{\text{Def}. 3.1.}$ Let $(\mathcal{C}, \omega)$ be a neutralized Tannakian category, and let S be a set of objects in $\mathcal{C}$. Denote by $\langle S\rangle^{\otimes}$ the smallest full subcategory of $\mathcal{C}$ containing all objects that are isom. to ~~some~~ those in S and is closed under taking subquotients, direct sums, tensor products and duals. We say that $\mathcal{C}$ is <u>$\otimes$-finitely generated</u>, if $\mathcal{C}=\langle S\rangle^{\otimes}$ for some finite set of objects S. This is equiv. to that $\mathcal{C}$ is $\otimes$-generated by a single object, e.g. $\bigoplus_{X\in S} X$. We call $X\in obj(\mathcal{C})$ a <u>tensor generator</u> of $\mathcal{C}$, if $\mathcal{C}=\langle X\rangle^{\otimes}$, i.e., every object in $\mathcal{C}$ is isom. to a subquotient of $P(X,X^\vee)$ for some polynomial $P(t_1,t_2)\in\mathbb{N}[t_1,t_2]$.

An affine k-group $G(=\varprojlim G_i)$ is <u>pro-reductive</u>, if all (or a cofinal family) of its algebraic quotients are reductive (equivalently, if $\pi_0(G):=\varprojlim \pi_0(G_i)$ and $R_u(G):=\varprojlim R_u(G_i)$ are both trivial).

• <u>Prop. 3.2. Let</u> G <u>be an affine</u> k-<u>group, and</u> $\mathcal{C}=Rep(G)$, <u>with unit object</u> $\mathbb{1}=(k,\text{triv.})$. <u>Then</u>

i). G <u>is diagonalizable</u> $\Leftrightarrow$ $\mathcal{C}$ <u>is semisimple, and simple objects are all</u> 1-<u>dim'l</u>;

ii). G <u>is unipotent</u> $\Leftrightarrow$ <u>simple objects in</u> $\mathcal{C}$ <u>are isom. to</u> $\mathbb{1}$;

iii). G *is finite* $\Leftrightarrow$ $\mathcal{C} = \langle X \rangle$ *for some* $X \in obj(\mathcal{C})$;

iv). G° *is pro-reductive* $\Leftrightarrow$ $\mathcal{C}$ *is semisimple*;

v). G/k *is of finite type* $\Leftrightarrow$ $\mathcal{C}$ *is* $\otimes$- *finitely generated*.

Pf: i) follows from (II, 2.7), and ii) is none other than the definition (II, 4.1).

iii). ($\Rightarrow$). Take X to be the regular rep. of G; then any f. dim'l G-rep. is a subrep. of X^n, for some n.

($\Leftarrow$). With notations in the proofs of (2.4) and (2.8), we have $k[G] = B = A_X^\vee$, which is finite-dim'l.

• *Lem. 3.2.1. Let G be an affine k-group. Then any f. dim'l G-representation (V, ρ) factors through some algebraic quotient group G_i of G*: (G_i *certainly depends on* (V, ρ))

$$\begin{tikzcd} G \arrow[rr, "\rho"] \arrow[dr] & & GL_V \\ & G_i \arrow[ur] & \end{tikzcd}.$$

Pf: Take a basis $\{e_i\}_{i=1}^n$ for V, $(V, \rho) \rightsquigarrow (V, \sigma)$, write

$$\sigma(e_i) = \sum_j e_j \otimes a_{ij}, \quad a_{ij} \in k[G].$$

Then let $k[G_i]$ be a Hopf subalgebra of $k[G]$ containing all the a_{ij}'s, and is of finite type over k (cf. I, 2.10). $\square$.

iv). ($\Rightarrow$). $\forall (V,\rho) \in \mathrm{obj}(\mathcal{C})$, by (3.2.1), it descends to an action of some alg. quotient G_i of G, so by (III, 4.3), it is G_i-semisimple, hence G-semisimple.

($\Leftarrow$). $\forall$ alg. quotient G_i of G, and $\forall$ G_i-rep. V, it is s.s. when viewed as a G-rep., hence is s.s.

v). ($\Leftarrow$). Assume that $\mathcal{C} = \langle X \rangle^{\otimes}$, $X = (V,\rho)$. Then (V,ρ) is a faithful G-rep., otherwise G acts on V as well as $V^{\vee}$, and hence all f.-dim'l G-rep., through $\overline{G} := G/\ker(\rho)$ (we did not discuss quotient groups of general affine groups; an ad hoc def'n is that $G/\ker(\rho)$ is the inverse limit of those G_i's such that the composite

$$\ker(\rho) \hookrightarrow G \longrightarrow G_i$$

is trivial), namely the quotient hom. $q : G \twoheadrightarrow \overline{G}$ induces an equivalence of neut. Tann. cat.

$$\omega^q : \mathrm{Rep}(\overline{G}) \xrightarrow{\sim} \mathrm{Rep}(G),$$

therefore, q is an isom. by (2.8.4).

Note that we cannot deduce directly that the injective morp. ρ of group functors is a closed immersion, since G is not known a priori to be an algebraic group (see the proof of (I, 4.6), where Chevalley's theorem on constructibility is crucial). By (3.2.1), ρ factors through some alg. quotient G_i of G:

$$\begin{tikzcd} G \arrow[r, hook, "\rho"] \arrow[dr] & GL_V \\ & G_i \arrow[u, dashed] \end{tikzcd},$$

then $G \to G_i$ has trivial kernel, hence $G \xrightarrow{\sim} G_i$ (why?) is an alg. group.

($\Longrightarrow$). Let $X = (V, \rho)$ be a f.-dim'l faithful rep. of G, by (I, 5.2). We claim that $\mathcal{C} = \langle X \rangle^{\otimes}$. For any $W \in \mathrm{ob}(\mathcal{C})$, the map $\sigma : W \to \omega^G(W) \otimes k[G]$ is $k[G]$-equiv. (by def'n of a coaction). (hence G-equiv.) and injective

$$\begin{tikzcd} W \arrow[r, "\sigma"] \arrow[dr, "\sim"'] & W \otimes k[G] \arrow[d, "id \otimes \varepsilon"] \\ & W \otimes k \end{tikzcd},$$

where $\omega^G(W)$ is the vector space W with trivial G-action. So $\omega^G(W) \otimes k[G] \approx k[G]^n$; let W_i be the image of

$$W \overset{\sigma}{\hookrightarrow} k[G]^n \xrightarrow{pr_i} k[G]; \quad (1 \leq i \leq n)$$

then $W \overset{\sigma}{\hookrightarrow} \prod W_i$. If all W_i's belong to $\langle X \rangle^{\otimes}$, so also does W, thus we may assume that W is a f. dim'l sub-G-rep. of $k[G]$.

Choose a basis $\{e_1, \cdots, e_n\}$ of V, and let a_{ij} be the image of T_{ij} under $k[GL_n] \twoheadrightarrow k[G]$, so that

$$\sigma(e_j) = \sum_{l=1}^{n} e_l \otimes a_{lj}.$$

Let (b_{ij}) be matrix coef. of $V^\vee$: $(b_{ij}) = {}^t(a_{ij})^{-1}$.

By Cramer's rule,

$$b_{ij} = \frac{1}{\det(a_{ij})}\,((i,j)\text{-minor of } (a_{ij})),$$

and

$$\frac{1}{\det(a_{ij})} = \det(b_{ij}),$$

so that

$$k[G] = k[a_{ij}, \tfrac{1}{\det(a_{ij})}] = k[a_{ij}, b_{ij} \mid 1 \le i, j \le n].$$

Up to enlarging W, we may assume that W has a basis consisting of monomials in a_{ij} and b_{ij}'s.

For instance, if $a_{i_1 j_1} \cdot a_{i_2 j_2} \in W$ is a basis vector,

since

$$\Delta(a_{i_1 j_1} \cdot a_{i_2 j_2}) = \Big(\sum_{l_1=1}^{n} a_{i_1 l_1} \otimes a_{l_1 j_1}\Big) \cdot \Big(\sum_{l_2=1}^{n} a_{i_2 l_2} \otimes a_{l_2 j_2}\Big)$$

$$= \sum_{l_1, l_2} a_{i_1 l_1} \cdot a_{i_2 l_2} \otimes a_{l_1 j_1} \cdot a_{l_2 j_2}$$

in $W \otimes k[G]$, and for $\sigma^{\otimes 2} : V^{\otimes 2} \longrightarrow V^{\otimes 2} \otimes k[G]$,

$$\sigma^{\otimes 2}(e_{j_1} \otimes e_{j_2}) = \sum_{l_1, l_2} e_{l_1} \otimes e_{l_2} \otimes (a_{l_1 j_1} \cdot a_{l_2 j_2}),$$

we may map $e_{j_1} \otimes e_{j_2}$ to $a_{i_1 j_1} \cdot a_{i_2 j_2}$ (note that $e_{l_1} \otimes e_{l_2}$ falls in the sub-$k[G]$-comodule of $V^{\otimes 2}$ generated by $e_{j_1} \otimes e_{j_2}$

$\iff a_{i_1 l_1} \cdot a_{i_2 l_2} \neq 0$, so that we may (and have to) map $e_{l_1} \otimes e_{l_2}$ to $a_{i_1 l_1} \cdot a_{i_2 l_2}$), realizing this sub-$k[G]$-comodule of W as a subquot. of $V^{\otimes 2}$. The general case is similar, so W is a subquot. of $P(X, X^\vee)$, for some $P(t_1, t_2) \in \mathbb{N}[t_1, t_2]$. $\square$.

• Exercises 3.3. Let G be an affine k-group (with k of char. 0, ~~as usual~~), and $\mathcal{C} = \mathrm{Rep}(G)$.

~~i). G is finite $\iff$ $\mathcal{C} = \langle X \rangle$ for some object X.~~

ii). G is connected $\iff$ $\forall$ non-trivial G-rep. X, the full subcategory $\langle X \rangle$ is not closed under $\otimes$.

(Hint: If $\langle X \rangle$ is closed under $\otimes$, then $\langle X \oplus X^\vee \rangle$ is closed under $\otimes$ and taking duals, hence get $\mathrm{Rep}(H) \simeq \langle X \oplus X^\vee \rangle \hookrightarrow \mathcal{C}$ $\rightsquigarrow G \twoheadrightarrow H$. By i), H is finite ...). //.

• Ex. $\forall$ complex elliptic curve E, the Hodge stru. on $H_1(E, \mathbb{R})$ corresponds to the tautological rep. $\mathbb{S} \hookrightarrow GL_{2,\mathbb{R}}$ (up to conjugation). //.

§4. The classification.

Let $\mathfrak{g}$ be a semisimple Lie algebra over k. Then $\mathrm{Rep}(\mathfrak{g})$, endowed with the usual notions of tensors and duals, as well as the forgetful functor $\omega^{\mathfrak{g}} : (V, \rho) \mapsto V$, becomes a neutralized Tannakian category over k (exercise). Hence by (2.8), it is associated with an affine k-group scheme

$$G(\mathfrak{g}) := \pi_1(\mathrm{Rep}(\mathfrak{g}), \omega^{\mathfrak{g}}).$$

• <u>Lem. 4.1.</u> $G(\mathfrak{g})$ <u>is of finite type over</u> k.

<u>Pf</u>: This follows from that $Rep(\mathfrak{g})$ is $\otimes$-finitely generated and (3.2 v). In fact, the representation ring $\mathcal{R}(\mathfrak{g})$ of $\mathfrak{g}$ (i.e. the Grothendieck ring of $Rep(\mathfrak{g})$) is isom. to the polynomial ring $\mathbb{Z}[\varpi_\alpha ; \alpha \in S]$ of the fundamental weights (cf. Bourbaki, Groupes et Alg. de Lie, Ch. VIII, §7, n°7, Th. 2 (i)). $\square$

Note that, in contrast with alg. groups, it is not the case that any f. dim'l faithful $\mathfrak{g}$-rep. $\otimes$-generates $Rep(\mathfrak{g})$; think of the 1-dim'l (abelian) Lie algebra $\mathfrak{g} = k$.

Next, we construct a hom. of Lie algebras

$$\eta : \mathfrak{g} \longrightarrow Lie(G(\mathfrak{g})).$$

Note that, by (II, 3.7),

$$Lie(G(\mathfrak{g})) = \ker\left(G(\mathfrak{g})(k[\varepsilon_1]) \xrightarrow{\pi} G(\mathfrak{g})(k)\right)$$
$$= \ker\left(Aut^{\otimes}(\omega^{\mathfrak{g}}_{k[\varepsilon_1]}) \longrightarrow Aut^{\otimes}(\omega^{\mathfrak{g}})\right).$$

Given $x \in \mathfrak{g}$, we associate, to each $(V, \rho) \in ob(Rep(\mathfrak{g}))$, the linear automorphism of the $k[\varepsilon_1]$-module $V_{k[\varepsilon_1]}$

$$id_{V_{k[\varepsilon_1]}} + \varepsilon_1 \cdot \rho(x) : v + \varepsilon_1 \cdot w \longmapsto v + \varepsilon_1 (w + \rho(x) \cdot v).$$

One verifies immediately that it is in $Aut^{\otimes}(\omega^{\mathfrak{g}}_{k[\varepsilon_1]})$, and reduces to $\mathrm{id}_{\omega\mathfrak{g}}$ mod. ε_1. It is also clear that $x \mapsto \eta(x)$ is a hom. of Lie algebras.

- Exercise 4.2. Show that the composite

$$\mathrm{Rep}(\mathfrak{g}) \xrightarrow[(2.8)]{\sim} \mathrm{Rep}(G(\mathfrak{g})) \longrightarrow \mathrm{Rep}(\mathrm{Lie}\, G(\mathfrak{g})) \xrightarrow[\eta^*]{} \mathrm{Rep}(\mathfrak{g})$$

is isom. to the identity functor. //.

- Cor. 4.3. $G(\mathfrak{g})$ is connected.

