MATH 731: ALGEBRAIC $\mathscr{D}$-MODULES

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These are course notes for MATH731 "Algebraic $\mathscr{D}$-modules" taught by Professor Bhargav Bhatt, taken by Zhan Jiang, who is responsible for any and all errors. Please email zoeng@umich.edu with any corrections. All texts in blue are comments by myself.

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Compiled on February 3, 2022.

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1. 08/31/2020

1.1. **Introduction.** Let X be a smooth projective variety over $\mathbb{C}$. Then $X^{\text{an}} = X(\mathbb{C})$ is a compact complete manifold. We have four categories:

(1) $\mathbf{Rep}_{\mathbb{C}}(\pi, (X^{\text{an}}))$: finite dimensional $\mathbb{C}$-representations of $\pi_1(X^{\text{an}})$.
(2) $\mathbf{Loc}_{\mathbb{C}}(X^{\text{an}})$: locally constant sheaves of finite dimensional $\mathbb{C}$-vector spaces.
(3) $\mathbf{Vect}^{\nabla}(X^{\text{an}})$
$$= \{(\mathscr{E}, \nabla) \,|\, \mathscr{E} \in \mathbf{Vect}(X^{\text{an}}), \nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega'_{X^{\text{an}}} \text{ such that } \nabla \text{ is a derivation}, \nabla^2 = 0.\}$$
where **Vect** is the category of holomorphic vector bundles.
(4) $\mathbf{Vect}^{\nabla}(X)$ is similar to above.

Theorem 1.1. *All 4 categories are equivalent to each other*

Proof. (1) $\Leftrightarrow$ (2): $V \in \mathbf{Rep}_{\mathbb{C}}(\pi, (X^{\text{an}})) \mapsto L(V) \in \mathbf{Loc}_{\mathbb{C}}(X^{\text{an}})$ where $L(V)$ is the descent to X^{an} of the constant sheaf $\underline{V}$ on $\widetilde{X}$ = universal cover of X, π_1-equivariantly.

(2) $\Leftrightarrow$ (3): $L \in \mathbf{Loc}_{\mathbb{C}}(X^{\text{an}}) \mapsto \mathscr{E}(L) = L \otimes_{\mathbb{C}} \mathcal{O}_{X^{\text{an}}} \in \mathbf{Vect}(X^{\text{an}})$ with $\nabla = \text{id} \otimes \text{d}$ flat connection. $(\mathscr{E}, \nabla) \mapsto \mathscr{E}^{\nabla=0} := \text{Ker}(\mathscr{E} \to \mathscr{E} \otimes \Omega^1_{X^{\text{an}}})$.

(3) $\Leftrightarrow$ (4): $(\mathscr{E}, \nabla) \in \mathbf{Vect}^{\nabla}(X) \mapsto (\mathscr{E}^{\text{an}}, \nabla^{\text{an}}) \in \mathbf{Vect}^{\nabla}(X^{\text{an}})$

Use GAGA in other direction. $\square$

Now $\overline{\mathbf{Loc}_{\mathbb{C}}(X^{\text{an}})}$ is the closure of theory of local systems under standard geometric operations (ex: pushforward, pullback, extension by 0, $\mathbb{D}(-)$, ...) = theory of constructive sheaves of $\mathbb{C}$-vector spaces.

Analogously, $\overline{\mathbf{Vect}^{\nabla}(X)}$ is the closure of theory under geometric operations = theory of holonomic $\mathscr{D}$-modules.

In fact, there exists an equivalence $\text{D}^{\flat}_{\text{cons}}(X^{\text{an}}) \overset{\text{RH}}{\cong} \text{D}^{\flat}_{\text{reg.hol}}(\mathscr{D}_X)$.

Why do we care about this?

(1) A holonomic $\mathscr{D}_X$-module has an underlying quasi-coherent sheaf. So nontrivial theorem on constructible sheaves $\overset{\text{RH}}{\Rightarrow}$ nontrivial theorems for quasicoherent sheaves.
(2) $\mathscr{D}_X$ =“universal enveloping algebra of $T_{X/\mathbb{C}}$” $\Rightarrow$ for $X = G/B$ where G is a complex Lie group, $B \subseteq G$ Borel subgroup, we get a close link between {reps of $\mathfrak{g}$} and $\mathscr{D}_X$-modules (where $\mathfrak{g} = \text{Lie}(G)$. (ex: Kazhdan-Lusfzig conjecture by B-B)
(3) Holonomic $\mathscr{D}_X$-modules give quasi-coherent sheaves with “hidden finiteness” $\Rightarrow$ applications in commutative algebra.

1.2. **Review of differential forms.** Fix a map $k \to T$ of commutative rings

(1) For any $M \in \mathbf{Mod}_R$, a *derivation* $\text{d} : R \to M$ is a k-linear map such that $\text{d}(fg) = f\text{d}(g) + g\text{d}(f) \rightsquigarrow \text{Der}_k(R, M)$ = R-linear module of all derivations.
(2) The map $\text{d} : R \to \Omega^1_{R/k}, f \mapsto \text{d}f$ is the universal derivation $\Rightarrow \text{Hom}_R(\Omega^1_{R/k}, M) = \text{Der}_k(R, M)$.
(3) Assume $k \to R$ is finitely presented. Then R is smooth over $k \Leftrightarrow \Omega^1_{R/k}$ is locally free of “correct rank” (use: $\Omega^1_{R/k} \otimes_k k' = \Omega^1_{R'/k'}$ where $R' = R \otimes_k k'$).
(4) In (2), take $M = R \Rightarrow$
$$\text{T}_{R/k} := \text{Hom}_R(\Omega^1_{R/k}, R) = \text{Der}_k(R, R) \subseteq \text{End}_k(R)$$

Exercise 1.2. $\text{T}_{R/k}$ is closed under $[-,-]$ in $\text{End}_k(R)$. (i.e.: $a, b \in \text{T}_{R/k}$, then $ab - ba \in \text{T}_{R/k}$ where ab is the product in $\text{End}_k(R)$.

$[a,b]$ is certainly an element in $\mathrm{End}_k(R)$. So we need to verify that $[a,b]$ is a derivation, i.e., $[a,b](fg) = f[a,b](g) + g[a,b](f)$ for any $f, g \in R$. Note that

$$\begin{aligned} ab(fg) &= a(fb(g) + gb(f)) \\ &= fab(g) + a(f)b(g) + gab(f) + a(g)b(f) \\ ba(fg) &= b(fa(g) + ga(f)) \\ &= fba(g) + b(f)a(g) + gba(f) + b(g)a(f) \\ [a,b](fg) &= f[a,b](g) + g[a,b](f). \end{aligned}$$

So $[a,b] \in \mathrm{Der}_k(R,R)$.

So one may think of $\mathrm{T}_{R/k}$ as a Lie algebra over k.

Example 1.3. $R = k[x_1, \dots, x_n]$. $\Omega^1_{R/k} = R\mathrm{d}x_1 \oplus \cdots \oplus R\mathrm{d}x_n$ and $\mathrm{T}_{R/k} = R\partial_1 \oplus \cdots \oplus R\partial_n$ where $\partial_i = \frac{\partial}{\partial x_i} \in \mathrm{End}_k(R)$. $[\partial_i, \partial_j] = 0$ for any i, j. Note that $[-,-] \neq 0$, for example $[\partial_1, x_1\partial_2] = \partial_2$.

1.3. (Grothendieck's) Ring of differential operators.

Fix a map $k \to R$ between commutative rings.

Definition 1.4. Define an increasing family $\mathrm{F}_i\mathscr{D}(R/k) \subseteq \mathrm{End}_k(R)$ for $i \geqslant 0$ as follows:

- $\mathrm{F}_0\mathscr{D}(R/k) = R \subseteq \mathrm{End}_k(R)$ are all the scalar multiplications.
- $a \in \mathrm{F}_i\mathscr{D}(R/k) \Leftrightarrow [a,f] \in \mathrm{F}_{i-1}\mathscr{D}(R/k)$ for any $f \in R = \mathrm{F}_0\mathscr{D}(R/k)$.

$\Rightarrow$ the ring of differential operators $\mathscr{D}(R/k) = \cup_i \mathrm{F}_i\mathscr{D}(R/k) \subseteq \mathrm{End}_k(R)$. Here F_i are called *order filtration*.

Example 1.5.
(1) For any $f \in R$, the map $g \mapsto fg$ is in $\mathrm{F}_0\mathscr{D}(R/k)$.
(2) $R = k[x]$, so the map $f(x) \mapsto \left(\frac{\partial}{\partial x}\right)^n (f(x))$ lies in $\mathrm{F}_n\mathscr{D}(R/k)$.

Lemma 1.6. *Fix a map $k \to R$ as before. Fix $a \in \mathrm{F}_m\mathscr{D}(R/k), b \in \mathrm{F}_n\mathscr{D}(R/k)$.*

(i) $ab \in \mathrm{F}_{m+n}\mathscr{D}(R/k) \Rightarrow \mathscr{D}(R/k)$ is an associative k-subalgebra of $\mathrm{End}_k(R)$ and

$$\mathrm{gr}_* \mathscr{D}(R/k) = \oplus_{m \geqslant 0} \mathrm{F}_{m+1}\mathscr{D}(R/k)/\mathrm{F}_m\mathscr{D}(R/k)$$

is a ring.