Pf: By (4.2), $\mathrm{Rep}(G(\mathfrak{g})) \longrightarrow \mathrm{Rep}(\mathrm{Lie}\, G(\mathfrak{g}))$ is fully faithful. If $\pi_0(G(\mathfrak{g})) \neq \mathbb{1}$, let V be an irred. non-triv. rep. of $\pi_0(G(\mathfrak{g}))$. Then $\mathrm{Hom}_{G(\mathfrak{g})}(k, V) = 0$ but $\mathrm{Hom}_{\mathrm{Lie}\, G(\mathfrak{g})}(k, V) \overset{}{\underset{(\mathrm{III}, 3.3.2)}{=\!=}} \mathrm{Hom}_{G(\mathfrak{g})^0}(k, V)$

$= V$. □.

By (3.2 iv) and (III, 1.3.6), we see that the conn. alg. group $G(\mathfrak{g})$ is reductive. By (III, 4.2), the composite

$$R(G(\mathfrak{g})) \hookrightarrow G(\mathfrak{g}) \twoheadrightarrow G(\mathfrak{g})^{ab} \quad (:= G(\mathfrak{g})/\mathcal{D}G(\mathfrak{g}))$$

is an isogeny, hence induces an isom. on Lie alg.; in particular, $\mathfrak{r}(\mathrm{Lie}\, G(\mathfrak{g}))$ (which is also the center $\mathfrak{z}(\mathrm{Lie}\, G(\mathfrak{g}))$) is a direct factor of $\mathrm{Lie}\, G(\mathfrak{g})$. In fact, the mult. map

$$R(G(\mathfrak{g})) \times \mathcal{D}G(\mathfrak{g}) \longrightarrow G(\mathfrak{g})$$

is an isogeny, inducing an isom. $\mathrm{Lie}\, G(\mathfrak{g}) \simeq \mathfrak{z} \times \mathfrak{s}$, where $\mathfrak{z}$ is abelian and $\mathfrak{s}$ is semisimple.

• Cor. 4.4. $\eta : \mathfrak{g} \to \mathrm{Lie}\, G(\mathfrak{g})$ is an isomorphism; in particular, $G(\mathfrak{g})$ is semisimple, with Lie algebra $\mathfrak{g}$.

Pf: By (4.2), $\eta^* : \mathrm{Rep}(\mathrm{Lie}\, G(\mathfrak{g})) \longrightarrow \mathrm{Rep}(\mathfrak{g})$ is essentially surjective, i.e. any $\mathfrak{g}$-rep. can be factored through η. By taking a faithful $\mathfrak{g}$-rep. (Ado's thm.), we see that η is injective

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\eta"] \arrow[dr, hook] & \mathrm{Lie}\, G(\mathfrak{g}) \arrow[d, dashed] \\ & \mathfrak{gl}_V \end{tikzcd}$$

As $\mathfrak{g} \xrightarrow{\eta} \mathrm{Lie}\, G(\mathfrak{g}) \xrightarrow{pr_1} \mathfrak{z}$ is zero, η injects $\mathfrak{g}$ into the subalgebra $(0) \times \mathfrak{s} \subset \mathrm{Lie}\, G(\mathfrak{g})$.

It is clear that $\mathfrak{z} = 0$ ($\Leftrightarrow G(\mathfrak{g})^{ab} = \mathbb{1}$): if not, let χ be a non-trivial character of $G(\mathfrak{g})^{ab}$, $\chi_0 =$ the triv. character; then

$$\mathrm{Hom}_{G(\mathfrak{g})}(\chi_0, \chi) = 0 \quad \text{but} \quad \mathrm{Hom}_{\mathfrak{g}}(k, k) = k, \text{ contradicting}$$

with that

$$\mathrm{Rep}(G(\mathfrak{g})) \longrightarrow \mathrm{Rep}(\mathrm{Lie}\, G(\mathfrak{g})) \xrightarrow{\eta^*} \mathrm{Rep}(\mathfrak{g})$$

is an equivalence.

Now $\eta : \mathfrak{g} \hookrightarrow \mathfrak{s} = \mathrm{Lie}\, G(\mathfrak{g})$. Let

$$\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_m \quad , \quad \mathfrak{s} = \mathfrak{a}_1 \times \cdots \times \mathfrak{a}_n$$

be decomposition into min. ideals. Then the $\mathfrak{g}_i$'s are all the irred. sub-$\mathrm{ad}_{\mathfrak{g}}$-modules of $\mathfrak{g}$. By (III, 3.3.1), the $\mathfrak{a}_i$'s are all the irred. sub-$\mathrm{Ad}_{G(\mathfrak{g})}$-modules of $\mathfrak{s}$, thus each $\mathfrak{a}_i$ is also an irred. $\mathrm{ad}_{\mathfrak{s}}|_{\mathfrak{g}}$-module (since

$$\mathrm{Rep}(G(\mathfrak{g})) \longrightarrow \mathrm{Rep}(\mathfrak{s}) \xrightarrow{\eta^*} \mathrm{Rep}(\mathfrak{g})$$

is an equiv.; or just use the full faithfulness and Schur's lemma: $\dim_k \mathrm{End}_{\mathfrak{g}}(\mathfrak{a}_i) = \dim_k \mathrm{End}_{G(\mathfrak{g})}(\mathfrak{a}_i) = 1$). As the $\mathfrak{g}$-rep. $(\mathfrak{s}, \mathrm{ad}_{\mathfrak{s}}|_{\mathfrak{g}})$ is semisimple, containing $(\mathfrak{g}, \mathrm{ad}_{\mathfrak{g}})$ as a subrep. via $\eta : \mathfrak{g} \hookrightarrow \mathfrak{s}$, each $\mathfrak{g}_i$ is a simple ideal $\mathfrak{a}_j$ of $\mathfrak{s}$, and $\mathfrak{g}$ is a direct sum of some of the $\mathfrak{a}_j$'s; in particular, $\mathfrak{g} \subset \mathfrak{s}$ is an ideal. If $\mathfrak{g} \subsetneq \mathfrak{s}$, as before, one can find an $\underbrace{\text{irred.}}_{\text{non-triv.}}$ $G(\mathfrak{g})$-rep. on which $\mathfrak{g}$ acts trivially (namely, a simple ideal $\mathfrak{a}_i$ not contained in $\mathfrak{g}$). Therefore, $\mathfrak{g} = \mathfrak{s}$. $\square$

• <u>Prop. 4.5</u>. $G(\mathfrak{g})$ <u>is universal in the sense that</u>, $\forall$ <u>alg.</u> group H, $\forall$ <u>hom.</u> of <u>Lie algebras</u> $\varphi : \mathfrak{g} \to \mathrm{Lie}(H)$, $\exists!$ <u>hom.</u> <u>of alg. groups</u> $\tilde{\varphi} : G(\mathfrak{g}) \to H$ <u>making the diagram commute</u>

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\eta"] \arrow[dr, "\varphi"'] & \mathrm{Lie}\, G(\mathfrak{g}) \arrow[d, "d\tilde{\varphi}"] \\ & \mathrm{Lie}(H) \end{tikzcd}$$

<u>Pf</u>: Consider the composite functor

$$\begin{tikzcd} \mathrm{Rep}(H) \arrow[r] \arrow[drr, "\omega^H"'] & \mathrm{Rep}(\mathrm{Lie}(H)) \arrow[r, "\varphi^*"] & \mathrm{Rep}(\mathfrak{g}) \arrow[d, "\omega^{\mathfrak{g}}"] \\ & & \mathrm{Vec}_k \end{tikzcd}$$

By (2.8.4), $\exists!$ hom. $\tilde{\varphi} : G(\mathfrak{g}) \to H$ inducing it, hence $\tilde{\varphi}$ makes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\eta"] \arrow[d, "\varphi"'] & \mathrm{Lie}\, G(\mathfrak{g}) \arrow[dl, "d\tilde{\varphi}"] \\ \mathrm{Lie}(H) & \end{tikzcd}$$

commute (why?). It is unique because η is an isom. (4.4):

$$d\tilde{\varphi}_1 \circ \eta = \varphi = d\tilde{\varphi}_2 \circ \eta \Rightarrow d\tilde{\varphi}_1 = d\tilde{\varphi}_2 \xrightarrow[(4.3)]{(\mathrm{II}, 2.8.3)} \tilde{\varphi}_1 = \tilde{\varphi}_2 : G(\mathfrak{g}) \to H^\circ. \qquad \square$$

In particular, if $\varphi : \mathfrak{g} \to \mathrm{Lie}(H)$ is an isom., then $\tilde{\varphi} : G(\mathfrak{g}) \to H^\circ$ is an isogeny (II, 2.8.2). We say that $G(\mathfrak{g})$ is <u>simply-connected</u>, in the sense that it has no connected central isogenous covers (other than isom. $G' \xrightarrow{\sim} G(\mathfrak{g})$).

To proceed, we shall briefly review the rep. theory of semisimple Lie algebras. Let a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ be chosen, giving rise to a root system $R \subset \mathfrak{h}^\vee$, with root lattice Q and weight lattice P. Let $S \subset R$ be a base, giving rise to a <u>Borel subalgebra</u> (i.e. max. solvable subalg.)

of $\mathfrak{g}$ containing $\mathfrak{h}$:

$$\mathfrak{b} = \mathfrak{b}(S) := \mathfrak{h} \oplus \bigoplus_{\substack{\alpha > 0 \\ \alpha \in R}} \mathfrak{g}_\alpha ,$$

as well as the <u>fundamental weights</u> ϖ_α $(\alpha \in S)$, i.e. the $\mathbb{Z}$-basis of $P = P(R)$ dual to the $\mathbb{Z}$-basis $\{\alpha^\vee ; \alpha \in S\}$ of $Q(R^\vee)$:

$$\langle \varpi_\alpha , \beta^\vee \rangle = \delta_{\alpha\beta} \quad , \forall \alpha, \beta \in S .$$

Let $P_{++} = P_{++}(R)$ be the monoid generated by the fundamental weights:

$$P_{++} = \sum_{\alpha \in S} \mathbb{N} \cdot \varpi_\alpha \quad \Big(= \text{free monoid } \bigoplus_{\alpha \in S} \mathbb{N} \cdot \varpi_\alpha \Big)$$

$$= \{ v \in \mathfrak{h}^\vee \,|\, \langle v, \alpha^\vee \rangle \in \mathbb{N}, \ \forall \alpha \in S \} \subset P .$$

Weights in P_{++} are called <u>dominant weights</u>.

There is a one-to-one correspondence between irred. $\mathfrak{g}$-modules V and dominant weights $\varpi \in P_{++}$, such that, if V corresponds to ϖ (we denote $\varpi = \varpi_V$ and $V = V_\varpi$), then $\exists !$ $\mathfrak{b}$-stable line $L \subset V$; moreover, this line L, being also $\mathfrak{h}$-stable, is the full $\mathfrak{h}$-eigenspace in V with eigen-weight ϖ, and for any other weight ϖ' for $\mathfrak{h}$ in V, we have

$$\varpi' \in \varpi - \sum_{\alpha \in S} \mathbb{N} \cdot \alpha .$$

Let $M(P_{++})$ be the free abelian group gen. by P_{++} modulo the relations gen. by

$$\varpi = \varpi_1 \text{"}+\text{"} \varpi_2 \quad \text{whenever } V_\varpi \subset V_{\varpi_1} \otimes V_{\varpi_2} .$$

(formal sum, not addition in P_{++})

So $\varpi_1 \underset{\text{(add. in } P_{++})}{+} \varpi_2 = \varpi_1 \text{"}+\text{"} \varpi_2$ in $M(P_{++})$, showing that $P_{++} \to M(P_{++})$ is a hom. of monoids. For $\varpi \in P_{++}$, let $\varpi^\vee$ be the dominant weight of the dual rep. $V_\varpi^\vee$; then $V_\varpi \otimes V_{\varpi^\vee}$ contains the trivial rep. of $\mathfrak{g}$, showing that $\varpi \text{"}+\text{"} \varpi^\vee = 0$ in $M(P_{++})$, hence $P_{++} \to M(P_{++})$ is surjective. One shows that for ϖ and ϖ' in P_{++}, they have the same image in $M(P_{++})$ if and only if V_ϖ and $V_{\varpi'}$ both occur in $V_{\varpi_1} \otimes \cdots \otimes V_{\varpi_n}$ for some $\varpi_1, \ldots, \varpi_n \in P_{++}$, if and only if $\varpi - \varpi' \in Q$ (the difference is taken in P). So

$$P/Q \xleftarrow{\sim} P_{++}/Q \cap P_{++} \xrightarrow{\sim} M(P_{++}) .$$

Let $(\mathcal{C}, \omega)$ be a neut. Tann. cat., and let M be a comm. group. An M-__grading__ on $\mathcal{C}$ is a decomposition

$$X = \bigoplus_{m \in M} X_m , \quad \forall X \in \mathrm{Ob}(\mathcal{C}),$$

functorial in X, compatible with the tensor structure:

$$(X \otimes Y)_m = \bigoplus_{m_1 + m_2 = m} X_{m_1} \otimes Y_{m_2} .$$

• **Lem. 4.6.** Giving an M-grading on $\mathscr{C}$ is equivalent to giving a hom of affine group schemes

$$D(M) \longrightarrow Z(\pi_1(\mathscr{C}, \omega)), \text{ center of } \pi_1(\mathscr{C}, \omega).$$

Pf: Let $\mathscr{C} = \mathrm{Rep}(G)$, $G = \pi_1(\mathscr{C}, \omega)$. $\forall X \in \mathrm{ob}(\mathscr{C})$, we have

$$\omega(X) = \bigoplus_{m \in M} \omega(X_m),$$

an M-graded vector space depending functorially on X, hence a functor

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[dr, "\omega"'] & M\text{-}\mathrm{Gr.Vec}_k \arrow[d] \\ & \mathrm{Vec}_k \end{tikzcd},$$

and it is a tensor functor. By (2.8.4), we have a hom. of affine groups

$$D(M) \xrightarrow{f} G$$

such that $F \cong \omega^f$. That the decomposition happens in $\mathscr{C}$ rather than just in Vec_k will force f to factor through $Z(G)$. Explicitly, f is given as follows: $\forall R \in k\text{-}\mathcal{A}lg.$,

$$\mathrm{Hom}(M, R^\times) \underset{\uparrow \atop (\mathrm{II}, 2.1)}{=} D(M)(R) \xrightarrow{f_R} G(R) = \mathcal{A}ut^{\otimes}(\omega_R)$$

$$(M \xrightarrow{t} R^\times) \longmapsto \left\{ \begin{array}{l} \varphi : X \in \mathrm{ob}(\mathscr{C}) \mapsto \varphi_X \in GL(\omega(X)_R), \\ \omega(X)_R = \bigoplus_M \omega(X_m)_R,\ \varphi_X \text{ is} \\ \text{scalar mult. by } t(m) \text{ on } \omega(X_m)_R \end{array} \right\}.$$

One verifies that φ is indeed in $\mathrm{Aut}^{\otimes}(\omega_R)$ (that t preserves multiplication / identity implies condition b) / c) in (II, 3.4); note that, by considering $\theta_X : X \xrightarrow{\sim} \mathbb{1}\otimes X$, one deduces that $\mathbb{1} = \mathbb{1}_0$, $0 \in M$ being the identity). It is also clear that φ is in the center: $\forall$ R-alg. R', $\forall \psi \in G(R')$, to show that $\varphi_X \circ \psi_X = \psi_X \circ \varphi_X$ on $\omega(X)_{R'} = \bigoplus_{m\in M} \omega(X_m)_{R'}$, note that each $\omega(X_m)_{R'}$ is stable under ψ_X (by naturality of ψ: $\psi_X|_{\omega(X_m)_{R'}} = \psi_{X_m}$), and we have

$$t(m)\circ\psi_{X_m} = \psi_{X_m}\circ t(m) \qquad \text{on } \omega(X_m)_{R'}.$$

Therefore, the hom. $f: D(M) \to G$ is central:

$$\begin{tikzcd} D(M) \arrow[rr, "f"] \arrow[dr] & & G \\ & Z(G) \arrow[ur, hook] & \end{tikzcd}.$$

Conversely, given a hom. $f: D(M) \to Z(G) \subset G$, we have a tensor functor

$$\mathrm{Rep}(G) \xrightarrow{\omega^f} M\text{-}\mathrm{Gr.Vec}_k,$$

given by $(V,\rho) \rightsquigarrow (V, \rho\circ f)$, $V = \bigoplus_{m\in M}$ (eigen spaces V_m).
(II, 2.7)

Each subspace $V_m \subset V$ is G-stable, since f is central, so the decomp. $V = \bigoplus V_m$ is a decomp. in $\mathcal{C}$:

$$(V,\rho) = \bigoplus (V_m, \rho_m). \qquad \square$$

Given $(V,\rho) \in \mathrm{Rep}(\mathfrak{g})$, the $\mathfrak{h}$-action decomposes it into eigenspaces

$$V = \bigoplus_{\alpha \in \mathfrak{h}^\vee} V_\alpha ,$$

and by the finite-dimensionality of V, only those V_α with $\alpha \in P$ may survive. Clearly, $V_\alpha \otimes W_\beta \subset (V \otimes W)_{\alpha+\beta}$, so we have a P-grading on $V = \omega^{\mathfrak{g}}(V, \rho)$, giving rise to a tensor functor

$$\mathrm{Rep}(\mathfrak{g}) \longrightarrow P\text{-}\mathrm{Gr.Vec}_k ,$$

hence a hom. $D(P) \longrightarrow G(\mathfrak{g})$ by (2.8.4). We claim that it is injective (hence an alg. subgroup (I, 4.6)), and after taking Lie algebras, one obtains $\mathfrak{h} \hookrightarrow \mathfrak{g}$.

Explicitly, the hom. is given as follows: $\forall R \in k\text{-}\mathcal{A}lg.$,

$$\begin{array}{ccc} D(P)(R) & \longrightarrow & G(\mathfrak{g})(R) \\ (\mathrm{II}, 2.1) \to \| & & \| \\ \mathrm{Hom}_{gr}(P, R^\times) & & \mathrm{Aut}^{\otimes}(\omega_R^{\mathfrak{g}}) \\ \left(P \xrightarrow{t} R^\times\right) & \longmapsto & \left\{ \begin{array}{l} \varphi : (V, \rho) \rightsquigarrow \varphi_\rho \in GL(V_R), \\ \varphi_\rho = \bigoplus_\alpha (\text{scalar } t(\alpha) \text{ on } (V_\alpha)_R) \end{array} \right\} \end{array}$$

If $\varphi_\rho = \mathrm{id}_{V_R}$, $\forall (V, \rho)$, then by taking (V, ρ) to be the fundamental rep. V_ϖ ($\varpi \in P_{++}$ fundamental weight), we see that $t(\varpi) = 1_R$ (because φ_ρ acts by $t(\varpi)$ on the highest weight line $L_\varpi \subset V_\varpi$). As the ϖ's form a $\mathbb{Z}$-basis of P, we deduce that $t = 1$, hence $D(P) \hookrightarrow G(\mathfrak{g})$. We may denote the image of $D(P)$ in $G(\mathfrak{g})$ by $T(\mathfrak{h})$, a sub-torus, to show the dependence on $\mathfrak{h}$.

Define a hom. (of abelian Lie algebras)

$$\begin{array}{ccccc} \mathfrak{h} & \longrightarrow & \mathrm{Lie}\, D(P) & \xlongequal{(\mathrm{III},\, 2.6.5)} & \mathrm{Hom}_{gr.}(P, k) \\ x & \longmapsto & & & \left(ev_x : \alpha \longmapsto \alpha(x)\right). \end{array}$$

It is an isom. because P is a lattice in $\mathfrak{h}^\vee$:

$$\mathrm{Hom}_{gr.}(P, k) = \mathrm{Hom}_k(P \otimes k, k) = \mathrm{Hom}_k(\mathfrak{h}^\vee, k).$$

• Lem. 4.7. The diagram

$$\begin{array}{ccc} \mathfrak{h} & \hookrightarrow & \mathfrak{g} \\ \downarrow \simeq & & \simeq \downarrow \eta \\ \mathrm{Lie}\, D(P) & \hookrightarrow & \mathrm{Lie}\, G(\mathfrak{g}) \end{array}$$

commutes. In particular, $D(P)$ is a maximal torus in $G(\mathfrak{g})$.

Pf. Straight forward:

$$\begin{array}{ccc} x \in \mathfrak{h} & \longmapsto & x \in \mathfrak{g} \\ \big\downarrow & & \big\downarrow \eta \\ & & id + x \cdot \varepsilon_1 \\ P \to k \simeq 1 + k \cdot \varepsilon_1 & & \\ \left(\alpha \longmapsto 1 + \alpha(x) \cdot \varepsilon_1\right) & \longmapsto & \varphi_\rho = \bigoplus_P \left(\begin{array}{l} \text{scalar mult. by } 1 + \alpha(x) \cdot \varepsilon_1 \\ \text{on } (V_\alpha)_{k[\varepsilon_1]} : \end{array}\right) \end{array}$$

$$v + w \cdot \varepsilon_1 \longmapsto v + \Big(w + \underbrace{\alpha(x) \cdot v}_{= \rho(x)(v)}\Big) \cdot \varepsilon_1$$

So $T(\mathfrak{h})$ cannot be properly contained in any larger torus in $G(\mathfrak{g})$, otherwise $\mathfrak{h}$ will not be self-normalizing. $\square$.

• <u>Prop. 4.8.</u> <u>The center</u> $Z(\mathfrak{g})$ <u>of</u> $G(\mathfrak{g})$ <u>is the image of</u>

$$D(P/Q) \hookrightarrow D(P) \hookrightarrow G(\mathfrak{g}).$$

<u>Pf</u>: $Z(\mathfrak{g})$ is a commutative finite alg. group, hence diagonalizable; let $Z(\mathfrak{g}) = D(M)$ for some comm. group M. Then

$$D(M) \hookrightarrow G(\mathfrak{g})$$

defines an M-grading on $\mathrm{Rep}(\mathfrak{g})$, by (4.6). Explicitly, for each simple $\mathfrak{g}$-module V_ϖ $(\varpi \in P_{++})$, which is no longer decomposable, there is an associated element $m(\varpi) \in M$ such that $V_\varpi = (V_\varpi)_{m(\varpi)}$; functoriality is redundant; as for tensor product, the M-grading on $V_{\varpi_1} \otimes V_{\varpi_2}$ $(\varpi_i \in P_{++})$ has to be in the single degree $m(\varpi_1) + m(\varpi_2)$, therefore, for any simple $V_\varpi \subset V_{\varpi_1} \otimes V_{\varpi_2}$, functoriality will force

$$m(\varpi) = m(\varpi_1) + m(\varpi_2);$$

that $m(0) = 0$ is also forced by $\theta_X : X \xrightarrow{\sim} \mathbb{1} \otimes X$. So the function $\varpi \in P_{++} \mapsto m(\varpi) \in M$ induces a group hom.

$$P/Q \simeq M(P_{++}) \longrightarrow M,$$

showing that

$$\begin{tikzcd} D(M) \arrow[rr, hook] \arrow[d] & & G(\mathfrak{g}) \\ D(P/Q) \arrow[r] & D(P) \arrow[ur] & \end{tikzcd}.$$

There are several ways to show that $D(P/Q)$ is central.

One way is to observe that, the P/Q-grading on $V = \omega^{\mathfrak{g}}(V, \rho)$ given by $D(P/Q) \hookrightarrow D(P) \hookrightarrow G(\mathfrak{g})$, which is

$$V = \bigoplus_{\alpha \in P} V_\alpha = \bigoplus_{[\alpha] \in P/Q} \Big(\bigoplus_{\alpha' \in [\alpha]} V_{\alpha'} \Big),$$

lifts to a P/Q-grading on $\mathrm{Rep}(\mathfrak{g})$, namely, each summand $\bigoplus_{\alpha' \in [\alpha]} V_{\alpha'}$ is $\mathfrak{g}$-stable: $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in R} \mathfrak{g}_\beta$ and

$$\mathfrak{g}_\beta \cdot V_\alpha \subset V_{\alpha+\beta}, \quad \forall \beta \in R,$$

therefore, $D(P/Q) \hookrightarrow G(\mathfrak{g})$ is central (4.6). Another way is to use (III, 3.4.1)

$$\begin{aligned} Z(\mathfrak{g}) &= \ker(\mathrm{Ad}: G(\mathfrak{g}) \to GL_{\mathfrak{g}}) \\ &= \ker(\mathrm{Ad}|_{D(P)}: D(P) \to GL_{\mathfrak{g}}). \end{aligned}$$

(since $Z(\mathfrak{g}) \subset D(P)$)

The decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ is taken w.r.t. $\mathrm{ad}|_{\mathfrak{h}}$, as well as $\mathrm{Ad}|_{D(P)}$ (by (4.7) and (III, 3.3.1)), and $D(P)$ acts on $\mathfrak{g}_\alpha$ via $\alpha: D(P) \to \mathbb{G}_m$. So

$$\ker(\mathrm{Ad}|_{D(P)}) = \bigcap_{\alpha \in R} \ker(\alpha: D(P) \to \mathbb{G}_m) = \mathrm{Spec}\, \frac{k[P]}{(\alpha - 1;\ \alpha \in R)}$$

$= \mathrm{Spec}\, k[P/Q] = D(P/Q)$. $\square$.

If G is a semisimple alg. group with Lie algebra $\mathfrak{g}$, then by (4.5), there is a unique isogeny $G(\mathfrak{g}) \to G$

inducing a specified isom. on their Lie algebras, hence $\mathrm{Rep}(G)$ can be identified with the full subcategory of $\mathrm{Rep}(G(\mathfrak{g}))$ consisting of $G(\mathfrak{g})$-rep. on which $\ker(G(\mathfrak{g}) \to G)$ acts trivially. How do we translate this condition to $\mathfrak{g}$-rep.?

• Th. 4.9. Let $((V, R), X)$ be a diagram (see p. 2), and let $\mathfrak{g}$ be the semisimple Lie algebra over k corresponding to (V, R). Let $\mathrm{Rep}(\mathfrak{g})^X$ be the full subcategory of $\mathrm{Rep}(\mathfrak{g})$ consisting of $\mathfrak{g}$-representations whose irreducible components all have highest weights in $X_{++} := X \cap P_{++}$. Then $\mathrm{Rep}(\mathfrak{g})^X$ is a Tannakian subcategory, endowed with a tensor functor

$$\begin{tikzcd} \mathrm{Rep}(\mathfrak{g})^X \arrow[rr] \arrow[dr] & \circlearrowright & X\text{-}\mathrm{Gr.Vec}_k \arrow[dl] \\ & \mathrm{Vec}_k & \end{tikzcd},$$

whose Tannaka dual $(G^X, T^X = D(X) \to G)$ is a semisimple alg. group together with a maximal torus, giving rise to the diagram $((V, R), X)$.

Pf: The subcategory $\mathrm{Rep}(\mathfrak{g})^X$ is closed under $\otimes$: for $\varpi_1, \varpi_2 \in X_{++}$, $V_{\varpi_1} \otimes V_{\varpi_2} = V_{\varpi_1 + \varpi_2} \oplus$ (some irred. $\mathfrak{g}$-modules of the form $V_{\varpi_1 + \varpi_2 - \sum_{\alpha \in S} n_\alpha \cdot \alpha}$), with $n_\alpha \geq 0$, and $\alpha \in S \subset X$, so all highest weights occurring in $V_{\varpi_1} \otimes V_{\varpi_2}$ belong to X.