(ii) $[a,b] \in \mathrm{F}_{m+n-1} \Rightarrow \mathrm{gr}_* \mathscr{D}(R/k)$ is a commutative ring.

Proof. For 1.6.(i): Want $ab \in F_{m+n} \Leftrightarrow [ab, f] \in F_{m+n-1}$ for any $f \in R$. Prove by induction on $m+n$:

- $m = n = 0$: clear because R is commutative,
- $[ab,f] = abf - fab = a(fb + [b,f]) - (af - [a,f])b = \cancel{afb} + a[b,f] - \cancel{afb} + [f,a]b$. The remaining terms are both in F_{m+n-1}. Hence the left-hand side is in F_{m+n-1}.

□

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2.1. (Grothendieck's) Ring of differential operators (continued).

Proof of Lemma 1.6.(ii). $a \in \mathrm{F}_m, b \in \mathrm{F}_n$, we want to show that $[a,b] \in \mathrm{F}_{m+n-1}$.

So we want to show that $\forall f \in R$, we have $[[a,b],f] \in \mathrm{F}_{m+n-2}$.

- If $m = 0$ or $n = 0$, then this is clear
- In general, Jacobi identity says

$$[[a,b],f] = [a,[b,f]] - [b,[a,f]]$$

By induction , both terms are in $\mathrm{F}_{m+n-1-1}$. So left-hand side is in F_{m+n-2}.

□

Remark 2.1. (1) A *filtered ring* is an associated ring R equipped with an increasing multiplicative exhaustive $\mathbb{N}$-graded filtration $\{\mathrm{F}_i R\}_{i\geqslant 0}$. Such an R is *almost commutative* if $\mathrm{gr}_*(R)$ is commutative.

Exercise 2.2. If $\mathrm{gr}_*(R)$ is noetherian, so is R.

Consider the corresponding ideal $\mathrm{gr}_*(I) = \oplus_{i=1}^{\infty}(I \cap \mathrm{F}_{k+1}R)/(I \cap \mathrm{F}_k R)$ of I in $\mathrm{gr}_*(R)$. It is not hard to see that this is an ideal of $\mathrm{gr}_*(R)$. Since $\mathrm{gr}_*(R)$ is noetherian, the ideal $\mathrm{gr}_*(I)$ is finitely generated. Since $\mathrm{gr}_*(I)$ is a homogeneous ideal, we can assume that it is generated by homogeneous elements $\overline{a}_1, \dots, \overline{a}_n$ where $a_i \in R$.

Now we claim that $a_1, \dots, a_n$ generates I in R. For any $f \in I$, there is some k such that $f \in \mathrm{F}_k R$ and $f \notin \mathrm{F}_{k-1}R$. We have

$$\overline{f} = \overline{a_1}u_1 + \dots + \overline{a_n}u_n$$

for some elements $\overline{u}_1, \dots, \overline{u}_n$ in $\mathrm{gr}_*(R)$. Suppose that each $\overline{u_i}$ is $\sum_{j=1}^{k_i} u_{ij}$ where u_{ij} are images of $v_{ij} \in R$. Then $f - (\sum_{i=1}^n \sum_{j=1}^{k_i} a_i v_{ij}) \in I \cap \mathrm{F}_{k-1}R$. Therefore by induction, we can find elements $w_1, \dots, w_n$ in R such that $f - (\sum_{i=1}^n a_i w_i) = 0$.

(2) For $k \to R$ as above, have a natural map

$$R \oplus \mathrm{T}_{R/k} \to \mathrm{F}_1\mathscr{D}(R/k) \subseteq \mathscr{D}(R/k)$$

Check: This is actually an isomorphism. (Key point: for any $a \in \mathrm{T}_{R/k}, f \in R$ we have $[a, f] = a(f)$. Implicitly $\mathrm{T}_{R/k} = \mathrm{Der}_k(R, R)$.)

This is trivially an injection. So we only need to show surjectivity. We first note that $h \in \mathrm{F}_1\mathscr{D}(R/k) \smallsetminus \mathrm{F}_0\mathscr{D}(R/k) \Rightarrow h(1) = 0$. This is because $[h, 1] \in R$ and $[h, 1](f) = h(f) - h(f) = 0$ for any $f \in R$. Then $[h, 1] = 0$. Then we need to show that $[h, f] \in R$ for any $f \in R$, i.e., $h(fg) - fh(g) \in R$ for any $g \in R$. Note that $[h, f] \in R$ is R-linear, so

$$\begin{aligned}[h, f](g) &= g[h, f](1)\\ &= g(h(f) - fh(1))\\ &= gh(f).\end{aligned}$$

Thus we have $h(fg) - fh(g) = gh(f)$, which shows that $h \in \mathrm{T}_{R/k}$.

(3) Left/right multiplication makes $\mathscr{D}(R/k)$ a (R, R)-bimodule ($\Leftrightarrow R \otimes_k R$-module). Moreover, $\mathrm{F}_i\mathscr{D}(R/k) \subseteq \mathscr{D}(R/k)$ is a sub-bimodule (by Lemma 1.6).

Proposition 2.3. *Fix $k \to R$ as before. Regard $R \otimes_k R$ as an R-algebra via the first factor. Then we have a map*

$$\mathscr{D}(R/k) \subseteq \mathrm{Hom}_k(R, R) \cong \mathrm{Hom}_R(R \otimes_k R, R)$$

identifies $\mathrm{F}_n\mathscr{D}(R/k) \cong \mathrm{Hom}_R(R \otimes_k R/I_\Delta^{n+1}, R)$ *where* $I_\Delta = \mathrm{Ker}(R \otimes_k R \to R)$.

So $\mathscr{D}(R/k) = \varinjlim_n \mathrm{Hom}_R(R \otimes_k R/I_\Delta^{n+1}, R) := \mathrm{Hom}_R(\{R \otimes_k R/I_\Delta^{n+1}\}, R)$.

Proof. Note that we have

(1) For any $a \in \mathrm{Hom}_k(R, R)$, write $\mathrm{ad}(a) = [-, a]$ (operator on $\mathrm{Hom}_k(R, R)$).
Check: for any $b \in \mathrm{F}_n\mathscr{D}(R/k) \Leftrightarrow \mathrm{ad}(f_0)\cdot\mathrm{ad}(f_1)\cdots\mathrm{ad}(f_n)\cdot b = 0$ for any $f_0, \dots, f_n \in R$. Note that $\mathrm{ad}(f_0)\cdot\varphi(r) = [f_0, \varphi](r) = \widetilde{\varphi}((1 \otimes f_0 - f_0 \otimes 1)\cdot(1 \otimes r))$.

(2) The isomorphism $\mathrm{Hom}_k(R, R) \cong \mathrm{Hom}_R(R \otimes R, R)$ carries $\mathrm{ad}(f)$ ($f \in R$) on the left-hand side to scalar multiplication by $1 \otimes f - f \otimes 1$. One direction of this isomorphism is given by $\varphi \mapsto (\sum a \otimes b \mapsto \sum a\varphi(b))$.

(3) The isomorphism in (2) is a (R, R)-bimodule isomorphism.

Combine the above to see:

$$\mathrm{F}_n\mathscr{D}(R/k) \cong \{f \in \mathrm{Hom}_R(R \otimes R, R) \quad | \quad f(I_\Delta^{n+1}) = 0\}$$

where $I_\Delta = \mathrm{Ker}(R \otimes R \to R) = (1 \otimes f - f \otimes 1)_{f \in R}$. So the proposition is proved. ∎

This is sort of arranging all the "twist" first, and then apply the map. Higher-order elements in $\mathrm{F}_n\mathscr{D}(R/k)$ are those commuting with longer "twists".

Notation:

(1) $\mathrm{P}^n(R) = R\otimes R/I_\Delta^{n+1}$, regarded as (R,R)-bimodule (or as R-module via left action).
(2) For a k-scheme X, write $\mathrm{P}^n(\mathcal{O}_X) = \mathrm{pr}_{1,*}(\mathcal{O}_{X\times_k X}/I_\Delta^{n+1}) \in \mathbf{QCoh}(X)$.

Corollary 2.4. *Let k be a noetherian ring and consider X/k finite type scheme. The assignments $(\mathrm{Spec}(R)\subseteq X)\mapsto \mathrm{F}_n\mathscr{D}(R/k)$ ($\mathscr{D}(R/k)$, resp.) are quasi-coherent sheaves $\mathrm{F}_n\mathscr{D}_{X/k}$ ($\mathscr{D}_{X/k}$, resp.) on X.*

In fact,

$$\mathrm{F}_n\mathscr{D}_{X/k} = \mathscr{H}om_X(\mathrm{P}^n(\mathcal{O}_X),\mathcal{O}_X),$$
$$\mathscr{D}_{X/k} = \mathrm{Colim}_n\,\mathscr{H}om_X(\mathrm{P}^n(\mathcal{O}_X),\mathcal{O}_X).$$

Proof. Left as an exercise, use $\mathscr{H}om(\mathbf{Coh},\mathbf{Coh}) = \mathbf{Coh}$. □

Exercise 2.5. Use Corollary 2.4 to show: if $R\to S$ is an étale map, then there exists a natural isomorphism:

$$S\otimes_R \mathscr{D}(R/k)\cong \mathscr{D}(S/k)$$

and the same for $\mathrm{F}_n\mathscr{D}(-/k)$.