It is also closed under taking duals: the highest weight $\varpi^\vee$ of $V_\varpi^\vee$ is equal to $-w(\varpi)$ for some $w \in W$, and clearly $w(x) \in x + \sum_{\alpha \in R} \mathbb{Z} \cdot \alpha$ for $x \in P(R)$, so

$$\varpi \in X \implies -w(\varpi) \in X.$$

Hence $\mathrm{Rep}(\mathfrak{g})^X$ is Tannakian. For $(V, \rho) \in \mathrm{Rep}(\mathfrak{g})^X$, the $\mathfrak{h}$-eigenspace decomposition of V becomes an X-grading, whence

$$\begin{tikzcd} \mathrm{Rep}(\mathfrak{g})^X \arrow[rr] \arrow[dr] & & X\text{-}\mathrm{Gr.Vec}_k \arrow[dl] \\ & \mathrm{Vec}_k & \end{tikzcd}.$$

For a $G(\mathfrak{g})$-rep. (V, ρ) to be in this subcategory, it is necessary and sufficient that, in the $T(\mathfrak{h})$-eigenspace decomposition

$$V = \bigoplus_{\alpha \in P} V_\alpha,$$

we have $V_\alpha = 0$ for all $\alpha \notin X$, i.e. the $T(\mathfrak{h})$-action $\rho|_{T(\mathfrak{h})}$ factors through $D(X)$:

$$\begin{tikzcd} D(P/X) \arrow[r, hook] & D(P) \arrow[r, two heads] \arrow[dr, "\rho|_{D(P)}"'] & D(X) \arrow[d, dashed] \\ & & GL_V \end{tikzcd},$$

which is equiv. to that $D(P/X)$ acts trivially on V. As $D(P/X) \subset D(P/Q)$ is central in $G(\mathfrak{g})$, this is equiv. to that the $G(\mathfrak{g})$-action on V descends to $G(\mathfrak{g})/D(P/X)$. Therefore, the Tannaka dual G^X of $\mathrm{Rep}(\mathfrak{g})^X$ is $G(\mathfrak{g})/D(P/X)$, again semisimple with Lie algebra $\mathfrak{g}$, and $D(X) = D(P)/D(P/X)$ is the image of $D(P)$ in G^X, hence $D(X) \hookrightarrow G^X$ is a max. torus T^X in G^X, with character group X. $\square$.

This shows the surjectivity of the correspondence

$$(G,T) \longmapsto ((V,R), X) \qquad \text{(up to isom.)}.$$

For the injectivity, let us prove that $(G,T) \simeq (G^X, T^X)$. Given (G,T), by (4.5) we have an isogeny $G(\mathfrak{g}) \to G$, which is central (i.e. the kernel is contained in $Z(\mathfrak{g})$) because the conjugation action of $G(\mathfrak{g})$ (connected) on the kernel (finite) is trivial by rigidity.

$$\begin{tikzcd}
 & G(\mathfrak{g}) \arrow[r, two heads] & G \\
D(P/X) \arrow[r, hook] & D(P) \arrow[u, hook] \arrow[r, two heads] & T = D(X) \arrow[u, hook] & (X \subset P).
\end{tikzcd}$$

Hence $\ker(G(\mathfrak{g}) \to G) = \ker(D(P) \subset G(\mathfrak{g}) \to G) = D(P/X)$, i.e. $G^X \xrightarrow{\sim} G$ and $D(P)/D(P/X) = T^X \xrightarrow{\sim} T$. Since $G(\mathfrak{g}) \to G$ is a central isogeny, one deduces (exercise) that the center $Z(G)$ is the image of $Z(\mathfrak{g})$, namely

$$Z(G) = D(P/Q)/D(P/X) \simeq D(X/Q).$$

(Or, one can prove that $Z(G) = D(X/Q)$ directly as in the case with $G(\mathfrak{g})$, showing $M(X_{++}) = X/Q$ and give an X/Q-grading on $\mathrm{Rep}(\mathfrak{g})^X$.) In fact, one can prove a "functoriality" result.

• Th. 4.10. Let (G,T) and (G',T') be semisimple alg. groups with max. tori, corresponding to diagrams $((V,R),X)$ and $((V',R'),X')$, respectively. Then for any isomorphism

$$\varphi : (V,R) \xrightarrow{\sim} (V',R')$$

of root systems such that $\varphi(X) \subset X'$, there is a unique isogeny $G' \xrightarrow{f} G$ inducing φ (in particular, $f(T') = T$).

Pf: From the classification of s.s. Lie algebras in terms of root systems, we know that φ induces an isom. of Lie algebras

$$(\mathfrak{g}, \mathfrak{h}) \simeq (\mathfrak{g}', \mathfrak{h}'),$$

hence an equivalence

$$\begin{tikzcd} \mathrm{Rep}(\mathfrak{g}') \arrow[rr, "\sim"] \arrow[dr, "\omega^{\mathfrak{g}'}"'] & & \mathrm{Rep}(\mathfrak{g}) \arrow[dl, "\omega^{\mathfrak{g}}"] \\ & \mathrm{Vec}_k & \end{tikzcd},$$

under which $\mathrm{Rep}(\mathfrak{g})^X$ corresponds to $\mathrm{Rep}(\mathfrak{g}')^{\varphi(X)}$, which is contained in $\mathrm{Rep}(\mathfrak{g}')^{X'}$. So by (2.8.4), $\exists !$ hom. (necessarily an isogeny since it induces $\mathfrak{g} \simeq \mathfrak{g}'$)

$$\begin{array}{ccc} G' = \text{"}G^{X'}\text{"} & \longrightarrow & \text{"}G^{X}\text{"} = G \\ \cup & & \cup \\ T' = D(X') & \twoheadrightarrow & D(X) = T \end{array}$$

$(\varphi|_X : X \hookrightarrow X') \rightsquigarrow$

inducing them. □

• Def. 4.11. The Spin group Spin_n, is $G(\mathfrak{so}_n)$.

Course notes of « Linear Algebraic Groups »

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Ch. V. Examples of algebraic groups in Algebraic Geometry.

In algebraic geometry, the Mumford-Tate group of a $\mathbb{Q}$-Hodge structure (e.g. the Betti cohomology of a complex algebraic variety), and the monodromy group of a ~~the~~ Galois representation (e.g. the l-adic cohomology of an alg. variety) are both examples of linear alg. groups. They are algebraic quotient groups of the conjectural motivic Galois group, which should be an affine group, Tannaka dual of the Tannakian category of motives. We will give a brief exposition of these concepts. In this chapter, we allow ourselves to state results without proof.

§1. Hodge structures.

Let V be a f. dim'l vector space over a base field k, and let $T^{m,n}(V)$ be $V^{\otimes m} \otimes (V^\vee)^{\otimes n}$, for $m, n \geq 0$, whose elements could be called <u>tensors of type (m,n) on V</u>.

Given a family $\{t_i\}_I$ of tensors on V of various types, one can associate an alg. subgroup $G \subset GL_V$, consisting of all operators that fix each t_i. If one wants to consider operators that preserve the t_i's up to scalar, one can consider a refined alg. subgroup of $GL_V \times \mathbb{G}_m$ to record the extra scalar. Conversely, given an alg. subgroup $G \subset GL_V$, one can consider, for each type (m,n), the set of tensors fixed by G of type (m,n) : $T^{m,n}(V)^G$. If we then consider the alg. subgroup $\widetilde{G} \subset GL_V$ of operators fixing all tensors that are fixed by G, we then have

$$G \subset \widetilde{G},$$

which is not necessarily an equality.

• <u>Exercise 1.1</u>. Assume that either $G \subset GL_V$ is a linearly reductive alg. subgroup, or $\mathrm{Hom}(GL_V, \mathbb{G}_m) \twoheadrightarrow \mathrm{Hom}(G, \mathbb{G}_m)$, then $G = \widetilde{G}$. (<u>Hint</u>: Use Chevalley's theorem (I, 5.3).) //.

Now let $k = \mathbb{Q}$, and let V be a $\mathbb{Q}$-Hodge structure (pure). Then for any integers m, n, p $(m, n \geq 0)$, (Tate Hodge struc.)

$$T^{m,n,p}(V) := V^{\otimes m} \otimes (V^{\vee})^{\otimes n} \otimes \mathbb{Q}(p)$$

is a $\mathbb{Q}$-H.S. By a <u>Hodge tensor on V</u>, we mean a rational tensor of Hodge type $(0,0)$ in some $T = T^{m,n,p}(V)$, i.e.

$$t \in T \cap T_{\mathbb{C}}^{0,0}.$$

Define a hom. of $\mathbb{C}$-alg. groups

$$\mu = \mu_V : \mathbb{G}_{m,\mathbb{C}} \longrightarrow GL_{V_\mathbb{C}}$$

$$\lambda \in \mathbb{C}^\times \longmapsto (\text{scalar mult. } \lambda^{-p} \text{ on } V_\mathbb{C}^{p,q})$$

(description on R-pts., for $R \in \mathbb{C}$-Alg., is similar). Then $\bar{\mu} \in \mathrm{Hom}_{\mathbb{C}\text{-gr.}}(\mathbb{G}_{m,\mathbb{C}}, GL_{V_\mathbb{C}})$ is given by

$$\lambda \in \mathbb{C}^\times \longmapsto (\bar{\lambda}^{-q} \text{ on } V_\mathbb{C}^{p,q}). \qquad \text{(exercise)}$$

Then $\mathbb{C}$-hom. $\mu + \bar{\mu} : \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \to GL_{V_\mathbb{C}}$ (note that $\mu(\lambda)$ commutes with $\bar{\mu}(\lambda')$, $\forall \lambda, \lambda'$) descends to an $\mathbb{R}$-hom.

$$h = h_V : \mathbb{S} \longrightarrow GL_{V_\mathbb{R}}$$

(on $\mathbb{R}$-pts): $\lambda \in \mathbb{C}^\times \longmapsto \lambda^{-p} \cdot \bar{\lambda}^{-q}$ on $V_\mathbb{C}^{p,q} \cap V_\mathbb{R}$

(on $\mathbb{C}$-pts): $(\lambda_1, \lambda_2) \in \mathbb{C}^\times \times \mathbb{C}^\times \longmapsto \lambda_1^{-p} \cdot \lambda_2^{-q}$ on $V_\mathbb{C}^{p,q}$.

For $V = \mathbb{Q}(1)$, we have that $\mu = \mathrm{id} : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}}$ and $h = \det : \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$, $\lambda \mapsto \lambda \cdot \bar{\lambda}$ (or $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a^2 + b^2$).

• <u>Def. 1.2.</u> Let V be a $\mathbb{Q}$-Hodge structure. Define its <u>Mumford–Tate group</u> $MT(V)$, to be the alg. subgroup of $GL_V \times \mathbb{G}_{m,\mathbb{Q}}$ (where $\mathbb{G}_{m,\mathbb{Q}}$ acts on $\mathbb{Q}(1)$ by $\lambda \mapsto \lambda$, hence on $\mathbb{Q}(p)$ by $\lambda \mapsto \lambda^{+p}$) fixing all Hodge tensors on V (namely, Hodge tensors in $T^{m,n,p}(V)$, $\forall m, n, p$).

Facts: $MT(V)$ is the smallest ($\mathbb{Q}$-) alg. subgroup G of $GL_V \times \mathbb{G}_m$ whose $\mathbb{C}$-points contains $\mathrm{Im}(\mu_{V\oplus\mathbb{Q}(1)})$, i.e.

$$\begin{tikzcd} \mathbb{G}_{m,\mathbb{C}} \arrow[rr, "\mu_{V\oplus\mathbb{Q}(1)}"] \arrow[dr, dashed] & & GL_{V_\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \\ & G_\mathbb{C} \arrow[ur, hook] & \end{tikzcd}$$

$$\lambda \mapsto (\lambda^{-p} \text{ on } V_\mathbb{C}^{p,q}, \lambda).$$

It is also the Tannaka dual group of the sub-Tannakian category $\langle V, \mathbb{Q}(1)\rangle^{\otimes}$ of $MHS_{\mathbb{Q}}^{split}$.

If V is a polarizable $\mathbb{Q}$-H.S. (pure), then $MT(V)$ is reductive. (Recall that a polarization of an $\mathbb{R}$-H.S. V, pure of weight k, is a **morphism** of $\mathbb{R}$-H.S., $(-1)^k$-symmetric,

$$Q : V\otimes V \longrightarrow \mathbb{R}(-k) := (2\pi i)^{-k}\cdot\mathbb{R},$$

such that $Q_C(u,v) := \underbrace{=}_{(2\pi i)^k} Q(u, Cv)$ on V, which is real-valued and symmetric, is positive-definite. Here $C = h(i)$ is the Weil operator, $h : \mathbb{S} \to GL_V$. ~~A pure $\mathbb{Q}$-H.S. V is said to be~~ A morphism of $\mathbb{Q}$-H.S., where V is a $\mathbb{Q}$-H.S. pure of wt. k,

$$Q : V\otimes V \longrightarrow \mathbb{Q}(-k)$$

is a polarization of V if $Q_{\mathbb{R}}$ is a polarization of $V_{\mathbb{R}}$.)

If a H.S. V is non-trivial, i.e. $\exists$ Hodge types (p,q) other than $(0,0)$ in $V_\mathbb{C} \simeq \bigoplus V_\mathbb{C}^{p,q}$, then clearly the p- and q-components of Hodge types in $\{V^{\otimes n} \mid n \geq 0\}$ are unbounded.

In particular, $\langle V \rangle$ is not closed under $\otimes$. By (IV, 3.3), we see that $MHS_{\mathbb{R}}$ (similarly, $MHS_{\mathbb{R}}^{split}$, $MHS_{\mathbb{Q}}$, $\cdots$) has connected Tannaka dual group. Hence, for instance, $MT(V)$ is a connected $\mathbb{Q}$-alg. group.

For a polarizable $\mathbb{Q}$-Hodge structure, we see that $MT(V)$ is commutative if and only if it is a torus.

• Examples 1.3.

a). For $V = \mathbb{Q}(1)$, $T^{m,n,p}(V) = \mathbb{Q}(m+p-n)$, on which $(a, \lambda) \in GL_V \times \mathbb{G}_m$ acts as $a^{m-n} \cdot \lambda^p$. Since $T^{m,n,p}$ has Hodge vectors if and only if $m+p=n$, the scalar $a^{m-n} \cdot \lambda^p$ should be 1 in this case, i.e. $a = \lambda$. So $MT(\mathbb{Q}(1)) = \mathbb{G}_m$, diagonally embedded into $GL_V \times \mathbb{G}_m$.

b). Let E be the complex elliptic curve $\mathbb{C}/\mathbb{Z}[i]$, and $V = H_1(E, \mathbb{Q}) = \mathbb{Q}(i)$, a polarizable $\mathbb{Q}$-H.S. of weight -1. Identify GL_V with GL_2 w. r. t. the basis $\{1, i\}$ of V. The polarization

$$Q: V \wedge V \xrightarrow{\sim} \mathbb{Q}(1)$$
$$1 \wedge i \longmapsto 2\pi i \qquad (\text{hence } i \wedge 1 \mapsto -2\pi i)$$

shows that $T^{0,2,1}(V)$ contains a Hodge vector, spanning the line $V^\vee \wedge V^\vee(1) \simeq \mathbb{Q}(0)$, on which $(A, \lambda) \in GL_V \times \mathbb{G}_m$ acts as multiplication by $\det({}^tA^{-1}) \cdot \lambda$. So $MT(V) \subset \{(A, \det(A));$
$A \in GL_2\}$.

Let $f : E \to E$ be the morf. "mult. by i", which induces an isom. of $\mathbb{Q}$-H.S. on $H_1(E, \mathbb{Q})$:

$$f_* : V \xrightarrow{\sim} V$$
$$1 \longmapsto i$$
$$i \longmapsto -1 .$$

$\left[1^* = \frac{1}{2\pi i} \cdot i \,,\; i^* = \frac{-1}{2\pi i} \cdot 1\right]$

Under the identification

$$\mathrm{Hom}(V, V) \simeq V^\vee \otimes V \overset{\mathbb{Q}}{\simeq} V \otimes V\,(-1)$$
$$f_* \longmapsto 1^* \otimes i - i^* \otimes 1 \longmapsto \frac{1}{2\pi i}(1 \otimes 1 + i \otimes i),$$

the morf. f_* of $\mathbb{Q}$-H.S. corresponds to a Hodge vector in $V \otimes V(-1)$. We know that $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda\right) \in GL_2 \times \mathbb{G}_m$ sends $\frac{1}{2\pi i}(1 \otimes 1 + i \otimes i)$ to

$$\frac{\lambda^{-1}}{2\pi i}\Big((a + ci) \otimes (a + ci) + (b + di) \otimes (b + di)\Big)$$

$$= \frac{\lambda^{-1}}{2\pi i}\Big((a^2 + b^2)\cdot(1 \otimes 1) + (ac + bd)\,(1 \otimes i + i \otimes 1) + (c^2 + d^2)(i \otimes i)\Big),$$

and solving

$$\begin{cases} a^2 + b^2 = c^2 + d^2 = \lambda = \det(A) = ad - bc \\ ac + bd = 0 \end{cases}$$

one obtains $a = d$, $b = -c$, therefore

$$MT(V) \subset \left\{\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, a^2 + b^2\right) \in GL_2 \times \mathbb{G}_m\right\}.$$

- Exercise: This is an equality. //.

The right hand-side, isomorphic to "Weil's restriction of scalars" $\mathrm{Res}_{\mathbb{Q}}^{\mathbb{Q}(i)}(\mathbb{G}_{m,\mathbb{Q}(i)}) : R \in \mathbb{Q}\text{-Alg.} \mapsto (\mathbb{Q}(i) \otimes_{\mathbb{Q}} R)^{\times}$, is a non-split $\mathbb{Q}$-torus of ~~rank~~ dim. 2.

- Def. 1.4. A polarizable $\mathbb{Q}$-Hodge structure V is of CM-type, if $MT(V)$ is a torus. A simple abelian variety A of dim. g over $\mathbb{C}$ is of CM-type, if the $\mathbb{Q}$-division algebra $\mathrm{End}(A)_{\mathbb{Q}}$ is commutative (hence is a field) of degree $2g$ over $\mathbb{Q}$. In particular, $H_1(A,\mathbb{Q})$ is an $\mathrm{End}(A)_{\mathbb{Q}}$-vector space of dim. 1.

Fact: A simple $\mathbb{C}$-abelian variety A is of CM-type if and only if the $\mathbb{Q}$-H.S. $V = H_1(A,\mathbb{Q})$ is of CM-type.

The full subcategory $PMHS_{\mathbb{Q}}^{split}$ of $MHS_{\mathbb{Q}}^{split}$, consisting of all $\mathbb{Q}$-MHS. whose pure components are all polarizable, is closed under $\otimes$ and taking duals, hence Tannakian. The abelianization $\pi_1(PMHS_{\mathbb{Q}}^{split})^{ab}$ of its Tannaka dual group, which is a $\mathbb{Q}$-protorus, projective limit of Mumford-Tate groups $MT(V)$ of CM. $\mathbb{Q}$-H.S. V, is called the connected Serre group, denoted S°. Its character group $X(S^{\circ})$ with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action can be identified with the group of locally constant functions

$$f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{Z}$$

(on which $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts by right translation) such that, letting c denote a (any) complex conjugation in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$,

— $f(\sigma c \tau) = f(c\sigma\tau)$, $\forall\, \sigma, \tau \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$,

— $f(\sigma) + f(\sigma c)$ is independent of σ.

When the Hodge structure V arises as the Betti cohomology group of an alg. variety $/\mathbb{C}$, one can also consider algebraic cycles in place of Hodge tensors. Let X be a projective smooth complex algebraic variety; then one can define, for each <u>alg. cycle Z of pure codimension</u> p on X (i.e. $\mathbb{Q}$-linear formal combination $\sum_{\text{finite}} a_i \cdot [Z_i]$ of irreducible subvarieties Z_i of X, all of codim. p), its <u>fundamental class</u> $cl(Z) \in H^{2p}_B(X^{an}, \mathbb{Q})$, a rational class of type (p,p). So one can view $cl(Z)$ as a morp. of $\mathbb{Q}$-Hodge structures

$$cl(Z) : \mathbb{Q}(0) \longrightarrow H^{2p}_B(X^{an}, \mathbb{Q})(p),$$

or, as we will do, write it as $cl(Z) \in H^{2p}_B(X^{an}, \mathbb{Q})(p)$.

(Recall that, one first defines this for divisors ($p=1$), using the exponential sequence

$$1 \longrightarrow \underset{\underset{\mathbb{Z}(1)}{\parallel}}{2\pi i \cdot \mathbb{Z}} \longrightarrow \mathcal{O}_{X^{an}} \longrightarrow \mathcal{O}^{\times}_{X^{an}} \longrightarrow 1,$$

then uses the "splitting principle" to define Chern classes of vector bundles on X, which can then be generalized to coherent $\mathcal{O}_X$-modules by locally free resolution, and finally one puts, for Z an irreducible subvariety of codim. p,

$$cl(Z) := \frac{1}{(p-1)!} \cdot c_p(\mathcal{O}_Z).\ )$$

The Hodge conjecture predicts that all Hodge vectors in $H_B^{2p}(X^{an}, \mathbb{Q})(p)$ (i.e. rational classes of Hodge type $(0,0)$) are fundamental classes of algebraic cycles.

As for Mumford-Tate groups, one can consider linear automorphisms of the Betti cohomology that preserve all the algebraic cycles, ~~and~~ not only on X but also on iterated self-products of X, via the Künneth formula.

Let us restrict ourselves to the case where X is an abelian variety. Note that for any m, n, we have

$$H_B^m(X^n) = \Lambda^m H_B^1(X^n), \quad H_B^1(X^n) = \bigoplus_n H_B^1(X),$$

so there is a natural action of $GL_{H_B^1(X)} \times \mathbb{G}_m$ on $H_B^m(X^n)(p)$, $\forall\, m, n, p$. Define $G_X^{alg} \subset GL_{H_B^1(X)} \times \mathbb{G}_m$ to be the (algebraic) subgroup of operators fixing all alg. cycles on self-products of X.

(As we switch from homology to cohomology, one may want to make $\mathbb{G}_m$ act on $\mathbb{Q}(1)$ by $\lambda \mapsto \lambda^{-1}$; this is not essential.)

• Example 1.5. If X is a non-CM elliptic curve, then G_X^{alg} is the group of $(A, \lambda) \in GL_2 \times \mathbb{G}_m$ preserving

$$cl(\text{point}) \in H_B^2(X)(1),$$

so

$$G_X^{alg} = \{ (A,\lambda) \mid det(A) = \lambda^{-1} \text{ (or } \lambda \text{, if you change the } \mathbb{G}_m\text{-action on } \mathbb{Q}(1) \text{ to } \lambda \mapsto \lambda^{-1}) \}$$

$$\simeq GL_{2,\mathbb{Q}}$$

If X is a C.M. elliptic curve, then G_X^{alg} is a subtorus in $GL_{2,\mathbb{Q}}$ of dimension 2; see Example 1.3, b). If $E = End(X)_{\mathbb{Q}}$, then it is a quadratic imaginary field, and

$$G_X^{alg} = Res_{\mathbb{Q}}^{E}(\mathbb{G}_{m,E}).$$

The group G_X^{alg}, or rather, the alg. cycles on X^n, puts some relations among the <u>abelian integrals</u>. To be precise, let X be an abelian variety defined over a subfield $k \subset \mathbb{C}$; then one has the <u>algebraic de Rham cohomology</u>

$$H_{dR}^i(X/k) := H^i(X, \text{de Rham complex } \Omega^{\bullet}_{X/k}).$$

We have

$$H_{dR}^i(X/k) \otimes_k \mathbb{C} \simeq H_{dR}^i(X_{\mathbb{C}}/\mathbb{C}) \simeq H_{dR}^i(X_{\mathbb{C}}^{an}, \mathbb{C})$$
$$\simeq H_B^i(X_{\mathbb{C}}^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

(These hold for general proj. smooth varieties.).

Let $\{\omega_i\}_{i=1}^{2\dim(X)}$ be a k-basis of $H^1_{dR}(X/k)$, and $\{a_i\}$ be a $\mathbb{Q}$-basis of $H^1_B(X^{an}_{\mathbb{C}}, \mathbb{Q})$. Then we can write

$$\omega_j = \sum_{i=1}^{2\dim(X)} p_{ij} \cdot a_i \quad , \quad p_{ij} \in \mathbb{C}, \text{ called } \underline{\text{periods}}.$$

Explicitly, if $\{[\gamma_i]\}$ is the dual basis to $\{a_i\}$ on $H_1(X^{an}_{\mathbb{C}}, \mathbb{Q})$, then

$$p_{ij} = \int_{\gamma_i} \omega_j \quad \text{(independent of the choice of } \gamma_i \in [\gamma_i]\text{)}.$$

- <u>Prop. 1.6.</u> We have

$$\text{tr. deg.}\left(k(p_{ij}; \forall i,j) / k\right) \leq \dim(G_X^{alg}). \quad \square.$$

- <u>Exercise 1.6.1.</u> For the elliptic curve $y^2 = x^3 - x$ defined over $\mathbb{Q}$, isomorphic to $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i$ after base change to $\mathbb{C}$, compute its periods p_{ij} w.r.t. $\omega_1 = \frac{dx}{y}$, $\omega_2 = \frac{x\,dx}{y}$, $\gamma_1 = [0,1] + \frac{i}{2}$, $\gamma_2 = [0,1]\cdot i + \frac{1}{2}$. //.

One can also define a third class of tensors, called <u>absolute Hodge cycles</u>: (for $X/\mathbb{C}$ proj. smooth)

$$\left\{\begin{matrix}\text{alg. cycles} \\ \text{on } X\end{matrix}\right\} \subset \left\{\begin{matrix}\text{abs. Hodge} \\ \text{cycles on } X\end{matrix}\right\} \subset \left\{\begin{matrix}\text{Hodge cycles} \\ \text{on } X\end{matrix}\right\},$$

those Hodge cycles in $H^{2k}_{dR}(X/\mathbb{C})$ that remain Hodge cycles in $H^{2k}_{dR}(X^\sigma/\mathbb{C})$ for any field automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$. (not necessarily continuous)

Note that X^{an} and $X^{\sigma, an}$ (where $X^{\sigma} = X \otimes_{\mathbb{C},\sigma} \mathbb{C}$) are not necessarily homeomorphic (there are such examples, by Serre, ...), hence the two $\mathbb{Q}$-structures of the $\mathbb{C}$-space:

$$\begin{array}{ccc} H_B^{2k}(X^{an}) , & H_B^{2k}(X^{\sigma, an}) & H_{dR}^{2k}(X/\mathbb{C}) \otimes_{\sigma} \mathbb{C} \simeq \\ \cap & \cap & \\ H_{dR}^{2k}(X/\mathbb{C}) & H_{dR}^{2k}(X^{\sigma}/\mathbb{C}) , & H_{dR}^{2k}(X^{\sigma}/\mathbb{C}) \end{array}$$

are a priori irrelevant. The Hodge conjecture implies that the subspace $H_B^{2k}(X^{an}) \cap F^k H_{dR}^{2k}(X/\mathbb{C})$ (where

$$F^k H_{dR}^n(X/\mathbb{C}) := \mathrm{Image}\,(H^n(X, \Omega_X^{\geq k}) \to H^n(X, \Omega_X^{\bullet})))$$

of $H_{dR}^{2k}(X/\mathbb{C})$, when $\otimes_{\sigma} \mathbb{C}$, coincides with the subspace

$$H_B^{2k}(X^{\sigma, an}) \cap F^k H_{dR}^{2k}(X^{\sigma}/\mathbb{C}) \subset H_{dR}^{2k}(X^{\sigma}/\mathbb{C}),$$

for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, namely Hodge cycles are absolute Hodge cycles.

- <u>Th. 1.7.</u> (Deligne, 1978). <u>For any abelian variety</u> $/\mathbb{C}$, <u>Hodge cycles are absolute Hodge cycles</u>. □.

§2. <u>Galois representations</u>.

Let Γ be an abstract group, $\rho : \Gamma \to GL(V)$ an F-linear rep. of Γ, for some f. dim'l F-vector space V. Then one may consider the Zariski closure of $\mathrm{Im}(\rho) \subset GL(V)$, which is an alg. subgroup $\overline{\mathrm{Im}(\rho)} \subset GL_V$.