2.2. Some calculations of $\mathscr{D}(R/k)$.

Proposition 2.6. *Say X/k is a smooth k-scheme*

- *Each $\mathrm{F}_n\mathscr{D}_{X/k}$ is a vector bundle over X (finite locally free sheaf on X).*
- *There exists a natural algebra isomorphism $\mathrm{gr}_*\,\mathscr{D}_{X/k}\equiv \Gamma_X^*(\mathrm{T}_{X/k}) = \oplus_{n\geqslant 0}\Gamma_X^n(\mathrm{T}_{X/k})$.*

In particular, if k is noetherian and a $\mathbb{Q}$-algebra, then $\mathrm{gr}_*(\mathscr{D}_{X/k}) = \mathscr{S}ym_X^*(\mathrm{T}_{X/k}) = \pi_*\mathcal{O}_{T^*X}$ where $\pi:\mathrm{T}^*X\to X$. So $\mathscr{D}_{X/k}$ is a sheaf of noetherian rings.

Proof. For (1), $\mathrm{F}_0\mathscr{D}_{X/k} = \mathcal{O}_X$ is a vector bundle. By induction, it suffices to show that $\mathrm{gr}_n\,\mathscr{D}_{X/k}$ is a vector bundle. Using Corollary 2.4, get $\mathrm{gr}_n\,\mathscr{D}_{X/k} = \mathscr{H}om_{\mathcal{O}_X}(I_\Delta^n/I_\Delta^{n+1},\mathcal{O}_X)$. Then we're done because X/k is smooth, therefore $X\hookrightarrow X\times_k X$ is a complete intersection.

In fact, $I_\Delta^n/I_\Delta^{n+1} = \mathscr{S}ym^n(I_\Delta/I_\Delta^2)$ by regularity of $X\hookrightarrow X\times X$, which is $\mathscr{S}ym^n(\Omega^1_{X/k})$. And $\Omega^1_{X/k}$ is a vector bundle because X/k is smooth.

So $\mathrm{gr}_n(\mathscr{D}_{X/k}) = \mathscr{H}om_X(\mathscr{S}ym^n(\Omega^1_{X/k}),\mathcal{O}_X) =: \Gamma_X^n(\mathrm{T}_{X/k})$. This also gives (2) by ignoring multiplication.

For multiplication: left as an exercise ([BO15, Section 2]). □

Remark 2.7. (1) Noetherionness of $\mathscr{D}(R/k)$ fails in char p
(2) Even in char 0, if R/k is not smooth, then $\mathscr{D}(R/k)$ might fail to be noetherian.

Exercise 2.8. Bernstein-Gelfand-Gelfand: $k=\mathbb{C}$, $R=\mathbb{C}[x,y,z]/(x^3+y^3+z^3)$.

The Weyl algebra: k field of characteristic 0, $R = k[x_1,\dots,x_n]$.

Set $\mathrm{A}_n(k) = R\otimes_k k[\partial_1,\dots,\partial_n]$ as a k-vector space. Then make it an algebra via

- $R\to \mathrm{A}_n(k), f\mapsto f\otimes 1$ is a ring map.
- $k[\partial_1,\dots,\partial_n]\to \mathrm{A}_n(k)$ is a ring map.
- Have $\partial_i x_j - x_j\partial_i = \delta_{ij}$ for any i,j.

Have a natural map

$$\begin{aligned} \mathrm{A}_n(k) &\to \mathscr{D}(R/k) \\ x_i &\mapsto x_i \\ \partial_i &\mapsto \frac{\partial}{\partial x_i} \end{aligned}$$

Claim. *This is an isomorphism (uses char 0).*

Have $\mathrm{A}_n(k) = \oplus_{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n} R\cdot\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$.

Set $\mathrm{F}_i\mathrm{A}_n(k) = \oplus_{\substack{\alpha\in\mathbb{N}^n \\ \sum_j \alpha_j\leqslant i}} R\cdot\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$.

Exercise 2.9. Let k be a perfect field of characteristic p, R a finite type k-algebra. Then

$$\mathscr{D}(R/k) \cong \bigcup_{e\geqslant 0} \mathrm{Hom}_{R^{p^e}}(R,R) \subseteq \mathrm{Hom}_k(R,R)$$

Hint: Use that projective systems $\{I_\Delta^n\}$ and $\{I_\Delta^{[p^e]}\}$, give the same pro-object.

Proposition 2.10. *Let k be a $\mathbb{Q}$-algebra, R a smooth k-algebra. Then $\mathscr{D}(R/k)$ is the free associative k-algebra generated by $\{\widetilde{f}\,|\,f\in R\}$ and $\{\widetilde{a}\,|\,a\in \mathrm{T}_{R/k}\}$ satisfying*

a) $f\mapsto\widetilde{f}$ is a ring map $R\to\mathscr{D}$.
b) $a\mapsto\widetilde{a}$ is a Lie algebra homomorphism $\mathrm{T}_{R/k}\to\mathscr{D}(R/k)$.
c) The map in (b) is an R-module map via the action of (a), i.e., $\widetilde{f\cdot a} = \widetilde{f}\cdot\widetilde{a}$.
d) For $f\in R, a\in\mathrm{T}_{R/k}$, $[\widetilde{a},\widetilde{f}] = \widetilde{a(f)}$.

In particular, $\mathscr{D}(R/k)$ is the k-subalgebra of $\mathrm{End}_k(R)$ generated by R and $\mathrm{T}_{R/k}$.

This statement is only true in characteristic 0, but not in characteristic p.

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3.1. Some calculations of $\mathscr{D}(R/k)$ (continued).

Proof of Proposition 2.10. Set $\mathscr{D}$ to be the free associated k-algebra satisfying (a) through (d). Then we get a natural map $\mathscr{D}\to\mathscr{D}(R/k)$ of associated k-algebras.

Set $\mathrm{F}_i\mathscr{D}$ to be the left R-submodule of $\mathscr{D}$ generated by $(\mathrm{T}_{R/k})^{\otimes_k j}$ where $j\leqslant i$. (Elements are R-linear combinations of monomials of degree $\leqslant i$ in $\mathrm{T}_{R/k}$).

Propisiton (d) $\Rightarrow$ $\mathrm{F}_i\mathscr{D}$ is stable under right multiplication by R. (becase $af = fa + a(f)$)

$\Rightarrow$ $\{\mathrm{F}_i\mathscr{D}\}$ is a multiplicative filtration (also increasing, exhaustive and $\mathbb{N}$-indexed)

So $\mathscr{D}$ is naturally a filtered ring.

Claim. *$\mathscr{D}$ is almost commutative, i.e., $a\in\mathrm{F}_i, b\in\mathrm{F}_j \Rightarrow [a,b]\in\mathrm{F}_{i+j-1}$.*

Proof. If $a,b\in\mathrm{F}_1\mathscr{D}$,: by (b) and (d)

In general, use $[ab,c] = a[b,c]+[a,c]b$ + induction. □

Also, the map $\mathscr{D}\to\mathscr{D}(R/k)$ is a filtered map, i.e., $\mathrm{F}_i\mathscr{D}\to\mathrm{F}_i\mathscr{D}(R/k)$ for any i.

So it sufficies (Exercise 3.1) to show that $\mathrm{gr}_*(\mathscr{D})\to\mathrm{gr}_*(\mathscr{D}(R/k))$ is an isomorphism.

Now use:

- $\operatorname{gr}_*(\mathscr{D}(R/k)) = \operatorname{Sym}^*_R(\underbrace{\operatorname{gr}_1 \mathscr{D}_{R/k}}_{\mathrm{T}_{R/k}})$.
- $\operatorname{gr}_* \mathscr{D}$ is a quotient of $\operatorname{Sym}^*_R(\operatorname{gr}_1 \mathscr{D})$.

Now $\operatorname{gr}_1 \mathscr{D}$ is a quotient of $\mathrm{T}_{R/k}$, get

$$\begin{array}{ccccc} & & \text{identity map} & & \\ \operatorname{Sym}^*_R(\mathrm{T}_{R/k}) & \twoheadrightarrow & \operatorname{Sym}^*_R(\operatorname{gr}_1 \mathscr{D}) & \longrightarrow & \operatorname{Sym}^*_R(\mathrm{T}_{R/k}) \\ & & \downarrow & & \downarrow \wr \\ & & \operatorname{gr}_*(\mathscr{D}) & \longrightarrow & \operatorname{gr}_*(\mathscr{D}(R/k)) \end{array}$$

Diagram chase $\Rightarrow$ all maps above are isomorphism. $\square$

Exercise 3.1. a map of $\mathbb{N}$-filtered exhaustive vector space is an isomorphism if gr_* is an isomorphism.

Exercise 3.2. Use similar reasoning to check that $k[x_1, \dots, x_n] \otimes_k k[\partial_1, \dots, \partial_n] \cong \mathscr{D}(\mathbb{A}^n_k)$.

3.2. **$\mathscr{D}$-modules.** Let k be a $\mathbb{Q}$-algebra, R a smooth k-algebra and $X = \operatorname{Spec}(R)$. For the associated algebra A, we have $\mathrm{L}\,\mathbf{Mod}_A$= left A-modules, $\mathrm{R}\,\mathbf{Mod}_A$ = right A-modules.