In what follows, Γ will be the absolute Galois group $\mathrm{Gal}(K^s/K)$ of some field K, and $F = \overline{\mathbb{Q}}_\ell$, ℓ a prime number $\neq \mathrm{char}(K)$.

For instance, let X be an algebraic variety over K. The theory of <u>ℓ-adic cohomology</u> associates, to each ~~...~~ integer $n \in [0, 2\dim X]$, a f. dim'l $\mathbb{Q}_\ell$-vector space

$$H^n(X_{K^s}, \mathbb{Q}_\ell)$$

together with a continuous action of $\mathrm{Gal}(K^s/K)$, functorial in X.

Let $\chi_\ell^{cycl} : \mathrm{Gal}(K^s/K) \to \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times$ be the <u>ℓ-adic cyclotomic character</u>, i.e., projective limit of the natural actions of $\mathrm{Gal}(K^s/K)$ on $\mu_{\ell^r}(K^s) \simeq \mathbb{Z}/\ell^r\mathbb{Z}$; so for any $\zeta \in \mu_{\ell^r}(K^s)$, $\sigma \in \mathrm{Gal}(K^s/K)$, we have

$$\sigma(\zeta) = \zeta^{\chi_\ell^{cycl}(\sigma) \pmod{\ell^r}}.$$

Usually we denote χ_ℓ^{cycl} by $\mathbb{Q}_\ell(1)$, the <u>Tate twist</u>, and $(\chi_\ell^{cycl})^{\otimes m}$ by $\mathbb{Q}_\ell(m)$.

For $X = \mathbb{G}_{m,K}$, we have that $H^1(\mathbb{G}_{m,K^s}, \mathbb{Q}_\ell)$ is a 1-dim'l $\mathrm{Gal}(K^s/K)$-rep., isomorphic to $(\chi_\ell^{cycl})^\vee = \mathbb{Q}_\ell(-1)$.

For X an abelian variety of dim. g over K, $H^1(X_{K^s}, \mathbb{Q}_\ell)$ is a $2g$-dim'l $\mathrm{Gal}(K^s/K)$-rep., dual to the <u>Tate module</u>

$$V_\ell(X) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \,, \quad \underset{\substack{\circlearrowright \\ \mathrm{Gal}(K^s/K)}}{T_\ell(X)} = \varprojlim_n \underset{\substack{\uparrow \\ (\ell^n\text{-torsion part})}}{X[\ell^n](K^s)},$$

and $H^n(X_{K^s}, \mathbb{Q}_\ell) = \Lambda^n H^1(X_{K^s}, \mathbb{Q}_\ell)$ as $\mathrm{Gal}(K^s/K)$-rep.,

and $H^{2g}(X_{K^s}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-g)$.

There are various "independence of ℓ" type of conjectures, at least for K of finite type over its prime field. For instance, are the $\mathbb{Q}_\ell$-alg. groups $\overline{\mathrm{Image}}$ have the same dimension? Same number of connected components? Even more optimistically, is there an alg. group over $\mathbb{Q}$ (or some finite extension of $\mathbb{Q}$) giving rise to all these $\mathbb{Q}_\ell$-alg. groups by base change (or to $\overline{\mathbb{Q}}_\ell$, resp.)?

More generally, let X be an alg. variety (connected) over a base field k, with a geometric pt. $\bar{x} \to X$ chosen. Grothendieck defined the <u>étale fundamental group</u> $\pi_1(X, \bar{x})$ of X. When X is integral and normal, $\pi_1(X, \bar{x})$ is the quotient group of $\mathrm{Gal}(k(X)^s/k(X))$, corresponding to the max. intermediate extension $L/k(X)$ unramified everywhere on X. An ℓ-<u>adic lisse sheaf</u> on X can be defined as a continuous ℓ-adic rep. of $\pi_1(X, \bar{x})$ ($\ell \neq \mathrm{char}(k)$ as always):

$$\rho_{\mathcal{F}} : \pi_1(X, \bar{x}) \longrightarrow GL(V), \quad V = \mathcal{F}_{\bar{x}}.$$

The $Gal(K^s/K)$-rep. on $H^n(X_{K^s}, \mathbb{Q}_\ell)$ we had before corresponds to the lisse sheaf $R^n a_* \mathbb{Q}_\ell$ on $Spec(K)$, where $a: X \to Spec(K)$ is the structural map.

For the $Gal(K^s/K)$-rep. on H^n, we introduce the following theorem of Serre.

• Th. 2.1. Let X be an elliptic curve over a number field K, such that $X_{\bar{K}}$ is not of C.M. type. Let

$$\rho_\ell : Gal(\bar{K}/K) \longrightarrow GL(T_\ell(X))$$

be the Galois rep. on the ℓ-adic Tate module, for each prime ℓ. Then, $Im(\rho_\ell)$ is open, for all ℓ. Moreover, $Im(\rho_\ell) = GL(T_\ell(X))$ for almost all ℓ. In particular, the Zariski closure of $Im(\rho_\ell)$ in $GL_{V_\ell(X)}$ is the full group $GL_{V_\ell(X)}$. □.

For the monodromy groups of lisse sheaves, we introduce a theorem of Grothendieck. Let the base field k be $\mathbb{F}_q$ (of characteristic p), and let X_0 be a normal alg. variety over $\mathbb{F}_q$, geometrically connected so that

$$\pi_1(X_0, \bar{x}) \longrightarrow \pi_1(Spec\, \mathbb{F}_q) \simeq Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q)$$

is surjective; its kernel can be identified with $\pi_1(X, \bar{x})$, where $X = X_0 \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$; see [SGA 1, IX, Th. 6.1].

Define the Weil group of $\mathbb{F}_q$ to be the subgroup of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \simeq \hat{\mathbb{Z}}$ generated by the Frobenius Frob_q :

$$\begin{array}{ccc} W(\overline{\mathbb{F}}_q/\mathbb{F}_q) & \subset & \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \\ \wr\| & & \wr\| \\ \mathbb{Z} & \subset & \hat{\mathbb{Z}} \end{array},$$

and define the Weil group of X_0 to be the inverse image of $W(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ in $\pi_1(X_0, \bar{x})$:

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(X, \bar{x}) \arrow[r] \arrow[d, equal] & W(X_0, \bar{x}) \arrow[r, "\deg"] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\square"] & \mathbb{Z} \arrow[r] \arrow[d, phantom, "\cap"] & 0 \\ 0 \arrow[r] & \pi_1(X, \bar{x}) \arrow[r] & \pi_1(X_0, \bar{x}) \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 0 \end{tikzcd},$$

topologized such that the subgroup $\pi_1(X, \bar{x})$ is open.

- Lem. 2.2. Let X_0 be a smooth $\mathbb{F}_q$-curve. Then the image of $\pi_1(X, \bar{x})$ in the abelianized Weil group $W(X_0, \bar{x})^{ab}$ ($:= W(X_0, \bar{x})$ modulo the closure of its derived group) is the product of a finite group and a pro-p-group (i.e. a profinite group all of whose finite quotient groups are p-groups).

Pf: By class field theory, if K denotes the function field $\mathbb{F}_q(X_0)$ of X_0 and $\mathbb{A}$ denotes the ring of adèles of K, then there is a natural isomorphism (the Artin reciprocity map) of topological groups

$$\mathbb{A}^\times / K^\times \cdot \prod_{v \notin S} \mathcal{O}_{K_v}^\times \xrightarrow{\ \sim\ } W(X_0, \bar{x})^{ab},$$

where S is the complement of X_0 in its smooth compactification $\overline{X}_0$. Moreover, the ~~subgroup~~ image of $\pi_1(X, \bar{x})$ in $W(X_0, \bar{x})^{ab}$ corresponds to $\mathbb{A}^1 / K^\times \cdot \prod_{v \notin S} \mathcal{O}_{K_v}^\times$, where $\mathbb{A}^1$ is the group of idèles of norm 1; see Weil [BNT, XIII, Th. 6]. From the exact sequence

$$0 \to \Big(\prod_{v \in S} \mathcal{O}_{K_v}^\times\Big) / K^\times \to \mathbb{A}^1 / K^\times \cdot \prod_{v \notin S} \mathcal{O}_{K_v}^\times \to \mathbb{A}^1 / K^\times \cdot \prod_{\text{all } v} \mathcal{O}_{K_v}^\times \to 0$$

$$\mathbb{A}^1 / K^\times \cdot \prod_{\text{all } v} \mathcal{O}_{K_v}^\times = \mathrm{Pic}^\circ(\overline{X}_0) = \mathrm{Jac}_{\overline{X}_0}(\mathbb{F}_q), \text{ a finite group.}$$

we see that $\mathbb{A}^1 / K^\times \cdot \prod_{v \notin S} \mathcal{O}_{K_v}^\times$ is the product of a finite group and a pro-p-group. $\square$.

The ~~rest~~ result also holds for any normal geom. connected alg. variety $X_0 / \mathbb{F}_q$. You may assume that X_0 is a curve in the sequel, if you want.

Let $\mathcal{F}_0$ be a lisse l-adic sheaf on X_0 (i.e. a cont. rep. $\rho_{\mathcal{F}_0} : \pi_1(X_0, \bar{x}) \to GL(\mathcal{F}_{0,\bar{x}})$), or just a <u>lisse Weil sheaf</u>, i.e. a cont. rep. $\rho_{\mathcal{F}_0} : W(X_0, \bar{x}) \to GL(V)$. Define the <u>geometric monodromy group</u> of $\mathcal{F}_0$ to be the Zariski closure of the image $\rho_{\mathcal{F}_0}(\pi_1(X, \bar{x}))$ in GL_V, denoted $G_{\mathcal{F}_0}^{geom}$ or G_ρ^{geom} ($\rho = \rho_{\mathcal{F}_0}$).

• <u>Cor. 2.3.</u> <u>Let</u> $\rho : W(X_0, \bar{x}) \to \overline{\mathbb{Q}}_\ell^\times$ <u>be a cont. character</u>.
<u>Then</u> $\rho(\pi_1(X, \bar{x}))$ <u>is a finite group</u>.

<u>Pf</u>: ρ factors through $W(X_0, \bar{x})^{ab}$, in which the image of $\pi_1(X, \bar{x})$ is (finite group) $\times$ (pro-p-group), whose image in $E_\lambda^\times$ (where E_λ is some finite extension of $\mathbb{Q}_\ell$) which is contained in $\mathcal{O}_\lambda^\times$ as $\pi_1(X, \bar{x})$ is compact, is finite, since $\ell \neq p$. $\square$.

Let $\rho : W(X_0, \bar{x}) \to GL(V)$ be a cont. ℓ-adic rep..
We will define an affine group G over $\overline{\mathbb{Q}}_\ell$ fitting into the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(X, \bar{x}) \arrow[r] \arrow[d] & W(X_0, \bar{x}) \arrow[r, "\deg"] \arrow[d] \arrow[dd, bend left, "\rho"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_\rho^{geom}(\overline{\mathbb{Q}}_\ell) \arrow[r] \arrow[dr, bend right] & G(\overline{\mathbb{Q}}_\ell) \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
 & & GL(V) & &
\end{tikzcd}$$
,

where the second row arises from an exact sequence of affine group schemes

$$0 \to G_\rho^{geom} \longrightarrow G \longrightarrow \underline{\mathbb{Z}} \to 0 .$$

Let $w \in W(X_0, \bar{x})$ be an element of degree 1; then the conjugation ~~int~~ inn(w) by w stabilizes the subgroup $\pi_1(X, \bar{x})$, hence ~~int~~ inn$(\rho(w))$ stabilizes the alg. subgroup $G_\rho^{geom} \subset GL_V$. Define G to be the semi-direct product

$$G = G_\rho^{geom} \rtimes \mathbb{Z}, \quad \xrightarrow[(g,n) \mapsto g \cdot \rho(w)^n]{} GL_V$$

where $1 \in \mathbb{Z}(\overline{\mathbb{Q}}_\ell)$ acts on G_ρ^{geom} by $\mathrm{inn}(\rho(w))$.

• <u>Th. 2.4.</u> (Grothendieck). <u>Let</u> $\mathcal{F}_0$ <u>be a lisse Weil sheaf on</u> X_0, <u>with geometric monodromy group</u> G^{geom}. <u>Then the radical of</u> $(G^{geom})^0$ <u>is unipotent</u>. <u>In particular, if</u> $\mathcal{F}_0$ <u>is geometrically semisimple</u> (<u>i.e. that</u> $\rho_{\mathcal{F}_0}|_{\pi_1(X,\bar{x})}$ <u>is a s.s. rep.</u>), <u>then</u> $(G^{geom})^0$ <u>is a semisimple alg. group</u>.

Pf: That $\mathcal{F}_0$ is geometrically s.s. is equiv. to that $G^{geom} \hookrightarrow GL_V$, the tautological rep., is s.s., thus is equiv., by (III, 4.3), to that $(G^{geom})^0$ is reductive.
Hence, the first claim implies the second.

Conversely, it suffices to prove the second claim.

Let $\mathcal{F}_0^\bullet$ be a Jordan-Hölder filtration of $\mathcal{F}_0$ (i.e. a filt. of $\rho_{\mathcal{F}_0}$ by subrep. V^i, s.t. each V^i/V^{i+1} is an irred. rep. of $W(X_0,\bar{x})$); then $Gr(\mathcal{F}_0^\bullet)$ is semisimple, hence geometrically semisimple (same proof as that of (II, 6.4)).
Let $P \subset GL_V$ $(V = \mathcal{F}_{0\bar{x}})$ be the alg. subgroup preserving the flag $(\mathcal{F}^i_{0\bar{x}} = V^i)_i$ (a ~~ ~~ <u>parabolic</u> subgroup $\begin{pmatrix} \urcorner & * \\ 0 & \llcorner \end{pmatrix}$), N the unipotent radical of P: $\begin{pmatrix} I & & & * \\ & I & & \\ & & I & \\ 0 & & & I \end{pmatrix}$, and $L := P/N$

the <u>Levi factor</u> : $\begin{pmatrix} \boxed{*} & & & 0 \\ & \boxed{*} & & \\ & & \boxed{*} & \\ 0 & & & \boxed{*} \end{pmatrix}$. Clearly, $\rho_{\mathcal{F}_0}(W(X_0,\bar{x}))$ is contained in $P(\overline{\mathbb{Q}}_\ell)$, hence $\rho_{\mathcal{F}_0}(\pi_1(X,\bar{x}))$, as well as $G^{geom}_{\mathcal{F}_0}$, is contained in P, and the image of $G^{geom}_{\mathcal{F}_0}$ in L is nothing other than $G^{geom}_{Gr(\mathcal{F}_0^\bullet)}$:

$$\begin{array}{ccccccccc} 1 & \to & N_1 & \to & G^{geom}_{\mathcal{F}_0} & \longrightarrow & G^{geom}_{Gr(\mathcal{F}_0^\bullet)} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & N & \to & P & \longrightarrow & L & \longrightarrow & 1, \end{array}$$

where N_1 is unipotent, by (II, 4.5). Assuming the second claim, $(G^{geom}_{Gr(\mathcal{F}_0^\bullet)})^\circ$ is semisimple, therefore, the radical of $(G^{geom}_{\mathcal{F}_0})^\circ$ is N_1.

So we may assume that $(G^{geom})^\circ$ is reductive. Let Z° be the connected center of $(G^{geom})^\circ$, and let T be $(G^{geom})^\circ/\mathcal{D}((G^{geom})^\circ)$; both are tori, and the natural map

$$Z^\circ \longrightarrow T$$

is an isogeny (see (III, 4.2)). We need to prove that T is trivial.

Let $\Sigma \subset X(Z^\circ)$ be the set of characters of Z° occuring in $Z^\circ \hookrightarrow GL_V$; it is a finite set generating the group $X(Z^\circ)$, because Z° acts on V faithfully. The conjugation action of $W(X_0, \bar{x})$ on $\pi_1(X, \bar{x})$ induces an action on G^{geom}, hence on Z^0 (which is a characteristic subgroup) and therefore on $X(Z^0)$. It preserves the subset Σ:

$$\rho(\sigma)^{-1} V_\chi = V_{\sigma * \chi}, \quad \sigma \in W(X_0, \bar{x}),\ \chi \in X(Z^0).$$

Therefore, $W(X_0, \bar{x})$ acts on ~~Z^0~~ Z^0 through a finite quotient group:

$$W(X_0, \bar{x}) \big/ \ker(W(X_0, \bar{x}) \to \mathrm{Aut}(\Sigma)).$$

Denote this kernel by W'; its image $\deg(W')$ in $\mathbb{Z}$ is a subgroup of finite index: $n\mathbb{Z} \subset \mathbb{Z}$. By Galois correspondence, this subgroup W' arises as the Weil group of a finite Galois étale covering

$$\begin{array}{ccc} X'_0 & \to & X_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,\mathbb{F}_{q^n} & \to & \mathrm{Spec}\,\mathbb{F}_q \end{array} :$$

$$\begin{array}{ccccccc} 1 \to \pi_1(\overbrace{X'_0 \otimes_{\mathbb{F}_{q^n}} \overline{\mathbb{F}_q}}^{X' :=}, \bar{x}') & \longrightarrow & W' & \longrightarrow & n\mathbb{Z} & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \longrightarrow \pi_1(X, \bar{x}) & \longrightarrow & W & \longrightarrow & \mathbb{Z} & \longrightarrow & 1, \end{array}$$

so $\pi_1(X', \bar{x}')$ is a normal subgroup of finite index in $\pi_1(X, \bar{x})$. Therefore, $[\overline{\rho(\pi_1(X, \bar{x}))} : \overline{\rho(\pi_1(X', \bar{x}'))}] < \infty$, and hence $(G^{geom}_{\rho|W'})^0 = (G^{geom}_\rho)^0$ (by (II, 1.8 (ii))). Replacing $X_0/\mathbb{F}_q$ by $X'_0/\mathbb{F}_{q^n}$ without changing $(G^{geom})^0$, we may assume that

$W(X_0, \bar{x})$ acts on Z° trivially. The "conjugation action" of $W(X_0, \bar{x})$ on $(G^{geom})^\circ$ then descends to an action on $(G^{geom})^\circ / Z^\circ$, which is a semisimple alg. group, for which $[\mathrm{Aut} : \mathrm{Inn}]$ is finite (III, 3.6). By replacing X_0 by a finite covering as before, we may assume that $W(X_0, \bar{x})$ acts on $(G^{geom})^\circ / Z^\circ$ by inner automorphisms, hence on $(G^{geom})^\circ$ by inner automorphisms: $\forall\, w \in W(X_0, \bar{x})$, $\exists\, g \in (G^{geom})^\circ(\overline{\mathbb{Q}}_\ell)$ s.t. $\mathrm{inn}(w) = \mathrm{inn}(\bar{g})$ on $(G^{geom})^\circ / Z^\circ$; then $\mathrm{inn}(w) = \mathrm{inn}(g)$ on $(G^{geom})^\circ$ because on $\mathrm{Lie}\,(G^{geom})^\circ = \mathrm{Lie}(Z^\circ) \times \mathrm{Lie}((G^{geom})^\circ / Z^\circ)$, they are both trivial on $\mathrm{Lie}(Z^\circ)$ and they agree on $\mathrm{Lie}((G^{geom})^\circ / Z^\circ)$; then apply (II, 2.8.3). Replacing X by a further covering $X' \to X$, arising from some $\mathbb{F}_{q^n}$-morp. $X'_0 \to X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, we may replace G^{geom} by $(G^{geom})^\circ$ (so $\pi_1(X', \bar{x}') \subset \pi_1(X, \bar{x})$ is the inverse image of $(G^{geom})^\circ \subset G^{geom}$) and hence assume that G^{geom} is connected. If $w \in W(X_0, \bar{x})$ is the element of degree 1 chosen before, $g_0 \in G^{geom}(\overline{\mathbb{Q}}_\ell)$ is such that $\mathrm{inn}(w) = \mathrm{inn}(g_0)$ on G^{geom} (where we denote the image of w under $W(X_0, \bar{x}) \to G(\overline{\mathbb{Q}}_\ell)$ again by w), then G is also the semi-direct product of G^{geom} and $\langle g_0^{-1} \cdot w \rangle$ (since $G = G^{geom} \cdot \langle g_0^{-1} w \rangle$ and $G^{geom} \cap \langle g_0^{-1} w \rangle = \mathbb{1}$), and inn $(g_0^{-1} \cdot w)$ on G^{geom} is trivial, so

$$G = G^{geom} \times \mathbb{Z}, \quad \mathbb{Z} = \langle g_0^{-1} \cdot w \rangle.$$

Now under the homomorphism

$$W(X_0, \bar{x}) \longrightarrow G(\bar{\mathbb{Q}}_\ell) \xrightarrow{pr_1} G^{geom}(\bar{\mathbb{Q}}_\ell) \twoheadrightarrow T(\bar{\mathbb{Q}}_\ell),$$

the image of the subgroup $\pi_1(X, \bar{x})$ is Zariski dense in $G^{geom}(\bar{\mathbb{Q}}_\ell)$, hence Zariski dense in $T(\bar{\mathbb{Q}}_\ell)$. But by (2.3), the image of $\pi_1(X, \bar{x})$ in $T(\bar{\mathbb{Q}}_\ell)$ is finite, so $T = \mathbb{1}$. $\square$.

§3. Motives.

Most, if not all, of what we can say in this section is conjectural.

Let k be a base field, and let $\mathcal{V}ar_k$ be the category of projective smooth k-alg. varieties (not necessarily connected). The idea of Grothendieck is that, there ought to be a $\mathbb{Q}$-linear semisimple Tannakian category $\mathcal{M}_k$, the category of k-motives (direct sums of pure ones), together with a functor

$$h : \mathcal{V}ar_k^{op} \longrightarrow \mathcal{M}_k$$

satisfying various expected properties (e.g. $h(X \amalg Y) = h(X) \oplus h(Y)$,

$h(X\times Y)=h(X)\otimes h(Y)$, etc.), and is universal among all "Weil cohomology theories", e.g. ℓ-adic cohomology groups $H^\bullet(X_{\bar{k}},\mathbb{Q}_\ell)$ with $\mathrm{Gal}(\bar{k}/k)$-action, Betti cohomology groups $H^\bullet_B(X^{\sigma,an},\mathbb{Q})$ with Hodge structure (for each embedding $k \overset{\sigma}{\hookrightarrow} \mathbb{C}$, if there is any).

We sketch the proposal of Grothendieck. Let

$$A^r(X)=\frac{\{\mathbb{Q}\text{-vector space gen. by subvarieties of codim. } r \text{ in } X\}}{\{\text{those that are numerically equiv. to } 0\}}.$$

For $X,Y\in \mathcal{V}ar_k$, with $X=\coprod_i X_i$ the decomp. into irred. components, define $C^r(X,Y)$ to be the subspace of $A^\bullet(X\times Y)$ consisting of classes of alg. cycles whose restrictions to $A^\bullet(X_i\times Y)$ are in $A^{r+\dim(X_i)}(X_i\times Y)$; so

$$C^r(X,Y)=\bigoplus_i A^{r+\dim(X_i)}(X_i\times Y)\subset A^\bullet(X\times Y).$$

One defines the "composition" law

$$\begin{aligned} C^r(X,Y)\underset{\mathbb{Q}}{\otimes} C^s(Y,Z) &\longrightarrow C^{r+s}(X,Z)\\ f\otimes g &\longmapsto (p_{XZ})_*(p_{XY}^*(f)\cdot p_{YZ}^*(g)).\end{aligned}$$

Then we have the <u>category of correspondences</u> $\mathcal{C}_k$, whose objects are symbols $h(X)$ (for $X\in\mathcal{V}ar_k$) and morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}_k}(h(X), h(Y)) = C^0(X, Y).$$

Put

$$h(X) \oplus h(Y) := h(X \amalg Y), \quad h(X) \otimes h(Y) := h(X \times Y);$$

both can be made into functors

$$\mathcal{C}_k \times \mathcal{C}_k \overset{\oplus}{\underset{\otimes}{\rightrightarrows}} \mathcal{C}_k .$$

$\boxed{\text{Exercise: Show that } h(X) \to h(X \amalg Y) \leftarrow h(Y) \text{ induced by } X \xleftarrow{(id_X,\, const)} X \amalg Y \xrightarrow{(const,\, id_Y)} Y \text{ is the coproduct in } \mathcal{C}_k. \;//.}$

We also have a functor

$$\begin{aligned} h : \mathcal{V}ar_k^{op} &\longrightarrow \mathcal{C}_k \\ X &\longmapsto h(X) \\ (f : Y \to X) &\longmapsto \text{graph } [\Gamma_f] \in C^0(X, Y). \end{aligned}$$

$\boxed{\Delta : X \hookrightarrow X \times X \rightsquigarrow \text{"cup product"}: h(X) \otimes h(X) \to h(X)}$

The $\mathbb{Q}$-linear category $\mathcal{C}_k$ is not abelian, and there is a universal way of making it "pseudo-abelian", by adjoining the images (equivalently, kernels) of all projectors (i.e., idempotent endomorphisms). Precisely, let $\mathcal{M}_k^{+}$, the false category of effective motives, be defined as follows. The objects are pairs $(h(X), p)$, $h(X) \in \mathrm{obj}(\mathcal{C}_k)$, $p \in \mathrm{End}_{\mathcal{C}_k}(h(X))$ a projector (i.e. $p \circ p = p$), and

$$\mathrm{Hom}((h(X), p), (h(Y), q)) = \frac{\left\{ f \in \mathrm{Hom}_{\mathcal{C}_k}(h(X), h(Y)) \;\middle|\; \begin{array}{ccc} h(X) & \xrightarrow{f} & h(Y) \\ {\scriptstyle p}\downarrow & \circlearrowleft & \downarrow{\scriptstyle q} \\ h(X) & \xrightarrow{f} & h(Y) \end{array} \right\}}{\{ f \mid f \circ p = 0 = q \circ f \}} ,$$

with composition law induced from that of $\mathcal{C}_k$. We have an additive functor $\mathcal{C}_k \to \dot{\mathcal{M}}_k^+$, $T \mapsto (T, id_T)$. Given an object (T, p) in $\dot{\mathcal{M}}_k^+$, it is the image of

$$(T, id_T) \xrightarrow{p} (T, id_T),$$

and $(T, id_T - p)$ is its kernel. Moreover, for any projector $f \in \mathrm{End}_{\dot{\mathcal{M}}_k^+}((T,p))$, the natural morphism its kernel exists, and

$$\ker(f) \oplus \underbrace{\mathrm{Im}(f)}_{\underset{\ker(id - f)}{\parallel}} \longrightarrow (T, p)$$

is an isomorphism. There is a tensor product structure

$$\otimes : \dot{\mathcal{M}}_k^+ \times \dot{\mathcal{M}}_k^+ \longrightarrow \dot{\mathcal{M}}_k^+$$
$$(T, p), (S, q) \longmapsto (T \otimes S, p \otimes q).$$

The composite

$$Var_k^{op} \xrightarrow{h} \mathcal{C}_k \longrightarrow \dot{\mathcal{M}}_k^+$$

is again denoted h. Let $\mathbb{1} := h(\mathrm{Spec}(k))$; we have

$$\mathbb{1} \otimes T \simeq T, \quad \forall\, T \in \mathrm{Obj}(\dot{\mathcal{M}}_k^+).$$

Let x be a (any) k-point on $\mathbb{P}^1_k$, and let

$$l_1 = [\mathbb{P}^1_k \times \{x\}], \quad l_2 = [\{x\} \times \mathbb{P}^1_k] \quad \text{in } A^1(\mathbb{P}^1_k \times \mathbb{P}^1_k).$$