Corollary 3.3. *Specifying a left $\mathscr{D}(R/k)$-module M is equivalent to giving*

(1) A left R-module M
(2) A Lie algebra map $\mathrm{T}_{R/k} \to \operatorname{End}_k(M), a \mapsto (\nabla_a : M \to M)$.

satisfying

(3) The map in (2) is R-linear continuous left R-actions on both sides, i.e., for any $f \in R, a \in \mathrm{T}_{R/k}$ and $m \in M$, $\nabla_{fa}(m) = f\nabla_a(m)$.
(4) for all $a \in \mathrm{T}_{R/k}$, the map $\nabla_a : M \to M$ is a derivation. (i.e., $\nabla_a(fm) = f\nabla_a(m) + a(f)m$ for all $f \in R, m \in M$.)

Proof. Use Proposition 2.10. $\square$

Exercise 3.4. Show that giving a $\mathscr{D}(R/k)$-module M is equivalent to giving an R-module M + k-linear map

$$\nabla : M \to \Omega^1_{R/k} \otimes_R M$$

satisfying

(1) ∇ is a connection: $\nabla(fm) = f\nabla(m) + \mathrm{d}f \otimes m$
(2) ∇ is flat: (a) gives a "curvature" map $\nabla^2 : M \to \Omega^2_{R/k} \otimes_R M = \operatorname{Hom}_R(\bigwedge^2 \mathrm{T}_{R/k}, M)$ which is R-linear, given by

$$m \mapsto (a \wedge b \mapsto (\nabla_{[a,b]} - [\nabla_a, \nabla_b])(m))$$

where $\nabla_a : \xrightarrow{\nabla} \Omega^1_{R/k} \otimes_R M \xrightarrow{a} M$. The condition is $\nabla^2 = 0$.

Exercise 3.5. Fix an R-module M,

(1) Giving a connection on $M \Leftrightarrow$ Giving $\eta_{a,b} : a^*M \cong b^*M$ (where $a, b : R \rightrightarrows R \otimes_k R/I^2_\Delta$ are the natural maps) reducting to id_M mod I_Δ.
(2) Giving a flat connection on $M \Leftrightarrow$ giving isomorphism $\eta_{a,b} : a^*M \cong b^*M$ as above such that $\eta_{b,c} \circ \eta_{a,b} = \eta_{a,c}$ as isomorphisms $a^*M \cong c^*M$, where $a, b, c : R \to R \otimes R \otimes R/I^2_\Delta$. (See Berthelot-Ogus)

$p^*_{23}(\eta_{ab}) \circ p^*_{12}(\eta_{ab}) = p^*_{13}(\eta_{ab})$ as isomorphisms $i^*_1 M \cong i^*_3(M)$ where $i_1, i_2, i_3 : R \to (R \otimes R \otimes R)/I^2_\Delta$.

Assume k noetherian.

Definition 3.6. A $\mathscr{D}(R/k)$ -module M is called $\mathcal{O}$-coherent if it is finitely generated over R.

Example 3.7. (1) R is $\mathscr{D}(R/k)$-moduel (Use: $\mathscr{D}(R/k) \hookrightarrow \mathrm{End}_k(R)$) and it is $\mathcal{O}$-coherent.

(2) If S is an étale R-algebra (ex: a localization), then $S \otimes_R \mathscr{D}(R/k) \cong \mathscr{D}(S/k)$ as before $\Rightarrow$ the map $\mathscr{D}(R/k) \to \mathscr{D}(S/k)$ is a ring map, so S is a $\mathscr{D}(R/k)$-module. Explicitly, since $R \to S$ is étale, we have $\mathrm{T}_{R/k} \otimes_R S \equiv \mathrm{T}_{S/k}$. So each $a \in \mathrm{T}_{R/k}$ has a natural lift $a \otimes 1 \in \mathrm{T}_{S/k}$. Use this to define the action. *Note*: S is $\mathcal{O}$-coherent $\Leftrightarrow$ S is finite étale over R.

(3) Say $I = (f_1, \dots, f_r) \subseteq R$ is an ideal.

Have

$$K = (R \to \prod_{i=1}^{r} R_{f_i} \to \prod_{i<j} R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_r})$$

(This is the argumented Cech complex)

The obesrvation is that K is a complex of $\mathscr{D}(R/k)$-modules. So each homology group $\mathrm{H}^i(K)$ $(= \mathrm{H}^i_I(R))$ is a $\mathscr{D}(R/k)$-module.

Example 3.8. $R = k[x]$, $I = (x)$ then $K = (R \overset{\text{can}}{\to} R_x) \underset{\text{qis}}{\to} (\frac{k[x,x^{-1}]}{k[x]})[-1]$.

So $\frac{k[x,x^{-1}]}{k[x]}$ is a $\mathscr{D}(R/k)$-module.

In fact, it is cyclic (use: $\frac{\mathrm{d}}{\mathrm{d}x}(\frac{1}{x^n}) = \frac{-n}{x^{n+1}}$)

(4) (Solutions of differential equations): $R = k[x]$

Example 3.9. (a) Consider $\mathcal{O}_X \cdot \mathrm{e}^x := \frac{\mathscr{D}(R/k)}{\mathscr{D}(R/k)\cdot(\frac{\partial}{\partial x} - 1)}$

Note that the element $1 \in \mathscr{D}$ gives a generator e^x in $\mathcal{O}_X \cdot \mathrm{e}^x$ satisfying $\frac{\partial}{\partial x}(\mathrm{e}^x) = \mathrm{e}^x$.

(b) $\mathcal{O}_X \cdot x^\lambda = \frac{\mathscr{D}(R/k)}{\mathscr{D}(R/k)\cdot(x\frac{\partial}{\partial x} - \lambda)}$ for $\lambda \in k$. Note: if $\lambda = n \in \mathbb{Z}_{\geqslant 0}$, then "$x^n$"$\in \mathrm{Hom}_{\mathscr{D}}(\mathcal{O}_X \cdot x^n, \mathcal{O}_X)$ where "x^n" means $\mathscr{D} \to \mathcal{O}_X, 1 \mapsto x^n$.

Proposition 3.10. *Say $M \in \mathscr{D}(R/k)$-module, let $a, b : R \rightrightarrows S$ such that $a \equiv b$ mod I where $I \subseteq S$ nilpotent. Then there exists a natural isomorphism $a^* M \underset{\eta_{a,b}}{\cong} b^* M$. Moreover, if $a, b, c : R \to S$ are all congruent mod I, then $\eta_{b,c} \circ \eta_{a,b} = \eta_{a,c}$ as isomorphisms $a^* M \cong c^* M$.*

Sketch. Step 1: reduce to the case $S = (R \otimes R)/I_\Delta^n$ for some n and $a, b : R \rightrightarrows S$ are the natural maps.

Step 2: Want $\eta_{a,b} : a^* M \to b^* M \Leftrightarrow \eta'_{a,b} : M \to a^* b_* M = (\frac{R \otimes_k R}{I_\Delta^n}) \otimes_R M =: \mathrm{P}^n(R) \otimes_R M$. (Needs an arugment) $\Rightarrow \mathrm{P}^n(R)^\vee \otimes_R M \to M \Leftrightarrow \mathrm{F}_n \mathscr{D}(R/k) \otimes_R M \to M$ and the last map is the $\mathscr{D}$-action on M. ☐

References are [BO15, Chapter 2] and [BD99, Section 7.10 - 7.14].

4. 09/14/2020

4.1. $\mathscr{D}$-modules (continued).

We want to discuss the $n = 2$ case of the crystal structure. Let k be an $\mathbb{Q}$-algebra, R a smooth k-algebra and M a left $\mathscr{D}(R/k)$ module.

The goal is that if $a, b : R \rightrightarrows (R \otimes_k R)/I_\Delta^2$ are two algebra maps, then we want $\eta_{ab} : a^* M \cong b^* M$.

$$\eta_{ab} \Leftrightarrow M \otimes_R (R \otimes_k R)/I_\Delta^2 \to (R \otimes R/I_\Delta^2) \otimes_R M \quad R \otimes R\text{-linear}$$

$$\Leftrightarrow M \overset{\eta}{\to} (R \otimes R/I_\Delta^2) \otimes_R M \quad R\text{-linear}$$

Here $\eta(m) = (1 \otimes m) + \nabla_M(m)$ (Use: $I_\Delta/I_\Delta^2 \cong \Omega^1_{R/k}$ $(f \otimes 1 - 1 \otimes f \leftarrow \mathrm{d}f)$ therefore $\nabla_M : M \to I_\Delta/I_\Delta^2 \otimes M$)

Check: This works

Key point:

$$\begin{aligned}\eta(fm) &= 1\otimes fm + \nabla_M(fm) \quad \in R\otimes (R/I_\Delta^2)\otimes_R M\\ &= (1\otimes f)\otimes m + (f\otimes 1 - 1\otimes f)\otimes m + f\cdot\nabla_M(m)\\ &= f\otimes 1\otimes m + f\otimes\nabla_M(m)\\ &= f\eta(m)\end{aligned}$$

Example 4.1. Say R is étale over $k[x]$. Fix $a, b : R \rightrightarrows S$ such that $a \equiv b$ module I (where I is nilpotent). Let M be a left $\mathscr{D}$-module.

Fact 4.2.

$$\begin{aligned}\eta_{ab} : M\otimes_{R,a} S &\cong M\otimes_{R,b} S\\ m\otimes 1 &\mapsto \sum_{i\geqslant 0}\frac{\partial}{\partial x}^{i}(m)\otimes\frac{(b(x)-a(x))^i}{i!}\in M\otimes_{R,b} S\\ &= \exp\left(\frac{\partial}{\partial x}(-)\otimes(b(x)-a(x))\right)(m)\end{aligned}$$

Corollary 4.3. *Assume k i a field, Say M is an $\mathcal{O}$-coherent $\mathscr{D}(R/k)$-module. Then M is finite projective over R.*

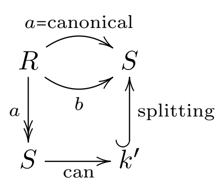
Proof. Small commutative algebra argument reduce to following:

for all $S = R/I$ where S artinian, the base change $M\otimes_R S$ is finite projective.

Set $k' = S_{\mathrm{red}}$, so k'/k is a finite extension ($\Rightarrow k'/k$ is étale because this is char 0).

Choose a k-algebra map $k'\to S$ splitting $S \twoheadrightarrow S_{\mathrm{red}} = k'$.

So we get two maps:



Crystal property $\Leftarrow M\otimes_R S = a^*M \cong b^*M$ where b^*M is finite free because k' is a field.

Note that the isomorphism $M\otimes_R S\cong$ free modules is "canonical". ☐

Remark 4.4. (1) $\mathcal{O}$-coherence is necessary for Corollary 4.3. (Ex: $k[x,x^{-1}]/k[x]$ over $k[x]$)

(2) $M = R/I$ only supports a $\mathscr{D}(R/k)$-module structure if $I = (e)R$ for e idempotent.

(3) $\mathbf{Vect}^{\nabla}(R)$= The category of vector bundles over R with flat connections is equivalent to the category of $\mathcal{O}$-coherent $\mathscr{D}(R/k)$-modules. The right-hand side category is obviously abelian. Thus the left-hand side is abelian. (This is saying that kernels, cokernels, images of maps in $\mathbf{Vect}^{\nabla}(R)$ Lie in $\mathbf{Vect}^{\nabla}(R)$).

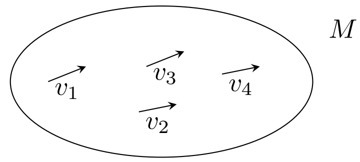
(4) If $M, N\in\mathbf{Vect}^{\nabla}(R)$, then $\mathrm{Hom}_{\mathbf{Vect}^{\nabla}(R)}(M,N)$ is a finite dimensional k-vector space.

Proof. By the proof of Corollary 4.3, for all $R\twoheadrightarrow k'$ where k' is a field, the map

$$\mathrm{Hom}_{\mathbf{Vect}^{\nabla}(R)}(M,N)\to\mathrm{Hom}_{k'}(M\otimes_R k', N\otimes_R k')$$

is injective (if $\mathrm{Spec}(R)$ is connected). Then conclude using finiteness of right-hand side. ☐

4.2. **Recollections on the Lie derivative.** Motivation:



For $v \in \Gamma(M, \mathrm{T}_M)$, we get a infinitesimal automorphism of M, hence, get an action of v on $\Omega^*(M)$. The goal is to make this explicit.

Construction 4.5. Let k be a $\mathbb{Q}$-algebra and R smooth k-algebra. Given $a \in \mathrm{T}_{R/k} = \mathrm{Der}_k(R,R) = \mathrm{Hom}_R(\Omega'_R, R)$,

Let $a : R \to R$ be the derivation and $i_a : \Omega'_R \to R$ be its linearization (so $i_a(\mathrm{d}f) = a(f)$)

Regard a as infinitesimal aut of $R \Leftrightarrow k[\varepsilon]$-linear automorphism of $R[\varepsilon]$ ($\varepsilon^2 = 0$)

Explicitly,

$$a \Leftrightarrow \varphi_a : R[\varepsilon] \to R[\varepsilon]$$

sending $f \in R$ to $f + \varepsilon a(f) = f + \varepsilon i_a(\mathrm{d}f)$.

ϕ_a^* gives a $k[\varepsilon]$-linear automorphism

$$\Omega^i(\phi_a) : \Omega^i_{R[\varepsilon]/k[\varepsilon]} \to \Omega^i_{R[\varepsilon]/k[\varepsilon]} \forall i$$

lifting the identity at $\varepsilon = 0$.

Set $\mathrm{Lie}_a(\omega) = \frac{\Omega^i(\phi_a)(\omega\otimes 1) - \omega \otimes 1}{\varepsilon} \in \Omega^i_{R/k} = \varepsilon \cdot \Omega^i_{R[\varepsilon]/k[\varepsilon]}$ where $\omega \in \Omega^i_{R/k}$.

So $\mathrm{Lie}_a : \Omega^i_{R/k} \to \Omega^i_{R/k}$ Lie derivative attached to $a \in \mathrm{T}_{R/k}$.

Properties of Lie_a:

(1) Action on functions: $f \in R = \Omega^0_{R/k}$: $\mathrm{Lie}_a(f) = a(f) = i_a(\mathrm{d}f) \in R$.

(2) de Rham differentials: For $\mathrm{d} : \Omega^i_{R/k} \to \Omega^{i+1}_{R/k}$ the de Rham differential, have $\mathrm{d} \circ \mathrm{Lie}_a = \mathrm{Lie}_a \circ \mathrm{d}$.

Proof. $\Omega^i(\phi_a), \mathrm{id}$ are both maps of complexes. □

(3) Multiplicativity: $\mathrm{Lie}_a : \Omega^*_{R/k} \to \Omega^*_{R/k}$ is a (non-signed) derivation:

$$\mathrm{Lie}_a(\omega_1 \wedge \omega_2) = \mathrm{Lie}_a(\omega_1) \wedge \omega_2 + \omega_1 \wedge \mathrm{Lie}_a(\omega_2).$$

Proof.

$$\begin{aligned}
\varepsilon \cdot \mathrm{Lie}_a(\omega_1 \wedge \omega_2) &= \Omega(\phi_a)(\omega_1 \wedge \omega_2) - \omega_1 \wedge \omega_2 \\
&= \Omega(\phi_a)(\omega_1) \wedge \Omega(\phi_a)(\omega_2) - \omega_1 \wedge \omega_2 \\
&= \Omega(\phi_a)(\omega_1) \wedge \Omega(\phi_a)(\omega_2) - \Omega(\phi_a)(\omega_1) \wedge \omega_2 + \Omega(\phi_a)(\omega_1) \wedge \omega_2 - \omega_1 \wedge \omega_2 \\
&= \Omega(\phi_a)(\omega_1) \wedge (\Omega(\phi_a)(\omega_2) - \omega_2) + (\Omega(\phi_a)(\omega_1) - \omega_1) \wedge \omega_2 \\
&= \Omega(\phi_a)(\omega_1) \wedge \varepsilon\,\mathrm{Lie}_a(\omega_2) + \varepsilon\,\mathrm{Lie}_a(\omega_1) \wedge \omega_2 \\
&= \omega_1 \wedge \varepsilon\,\mathrm{Lie}_a(\omega_2) + \varepsilon\,\mathrm{Lie}_a(\omega_1) \wedge \omega_2
\end{aligned}$$

□

(4) Cartan's magic formula: Extend $i_a : \Omega'_{R/k} \to R$ to a graded $\mathcal{O}_X$-linear map $\Omega^*_{R/k} \to \Omega^{*-1}_{R/k}$ by requiring $i_a(\omega_1 \wedge \omega_2) = i_a(\omega_1) \wedge \omega_2 + (-1)^{\deg(\omega_1)} \omega_1 \wedge i_a(\omega_2)$.

Explicitly,

$$i_a(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_k) = \sum_{i=1}^{k} (-1)^{i+1} i_a(\mathrm{d}x_i) \cdot \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_k$$

Then Cartan's formula: $\mathrm{Lie}_a = \mathrm{d} \circ i_a + i_a \circ \mathrm{d}$. In particular, $\mathrm{Lie}_a : \Omega^*_{R/k} \to \Omega^*_{R/k}$ is null homotopic. Hint: Reduce to checking statement for $f \in R$. Left as an exercise.

(5) Brackets: $\mathrm{Lie}_{[a,b]} = [\mathrm{Lie}_a, \mathrm{Lie}_b]$ for $a, b \in \mathrm{T}_{R/k}$. Hint: $[\varepsilon\,\mathrm{Lie}_a, \varepsilon\,\mathrm{Lie}_b] = [\Omega(\phi_a), \Omega(\phi_b)]$ (exercise)

(6) Scalar multiplication on left: Given $f \in R, \omega \in \Omega^i_{R/k}$,

$$\mathrm{Lie}_a(f\omega) = f\,\mathrm{Lie}_a(\omega) + a(f)\omega$$

Proof. Use (3) □

(7) Scalar multiplication on the right: Given $f \in R, \omega \in \Omega^n_{R/k}$ where n is the relative dimension,

$$\mathrm{Lie}_{fa}(\omega) = \mathrm{Lie}_a(fw)$$

Proof. Use Cartan's magic formula + $\mathrm{d}(\omega) = 0$. We get $\mathrm{Lie}_{fa}(\omega) = \mathrm{d}i_{fa}(\omega)$. So it suffices to show that $i_{fa}(-) = f \cdot i_a(-) = i_a(f(-))$. Check by hand using $\mathcal{O}$-linearity + multiplicativity of i_a. □

5. 09/16/2020

5.1. **Left-right switch.** Recall that a Lie derivative gave an action of $\mathrm{T}_{R/k}$ on $\Omega^*_{R/k}$. Today, we want to use the special case of $\omega_{R/k} = \Omega^n_{R/k}, n = \dim(R/k)$.