Then $[\Delta_{\mathbb{P}^1}] = l_1 + l_2$, hence we have a decomposition

$$h(\mathbb{P}^1_k) = (h(\mathbb{P}^1_k), [\Delta_{\mathbb{P}^1}]) = \underbrace{(h(\mathbb{P}^1_k), l_1)}_{\overset{?}{\parallel}\ \mathbb{L}} \oplus \underbrace{(h(\mathbb{P}^1_k), l_2)}_{\parallel\ \text{exercise}\ \mathbb{1}.},$$

where $\mathbb{L} = \ker(l_2 : h(\mathbb{P}^1) \to h(\mathbb{P}^1))$ is called the <u>Lefschetz motive</u>. Other suggestive notations for $\mathbb{L}$: $\underline{\mathbb{Q}}(-1)$, $\mathbb{1}(-1)$.
~~One has canonical isom. $\mathbb{L}^{\otimes a} \otimes \mathbb{L}^{\otimes b} \simeq \mathbb{L}^{\otimes(a+b)}$~~. One can show that the functor

$$T \mapsto T \otimes \mathbb{L} : \dot{\mathcal{M}}_k^+ \longrightarrow \dot{\mathcal{M}}_k^+$$

is fully faithful. So one can formally adjoin the $\otimes$-inverse of $\mathbb{L}$ to $\dot{\mathcal{M}}_k^+$, to obtain the <u>false category of motives</u> $\dot{\mathcal{M}}_k$. Precisely, objects are pairs $M = (T, n)$, $T \in \mathrm{Obj}(\dot{\mathcal{M}}_k^+)$, $n \in \mathbb{Z}$, to be thought of as $T \otimes \mathbb{L}^{\otimes n}$, also written as $M = T(-n)$, with

$$\mathrm{Hom}_{\dot{\mathcal{M}}_k}(T(-n), T'(-n')) := \mathrm{Hom}_{\dot{\mathcal{M}}_k^+}(T(-n-N), T'(-n'-N))$$
$$(\text{for } N \gg 0).$$

We identify $\dot{\mathcal{M}}_k^+$ as a subcategory of $\dot{\mathcal{M}}_k$ via

$$T \mapsto (T, 0).$$

The $\otimes$-structure extends to $\dot{\mathcal{M}}_k$:

$$T(n) \otimes T'(n') = (T \otimes T')(n+n').$$

Denote $(\mathbb{1}, -1)$ by $\mathbb{T}$, called the <u>Tate motive</u> ($\mathbb{T} = \underline{\mathbb{Q}}(1)$).
The category $\dot{\mathcal{M}}_k$ is clearly a $\mathbb{Q}$-linear tensor category. Jannsen proved (1992) that it is a semisimple abelian (rather than just pseudo-abelian) category. Moreover, dual objects exist: $(h(X), p, n)^\vee = (h(X), {}^t p, -d-n)$

for X of equidimension d, where ${}^t p$ is the transpose of the correspondence p. In particular, $h(X)^\vee \simeq h(X)(d)$ for X equidim. d, since ${}^t[\Delta_X] = [\Delta_X]$. One can verify that

$$M^{\vee\vee} = M \quad \text{and} \quad \mathrm{Hom}(M \otimes N, P) = \mathrm{Hom}(M, N^\vee \otimes P)$$

for $M, N, P \in \mathrm{Obj}(\dot{\mathcal{M}}_k)$. One can then define the internal Hom to be $\underline{\mathrm{Hom}}(M,N) := M^\vee \otimes N$. The tensor category $\dot{\mathcal{M}}_k$ is rigid.

- Exercise. Let X be a projective smooth curve $/k$ of genus g. Show that the rank of $h(X) \in \mathrm{Obj}(\dot{\mathcal{M}}_k)$ is $2-2g$. Recall that the rank of an object M in a rigid tensor category is the morphism

Hint: Show that $rk(h(X)) = [\Delta_X^2]$.

$$\mathbb{1} \xrightarrow{\delta_M} M^\vee \otimes M \xrightarrow{ev_M} \mathbb{1}$$

in $\mathrm{End}(\mathbb{1})$, and for $\dot{\mathcal{M}}_k$, $\mathrm{End}(\mathbb{1}) = \mathbb{Q}$. //.

Hence, $\dot{\mathcal{M}}_k$ has no fibre functor: $rk(h(X)) = rk(\omega(h(X))) \geqslant 0$. How to fix this?

Let $H^*(X)$ be a "Weil cohom. theory", e.g. one of the examples above. Consider the Künneth components π^i of the diagonal $cl(\Delta_X) \in H^{2d}(X \times X) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X)$, for $X \in \mathcal{V}ar_k$ connected of dim. d, i.e.,

$$cl(\Delta_X) = \sum_{i=0}^{2d} \pi^i \ , \quad \pi^i \in H^{2d-i}(X) \otimes H^i(X).$$

Then one $\overbrace{\text{form}}^{\text{weak}}$ of the Standard conjecture (of Lefschetz type), made by Grothendieck, says that the π^i's should be alg. cycles on $X \times X$. If we view cohomology classes on $X \times Y$ as linear maps between their cohomology groups, via Poincaré duality and Künneth formula (omitting Tate twists)

$$H^*(X \times Y) = H^*(X) \otimes H^*(Y) \simeq H^*(X)^\vee \otimes H^*(Y)$$
$$= \mathrm{Hom}(H^*(X), H^*(Y)),$$

then the classes π^i corresponds to the projectors

$$H^*(X) \twoheadrightarrow H^i(X) \hookrightarrow H^*(X).$$

We have $\pi^{2d-i} = {}^t(\pi^i)$.

If two alg. cycles Z_1 and Z_2 on X are homologically ~~numerically~~ equivalent ~~(i.e. $[Z_1] = [Z_2]$ in $A^r(X)$)~~ w.r.t. H^*, i.e. $cl(Z_1) = cl(Z_2)$ in $H^{2r}(X)$, then clearly they are numerically equivalent ($[Z_1] = [Z_2]$ in $A^r(X)$). An important consequence of the Standard conjectures is that the converse also holds. This consequence is even unknown for $k = \mathbb{C}$, $H^* =$ Betti cohomology; this special case is also a consequence of the Hodge conjecture.

Let us assume these conjectures for now. Then the decomposition

$$id_{h(X)} = \sum_{i=0}^{2d} [\pi^i]$$

into orthogonal projectors leads to a decomposition

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X) \ , \ h^i(X) = (h(X), [\pi^i])$$
$$= \mathrm{Im}([\pi^i]) .$$

This grading on $h(X) \in \mathrm{Obj}(\mathcal{C}_k)$ extends in an evident way to $T \in \mathrm{Obj}(\dot{\mathcal{M}}_k^+)$. By putting

$$T(m)^i = T^{i-2m} , \ T \in \mathrm{Obj}(\dot{\mathcal{M}}_k^+) , \ m, i \in \mathbb{Z},$$

the grading extends to objects $M \in \mathrm{Obj}(\dot{\mathcal{M}}_k)$. If

$$\dot{\tau}_{M,N} : M \otimes N \xrightarrow{\sim} N \otimes M \ , \ M, N \in \mathrm{Obj}(\dot{\mathcal{M}}_k)$$

is the canonical commutativity constraint on $\dot{\mathcal{M}}_k$, given as (by functoriality) $\dot{\tau}_{M,N} = \bigoplus_{i,j} \dot{\tau}_{M^i, N^j}$, then we define

$$\tau_{M^i, N^j} := (-1)^{ij} \cdot \dot{\tau}_{M^i, N^j} \ , \ \tau_{M,N} := \bigoplus \tau_{M^i, N^j} .$$

With this new commutativity constraint, the category $\dot{\mathcal{M}}_k$ is denoted by $\mathcal{M}_k$, called the (true) category of motives. One verifies that, for a genus g curve X, we now have

$$rk(h(X)) = 2 + 2g, \ h(X) \in \mathrm{Obj}(\mathcal{M}_k).$$

More generally, for $X \in \mathrm{Obj}(\mathcal{V}ar_k)$, one can show that

$$rk(h(X), p, n) = \sum_i \dim_p (H^i(X)) .$$

Deligne proved that it is a Tannakian category, neutralized over some extension field $F/\mathbb{Q}$.

For simplicity, let $k \subset \mathbb{C}$ be a subfield, and denote

$$H_B^*(X) = H_B^*(X_{\mathbb{C}}^{an}, \mathbb{Q}), \text{ for } X/k.$$

The Betti cohomology functor

$$\mathcal{V}ar_k^{op} \xrightarrow{H_B^\bullet} MHS_{\mathbb{Q}}^{split} \longrightarrow \mathcal{V}ec_{\mathbb{Q}}$$

factors through $h: \mathcal{V}ar_k^{op} \to \mathcal{M}_k$, also denoted $H_B^\bullet$:

$$\mathcal{M}_k \xrightarrow{H_B^\bullet} MHS_{\mathbb{Q}}^{split} \longrightarrow \mathcal{V}ec_{\mathbb{Q}}$$

$$(h(X), p, n) \longmapsto p(H_B^\bullet(X))(-n) \longmapsto p(H_B^\bullet(X)).$$

The well-definedness follows from the coincidence of the numerical equivalence and the (Betti) homological equivalence, hence from the Standard conj. or Hodge conjecture. So now $\mathcal{M}_k$ is neutralized over $\mathbb{Q}$ by the fiber functor $H_B^\bullet$. Let $G_{\mathcal{M}_k}$ be the associated Tannaka dual group of $(\mathcal{M}_k, H_B^\bullet)$; it is a $\mathbb{Q}$-affine group scheme, which, up to inner auto., is indep. of the choice of the embedding $k \overset{\sigma}{\hookrightarrow} \mathbb{C}$. For a motive $M \in Obj(\mathcal{M}_k)$, let G_M be the Tannaka dual group of $\langle M \rangle^{\otimes} \subset \mathcal{M}_k$. We have

$$G_{\mathcal{M}_k} = \varprojlim_{M} G_M .$$

As $\mathcal{M}_k$ is semisimple, by (IV, 3.2), $G^\circ_{\mathcal{M}_k}$ is pro-reductive, and the G°_M's are reductive.

Here are a few characterizations of the $\mathbb{Q}$-alg. groups G_M. We have

$$G_M \hookrightarrow GL_{H^\cdot_B(M)} ,$$

hence G_M acts on $T^{p,q}(H^\cdot_B(M))$. A vector $v \in T^{p,q} H^\cdot_B(M)$ is motivic if it is fixed by G_M, or equivalently, if it comes from a morphism

$$\mathbb{1} \longrightarrow T^{p,q}(M)$$

of motives. Then, by ($\cancel{}$ 1.1), G_M is the alg. subgroup of $GL_{H^\cdot_B(M)}$ fixing all motivic vectors of all types (p,q).

We also have ℓ-adic realizations of motives:

$$H^\cdot_\ell : \mathcal{M}_k \longrightarrow \mathrm{Rep}^{\mathrm{cont.}}_{\mathbb{Q}_\ell}(\mathrm{Gal}(\bar{k}/k)) \to \mathrm{Vec}_{\mathbb{Q}_\ell},$$

with comparison isomorphisms

$$H^\cdot_\ell(M_{\bar{k}}) \simeq H^\cdot_B(M) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell , \quad \forall M .$$

Then we have the following

- Conjecture (Grothendieck – Tate). The alg. group $G_M \otimes \mathbb{Q}_\ell$

is the Zariski closure of the image of the Galois rep.

$$\rho_M : \mathrm{Gal}(\bar{k}/k) \longrightarrow GL(H_\ell^\bullet(M_{\bar{k}})),$$

if k is of finite type (i.e. f.g. field ext.) over $\mathbb{Q}$ (so that $\mathrm{Gal}(\bar{k}/k)$ is large enough). //

- Conjecture. (Mumford-Tate). $G_M^\circ = MT(H_B^\bullet(M))$ (assuming $\mathbb{L} \leq M$ so that $\mathbb{Q}(1) \in \langle H_B^\bullet(M) \rangle^\otimes$).

- Examples.

i). $G_{\mathbb{1}} = \mathbb{1}$. (triv. group). (triv. motive $h(\mathrm{Spec}\, k)$).

ii). $G_{\mathbb{L}} = G_{\mathbb{T}} = \mathbb{G}_{m,\mathbb{Q}}$.

iii). Let E/k be an elliptic curve, $\mathrm{End}(E_{\mathbb{C}}) = \mathbb{Z}$, then $G_{h^1(E)} = GL_{2,\mathbb{Q}}$.

iv). More generally, let A/k be an abelian variety $/k$ of dimension g, $\mathrm{End}(A_{\mathbb{C}}) = \mathbb{Z}$. If $g = 2, 6$, or odd, then $G_{h^1(A)} = GSp_{2g,\mathbb{Q}}$.

We have

$$\mathcal{M}_k \longrightarrow \mathcal{M}_{\bar{k}} \quad (k \subset \bar{k} \subset \mathbb{C} \text{ an alg. closure})$$
$$M \longmapsto M_{\bar{k}}.$$

An Artin motive M/k is one such that $M_{\bar{k}}$ is trivial:

$$M_{\bar{k}} \simeq \bigoplus \mathbb{1}_{\bar{k}} .$$

By (IV, 2.8.4),

$$\left\{\begin{array}{l}\text{Artin motives}\\ \text{over } k\end{array}\right\} \subset \mathcal{M}_k \longrightarrow \mathcal{M}_{\bar{k}}$$

gives rise to homomorphisms of $\mathbb{Q}$-affine groups

$$\pi_1(\mathcal{M}_{\bar{k}}, H_B^\bullet) \longrightarrow \pi_1(\mathcal{M}_k, H_B^\bullet) \longrightarrow \pi_1\left(\begin{array}{c}\text{Artin}\\ \text{motives}/k\end{array}\right)$$

which is an exact sequence

$$1 \longrightarrow G^\circ_{\mathcal{M}_k} \longrightarrow G_{\mathcal{M}_k} \longrightarrow \underline{\mathrm{Gal}(\bar{k}/k)} \longrightarrow 1 .$$

$$G^\circ_{\mathcal{M}_k} = \pi_1(\mathcal{M}_{\bar{k}})$$

(because of Weil's restriction of scalars.)

The abelianization $(G^\circ_{\mathcal{M}_k})^{ab}$ is the connected Serre group S° (see p. 7-8).

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The End.