Observation: R is a left $\mathscr{D}$-module $\Rightarrow \mathrm{Hom}_k(R,k)$ is a right $\mathscr{D}$-module.

So if $\mathrm{Spec}(R)$ were compact manifold, then would expect

$$\omega_{R/k} \subseteq \mathrm{Hom}_k(R,k)$$
$$\omega \mapsto (f \mapsto \int f\omega)$$

is a right $\mathscr{D}$-submodule.

Proposition 5.1. *Giving a left $\mathscr{D}^{\mathrm{op}}$-module (right $\mathscr{D}$-module) is equivalent to giving:*

(a) An R-module M
(b) A map $\psi : \mathrm{T}_{R/k} \to \mathrm{End}_k(M)$ which is an anti-Lie algebra homomorphism (i.e.: $\forall a,b \in \mathrm{T}_{R/k}$, then $[\psi(a),\psi(b)] = -\psi([a,b])$).

satisfying

(3) For $f \in R, a \in \mathrm{T}_{R/k}$, we have $\psi(fa) = \psi(a)\cdot f$ in $\mathrm{End}_k(M)$
(4) For $f \in R, a \in \mathrm{T}_{R/k}$, we have $f\psi(a) = \psi(fa) + a(f)$ in $\mathrm{End}_k(M)$

Proof. Use presentation of $\mathscr{D}$ to get a presentation for $\mathscr{D}^{\mathrm{op}}$, check that $\psi : \mathrm{T}_{R/k} \to \mathrm{End}_k(M)$ extends to a ring map $\mathscr{D}^{\mathrm{op}} \to \mathrm{End}_k(M)$. □

Theorem 5.2 (left-right switch)**.** *Say M is a left $\mathscr{D}$-module. Then $\omega_{R/k} \otimes_R M$ is naturally a right $\mathscr{D}$-module via*

$$(x \otimes m)\cdot a = -\mathrm{Lie}_a(x) \otimes m - x \otimes a(m)$$

where $x \in \omega_{R/k}, m \in M, a \in \mathrm{T}_{R/k}$. Moreover, the functor $M \mapsto \Omega(M) = \omega_{R/k} \otimes_R M$ gives an equivalence between left and right $\mathscr{D}$-moduels with inverse given by $N \mapsto N \otimes_R \omega^{-1}_{R/k}$.

Proof. Check that $\psi(a) = (x \otimes m \mapsto (x \otimes m)\cdot a)$ gives a homomorphism $\mathscr{D}^{\mathrm{op}} \to \mathrm{End}_k(M)$ using previous proposition.

Why is formula well-definied?

$$\begin{aligned}
(x \otimes fm)\cdot a &:= -\mathrm{Lie}_a(x) \otimes fm - x \otimes a(fm) \\
&= -f\,\mathrm{Lie}_a(x) \otimes m - x \otimes a(f)m - x \otimes fa(m) \\
&= -(f\,\mathrm{Lie}_a(x) \otimes m + a(f)x \otimes m) - fx \otimes am \\
&= -(f\,\mathrm{Lie}_a(x) + a(f)x) \otimes m - fx \otimes m \\
\text{property (6) of Lie deriative } \to &= -\mathrm{Lie}_a(fx) \otimes m - fx \otimes m =: (fx \otimes m)\cdot a
\end{aligned}$$

□

Exercise 5.3. Say $R = k[x_1, \ldots, x_n], \mathscr{D} = R[\partial_1, \ldots, \partial_n]$ and $\omega_{R/k} \cong R\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. So if $M = R$ is a left $\mathscr{D}$-module, then $\omega_{R/k} \otimes M = R$ is a right $\mathscr{D}$-module.

Explicitly, if $P = \sum_\alpha a_\alpha(\underline{x})\underline{\partial}^\alpha \in \mathscr{D}$ and $f \in R$, then $f \cdot P = P^t(f)$ where $P^t = \sum(-\partial)^\alpha a_\alpha(x) \in \mathscr{D}$ [Key check: $-\mathrm{Lie}_{\frac{\partial}{\partial x}}(\mathrm{d}x) = \mathrm{d}i_{\frac{\partial}{\partial x}}(\mathrm{d}x) = \mathrm{d}(\frac{\partial}{\partial x}) = 0$]

Exercise 5.4. Have left $\mathscr{D}$-modules equivlent to right $\mathscr{D}$-module. This is given by $N \mapsto M \otimes_{\mathscr{D}} N$ where M is a $(\mathscr{D}^{\mathrm{op}}, \mathscr{D})$-bimodule $\Leftrightarrow$ a k-vecotr space with two left $\mathscr{D}^{\mathrm{op}}$-actions.

Show that $M = \Omega(\mathscr{D}) = \omega_{R/k} \otimes_R \mathscr{D}$ with two $\mathscr{D}^{\mathrm{op}}$-actions given by

(1) right multiplication by $\mathscr{D}$
(2) the formual in the theorem

5.2. **Operations on $\mathscr{D}$-modules.** Setup: $f : X \to Y$ map of smooth k-schemes ($k = \mathbb{Q}$-algebra)

$$\mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_X} = \text{all left } \mathscr{D}_X - \text{modules on } X$$
$$\mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X} = \{M \in \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_X} | M \text{ quasi coherent on } \mathcal{O}_X\}$$

In the case where X is affine, we have $\mathrm{L}\,\mathbf{Mod}^{\mathrm{ac}}_{\mathscr{D}_X} \cong \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}(\mathcal{O}(X))}$.

Pullback: Construct $f^* : \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_Y} \to \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_X}$ preserving quasicompact objects.

If M is a left $\mathscr{D}_Y$-module, set

$$\underbrace{f^*(M)}_{\text{as } \mathscr{D}-\text{modules}} = \underbrace{f^*(M)}_{\text{as } \mathcal{O}-\text{modules}} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M$$

as $\mathcal{O}$-modules

We obtain a flat connection $f^*M \overset{\nabla_{f^*M}}{\to} \Omega^1_X \otimes_{\mathcal{O}_X} f^*M$ as follows:

Given local sections $s \in \mathcal{O}_X, m \in M$ set

$$\nabla_{f^*M}(s \otimes m) = \mathrm{d}s \otimes m + s \cdot f^* \underbrace{\nabla(m)}_{f^*: f^*(\Omega^1_Y \otimes_{\mathcal{O}_Y} M) \to \Omega^1_X \otimes f^*M}$$

Sanity check: given a $r \in \mathcal{O}_Y$, have

$$\begin{aligned} \nabla_{f^*M}(sr \otimes m) &= \mathrm{d}(sr) \otimes m + srf^*\nabla_M(m) \\ &= (\mathrm{d}s)r \otimes m + s\mathrm{d}r \otimes m + srf^*\nabla_M(m) \\ &= \mathrm{d}s \otimes rm + s\,(\mathrm{d}r \otimes m + rf^*\nabla_M(m)) \\ &= \mathrm{d}s \otimes rm + s \cdot f^*\nabla_M(rm) \\ &= \nabla_{f^*M}(s \otimes rm) \quad \text{as wanted} \end{aligned}$$

Remark 5.5. 1) f^* preserves quasi-coherence (because we are also doing f^* on underlying $\mathcal{O}$-modules)
2) Explicitly, given local sections $s \in \mathcal{O}_X, m \in M, a \in \mathrm{T}_X$, how to describe $a \cdot (s \otimes m)$? Use

$$\mathrm{T}_X \to f^*\mathrm{T}_Y = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathrm{T}_Y$$
$$a \mapsto \sum_{i=1}^n t_i \otimes b_i$$

$$a \cdot (s \otimes m) = a(s) \otimes m + \sum_{i=1}^n st_i \otimes b_i(m)$$

3) (Pullback via transfer module): Set $\mathscr{D}_{X\to Y} = f^*\mathscr{D}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} \mathscr{D}_Y$, this is actually a $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule. Then

$$\begin{aligned} f^*M &= \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M \\ &= \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{D}_Y \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}M \\ &= \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}M \leftarrow \text{ left } \mathscr{D}_X\text{-module} \end{aligned}$$

Example 5.6. Consider $i : X \hookrightarrow Y$ where X is a point and $Y = \mathbb{A}^1 = \mathrm{Spec}(k[x])$.

What is $\mathscr{D}_{X\to Y}$?

$$\mathscr{D}_{X\to Y} = k \otimes_{k[x]} k[x,\partial] = \frac{k[x,\partial]}{x\cdot k[x,\partial]}$$

TO FIX LATER

Claim.

$$\frac{k[x,\partial]}{x\cdot k[x,\partial]} \cong \frac{k[x,x^{-1}]}{k[x]}$$

as $k[x]$-module via $\partial^{n-1} \mapsto x^{-n}$ for $n \geqslant 1$.

Have $x\partial - \partial x = 1$ in $k[x,\partial]$. So $\partial x = 1$ on left-hand side. Therefore $\partial^n x = \partial\cdot(\partial^{n-1}x) = \partial\cdot\partial^{n-2} = \partial^{n-1}$.

Given a $\mathbb{Q}$-algebra k, set $R = k[x]$ and $\mathscr{D} = \mathscr{D}(R/k) = R[\mathrm{d}]$ (where $\mathrm{d} = \mathrm{d}/\mathrm{d}x$). Recall that, thanks to the involution $\mathscr{D} \to \mathscr{D}^{\mathrm{op}}$ given by $P \mapsto P^t$ that we discussed today, any left $\mathscr{D}$-module M has a unique right $\mathscr{D}$-module structure given by $m * P = P^t(m)$ for $m \in M$ and $P \in \mathscr{D}$.

The correct statement that I should have stated and proved at the end of the lecture was the following:

Claim: The right $\mathscr{D}$-module $\mathscr{D}/x\mathscr{D}$ identifies with the right $\mathscr{D}$-module $R[x^{-1}]/R$ (where I turn the natural left $\mathscr{D}$-module structure on the target into a right $\mathscr{D}$-module structure via the left-right switch)

Proof: To give a map $\mathscr{D}/x\mathscr{D} \to R[x^{-1}]/R$, we must give an element of the target annihilated by right multiplication by x. For this element, I simply take the class of $x^{-1} \in R[x^{-1}]/R$, noting that $x^{-1} * x = x^t * x^{-1} = x * x^{-1} = 1 = 0$ (where I use $x^t = x$ and that $1 = 0 \in R[x^{-1}]/R$). Explicitly, the map sends $P \in D/xD$ to $P^t(1/x)$. The map is then seen to be surjective (as it carries $P = \mathrm{d}^n to (\mathrm{d}^n)^t(1/x) = (-1)^n\mathrm{d}^n(1/x) = n!1/x^{n+1}$ if I'm computing correctly) as well as injective (as a k-module basis for $\mathscr{D}/x\mathscr{D}$ is given by $\{1, d, d^2, d^3, \ldots.\}$ and this is mapped to a k-module basis for the target by the previous parenthetical).

6. 09/21/2020

Let k be a $\mathbb{Q}$-algebra.

6.1. Pushforwards.

Let $f : X \to Y$ be a map of smooth k-schemes of finite type. Given M a right $\mathscr{D}_X$-module, consider $M \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y} \in \mathrm{R}\,\mathbf{Mod}_{f^{-1}\mathscr{D}_Y}$.

Set $f_\dagger^{\mathrm{naive}}(M) = f_*(M \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}) \in \mathrm{R}\,\mathbf{Mod}_{\mathscr{D}_Y}$, this is the naive $\dagger$-pushforward of right $\mathscr{D}$-modules.

Remark 6.1. (1) One can show that $f_\dagger^{\mathrm{naive}}$ preserves quasi-coherence and all coproducts.

(2) This is called "naive" because it mixes a left-exact functor f_* with the right exact functor $-\otimes_{\mathscr{D}} \mathscr{D}_{X\to Y}$ ($\Rightarrow$ no good properties of $f_\dagger^{\mathrm{naive}}$).

(3) Using the left-right switch, obtain $f_\dagger^{\mathrm{naive}} : \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_X} \to \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_Y}$ via

$$\begin{aligned} f_\dagger^{\mathrm{naive}}(M) &= (f_*(\omega_X \otimes_{\mathcal{O}_X} M) \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}) \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \\ &= (r\to l) \circ f_\dagger^{\mathrm{naive}} \circ (l \to r)(M) \end{aligned}$$

Define $\mathscr{D}_{Y\leftarrow X} = (\omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_{X\to Y}) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1}$, which is a $(f^{-1}\mathscr{D}_Y, \mathscr{D}_X)$-bimodule on X.

Then $f_\dagger^{\text{naive}}(M) = f_*(\mathscr{D}_{Y\leftarrow X} \otimes_{\mathscr{D}_X} M) \in \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_Y}$ where M is a left $\mathscr{D}_X$-module.

Example 6.2. Consider $\iota : X \hookrightarrow Y$ where X is a point and Y is $\mathbb{A}^1$. Then

$$\begin{aligned}\iota_\dagger^{\text{naive}}(\underbrace{\omega_{\text{pt}}}_{\mathcal{O}\text{-coherent}}) &= i_*(\omega_{\text{pt}} \otimes_{\mathscr{D}_{\text{pt}}} \mathscr{D}_{X\to Y}) \\ &= i_*\mathscr{D}_{X\to Y} \\ &= \frac{k[x,\partial]}{x\cdot k[x,\partial]} \cong \underbrace{k[x,x^{-1}]/k[x]}_{\text{not } \mathcal{O}\text{-coherent}}\end{aligned}$$

Some examples:

de Rham Chomology: Assume X is a smooth curve over k.

Claim. *There exists a resolution*

$$\mathrm{dR}(\mathscr{D}_X) := (\mathscr{D}_X \xrightarrow{\mathrm{d}} \Omega_X^1 \otimes \mathscr{D}_X) \xrightarrow{\eta} \omega_X$$

in $\mathrm{R}\,\mathbf{Mod}_{\mathscr{D}_X}$ *where*

- d *is the flat connection for left* $\mathscr{D}_X$*-module* $\mathscr{D}_X$*.*
- $\mathrm{dR}(\mathscr{D}_X)$ *is a complex of right* $\mathscr{D}_X$*-modules through the right* $\mathscr{D}_X$*-action on itself.*
- η *is the right* $\mathscr{D}_X$*-action on* $\Omega_X^1 = \omega_X$ *(because* X *is a curve).*

Things to check:

(1) d, η are right $\mathscr{D}$-module maps
(2) $\eta \circ \mathrm{d} = 0$: $\mathrm{d}(1)$ = canonical element in $\Omega_X^1 \otimes \mathrm{T}_X \subseteq \Omega_X^1 \otimes \mathscr{D}_X$. So need $\eta(\mathrm{d}(1)) = 0$
    unwind identifications: need to show, given $\omega \in \Omega_X^1$, $a \in \mathrm{T}_X$ such that $\langle \omega, a\rangle = 1$. Have $\eta(\omega\otimes a) = 0$

    Proof. $\eta(\omega \otimes a) = \mathrm{Lie}_a(\omega) = \mathrm{d}(1) = 0$ (idea: $\omega = \mathrm{d}x$, $a = \frac{\partial}{\partial x}$, reduce to that case) □

(3) This gives a resolution: by étale localization, reduce to the case $X = \mathbb{A}^1$, where we get: Since $\omega_X = k[x]\mathrm{d}x$, the candidate resolution is

$$\left(k[x,\partial] \xrightarrow{P\mapsto \partial P} k[x,\partial]\right) \xrightarrow{xxx} k[x]$$

    One checks, by hand, that this is a resolution.

Remark 6.3. There exists an analogous resolution in higher dimensions

Example 6.4 (de Rham Coh)**.** X/k smooth affine curve, $f : X \to \mathrm{Spec}(k)$. Then $f_\dagger^{\text{naive}}(\omega_X) = f_*(\omega_X \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y})$. Now $\mathscr{D}_{X\to Y} = \mathcal{O}_X$ as Y is a point. So $f_\dagger^{\text{naive}} = f_*(\omega_X \otimes_{\mathscr{D}_X} \mathcal{O}_X)$.

Using $\mathrm{dR}(\mathscr{D}_X) \cong \omega_X$, for any left $\mathscr{D}_X$-module M, have $\omega_X \otimes_{\mathscr{D}_X} M = \mathrm{Coker}(M \xrightarrow{\nabla_M} \Omega_X^1 \otimes M)$.

$$\begin{aligned}f_\dagger^{\text{naive}}(\omega_X) &= f_*(\omega_X \otimes_{\mathscr{D}_X} \mathcal{O}_X) \\ &= f_*(\mathrm{Coker}(\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X^1)) \\ &= \mathrm{H}^1_{\mathrm{dR}}(X/k)(\underset{X \text{ affine}}{:=} \mathrm{H}^1(\mathcal{O}(X) \to \Omega_X^1(X)))\end{aligned}$$

Example 6.5 (Failure of non-derived Kashiwara)**.** Consider $i : X \hookrightarrow Y$ where X is a point and $Y = \mathbb{A}^1$ (they intersect at 0).

$i_\dagger : \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_{\text{pt}}} \to \mathrm{L}\,\mathbf{Mod}_{\mathbb{A}^1}$ and $i^* : \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_{\mathbb{A}^1}} \to \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_{\text{pt}}}$.

Claim. $i^* i_\dagger = 0$

Proof. $\mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_{\mathrm{pt}}} = \mathbf{Vect}(k)$ Need to show that $i^* i_\dagger(k) = 0$.

As k-modules,

$$\begin{aligned} i^* i_\dagger(k) &= i^*(k[x, x^{-1}]/k[x]) \\ &= k \otimes_{k[x]} k[x, x^{-1}]/k[x] \\ &= 0 (\text{because } k[x, x^{-1}]/k[x] \text{ is } X\text{-divisible}) \end{aligned}$$

□

Note $\mathrm{Tor}_1^{k[x]}(k, k[x, x^{-1}]/k[x]) \cong k$ So would have $\mathsf{L}i^* \circ \mathsf{R}i_\dagger \cong \mathrm{id}[1]$ once we derive.

6.2. **Quick review of derived categories.** Let A be an abelian category (ex: $A = \mathbf{Ab}, \mathbf{QCoh}(X), \mathbf{Mod}(X, \mathcal{O}_X)$ where X are schemes).

The category $\mathrm{Ch}(A)$ is the chain complexes of objects in A. (cohomologically graded)

recall: if $f : K \to L$ map in $\mathrm{Ch}(A)$ is called a *quasi-isomorphism* (qis) if $\mathrm{H}^*(K) \cong \mathrm{H}^*(L)$ via f.

Definition 6.6. $\mathsf{D}(A) = \mathrm{Ch}(A)[(\mathrm{qis})^{-1}]$.

So we get a functor $q : \mathrm{Ch}(A) \to \mathsf{D}(A)$ where $q(\mathrm{qis})$ =isomorphism and q is universal with this property.

Goal:

(1) Give more explicit description of $\mathsf{D}(A)$ via triangulated categories
(2) Understand $\mathsf{D}(A)$ as a functor of A.

References are [GM03], [Wei94, Chapter 10], [SG08, Appendix B, C], Stacks project, [Bei].

6.3. **Triangulated category.** Let $\mathcal{C}$ be an additive category. The *shift functor* $X \mapsto X[1]$ is an automorphism of $\mathcal{C}$.

Definition 6.7. A *triangle* in $\mathcal{C}$ is the datum of loops

$$X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} X[1]$$

The intuition is that a (exact) triangle as above is equivalent to the data of "$Z = Y/X$".

Definition 6.8. Assume $\mathcal{C}$ comes quipped with a collection of triangles, called exact triangles. Then $\mathcal{C}$ is *triangulated* if

(TR1) (Cones exists) Every map $u : X \to Y$ in $\mathcal{C}$ fits into an exact triangle

$$X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} X[1]$$

and

$$X \overset{\mathrm{id}}{\to} X \to 0 \to X[1]$$

is exact.

(TR2) (Rotational Symmetry) If $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} X[1]$ is exact, then

$$Y \overset{v}{\to} Z \overset{w}{\to} X[1] \overset{-u[1]}{\to} Y[1]$$

$$Z[-1] \overset{-w[-1]}{\to} X \overset{u}{\to} Y \overset{v}{\to} Z$$

are exact.

(TR3) (Cones are weakly functorial) Given $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$ two exact triangles and maps

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g'} \\ X' & \xrightarrow{u'} & Y' \end{array}$$

, then there exist a map of triangles from the first to the second extending f and g.

(TR4) (Octahedrel axiom, informal version): Given map $A \to B$ and $B \to C$, have $(C/A)/(B/A) \cong C/B$.

7. 09/23/2020

7.1. **Triangulated category (continued).** A functor $F : \mathcal{C} \to \mathcal{D}$ of triangulated categories is called *exact* if

(1) F commutes with $[1]$ (extra data)
(2) F preserves exact triangles.

Example 7.1. $F = [n]$ for any $n \in \mathbb{Z}$

Exercise 7.2. Say $\mathcal{C}$ is a triangulated category, and $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is an exact triangle.

(1) $vu = 0, wv = 0$.
(2) For any $T \in \mathcal{C}$, the natural maps gives a long exact sequence

$$\begin{array}{ccccc} \cdots & & \cdots & & \mathrm{Hom}(T, Z[-1]) \\ \mathrm{Hom}(T, X) & \longrightarrow & \mathrm{Hom}(T, Y) & \longrightarrow & \mathrm{Hom}(T, Z) \\ \mathrm{Hom}(T, X[1]) & \longrightarrow & \cdots & & \cdots \end{array}$$

(3) Z is unique up to non-unique isomorphism
(4) (May) (TR3) follows from the other axioms.

Remark 7.3. Given a triangulated category $\mathcal{C}$, here is typically no functor

$$\mathrm{Mor}(\mathcal{C}) \to \mathcal{C}$$
$$u : x \to y \mapsto \text{“cone of ”} u$$

7.2. **The homotopy category.** Let $\mathcal{A}$ be an abelian category (for example, $\mathbf{Mod}(X, \mathcal{O}_X)$.)

The homotopic category $\mathsf{K}(\mathcal{A})$ is the category where objects are $K \in \mathrm{Ch}(\mathcal{A})$ with morphisms being $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(K, L) = \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(K, L)/\sim$ where $\sim$ is the homotopic equivalence of maps: where $f, g : K \to L$ are maps in $\mathrm{Ch}(\mathcal{A})$ are homotopic ($f \sim g$) if there exists $s^i : K^i \to L^{i-1}$ in $\mathcal{A}$ such that $f - g = \mathrm{d}s + s\mathrm{d}$.

Check: This gives an additive category and the functors $K \in \mathrm{Ch}(\mathcal{A}) \mapsto \mathrm{H}^i(K) \in \mathcal{A}$ factor over $\mathrm{Ch}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$.

7.3. **Structures on $\mathsf{K}(\mathcal{A})$.** Shift functor:

$$K \mapsto K[1]$$
$$K[1]^i = K^{i+1}$$
$$\mathrm{d}_{K[1]} = -\mathrm{d}_K$$

So $\mathrm{H}^i(K[1]) = \mathrm{H}^{i+1}(K)$.

Exact triangles: Given a map $f : K \to L$ in $\mathrm{Ch}(\mathcal{A})$, set $C(f) \in \mathrm{Ch}(\mathcal{A})$: $C(f)^i = L^i \oplus K[1]^i = L^i \oplus K^{i+1}$. where $\mathrm{d}(x, y) = (\mathrm{d}x + f(y), -\mathrm{d}y)$.

So we get a triangle $K \xrightarrow{f} L \xrightarrow{\mathrm{can}} C(f) \xrightarrow{\mathrm{pr}_2} K[1]$.

A triangle in $K(\mathcal{A})$ is called *exact* if it is isomorphic to a triangle of the above form.

Theorem 7.4. *These structures turn $\mathsf{K}(\mathcal{A})$ into a triangulated category.*

Variants: For $x \in \{b, +, -, \varnothing\}$, have a full triangulated subcategory $\mathsf{K}^*(\mathcal{A}) \subseteq \mathsf{K}(\mathcal{A})$ defined as follows:

- $K \in \mathsf{K}^b(\mathcal{A}) \Leftrightarrow \mathrm{H}^i(K) = 0$ for any $i >> 0, i << 0$.
- $K \in \mathsf{K}^+(\mathcal{A}) \Leftrightarrow \mathrm{H}^i(K) = 0$ for any $i << 0$.
- $K \in \mathsf{K}^-(\mathcal{A}) \Leftrightarrow \mathrm{H}^i(K) = 0$ for any $i >> 0$.

Or $K \in \mathsf{K}^{[a,b]}(\mathcal{A}) \Leftrightarrow \mathrm{H}^i(K) = 0$ for any i outside $[a, b]$.

Example 7.5. In $\mathsf{K}(\mathbf{Ab})$, we have a quasiisomorphism

$$(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \xrightarrow{\varepsilon} \mathbb{Z}/2\mathbb{Z}$$

which is not an isomorphism in $\mathsf{K}(\mathbf{Ab})$.

7.4. **The derived category.** Let $\mathcal{A}$ be an abelian category and $f : K \to L$ in $\mathsf{K}(\mathcal{A})$ is a *quasiisomorphism* if $\mathrm{H}^*(f)$ is an isomorphism.

Theorem 7.6. *(1) The localization $q : \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})[(qis)^{-1}] =: \mathsf{D}(\mathcal{A})$ exists (i.e., there exists an initial object in the 2-category of maps $\mathsf{K}(\mathcal{A}) \to \mathcal{C}$ that invert quasiisomorphisms.)*
(2) The localization map $q : \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ induces a triangulated structure on $\mathsf{D}(\mathcal{A})$.
(3) (A calculus of fractions) For $K, L \in \mathsf{K}(\mathcal{A})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(K, L) &= \operatorname*{Colim}_{\substack{K' \to K \\ qis\ in\ \mathsf{K}(\mathcal{A})}} \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(K', L) \\ &= \operatorname*{Colim}_{\substack{L \to L' \\ qis\ in\ \mathsf{K}(\mathcal{A})}} \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(K, L')\end{aligned}$$

(4) If I^* is a bounded below complex of injectives, then

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(-, I^*) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(-, I^*)$$

So if $\mathcal{A}$ has enough injectives, then $\mathsf{K}^+(\mathrm{Inj}(\mathcal{A})) \cong \mathsf{D}^+(\mathcal{A})$.
(5) Dual statement for projectives.
(6) For $X, Y \in \mathcal{A} \subseteq \mathrm{Ch}(\mathcal{A})$, have

$$\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n]) \cong \mathrm{Ext}^n_{\mathcal{A}}(X, Y)$$

Example 7.7. The map $(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \xrightarrow{\varepsilon} \mathbb{Z}/2\mathbb{Z}$ is quasiisomorphism in $\mathsf{D}(\mathcal{A})$, we get a map

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\varepsilon^{-1}} & K = (\mathbb{Z} \xrightarrow{2} \mathbb{Z}) \\ & \searrow^{\alpha} & \downarrow \mathrm{pr}_1 \\ & & \mathbb{Z}[1] \end{array}$$

One checks that $\alpha \in \mathrm{Hom}_{\mathsf{D}(\mathbf{Ab})}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}[1])$ corresponds 1-1 to nonzero elements in $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$.

Remark 7.8. $\mathsf{K}(\mathsf{Inj}(\mathcal{A}))$ is a triangulated subcategory of $\mathsf{K}(\mathcal{A})$ but it does *NOT* map isomorphically to $\mathsf{D}(\mathcal{A})$ in general. But Spaltenstein(-Serpe) defined a notion of "K-injective" complexes in $\mathsf{K}(\mathcal{A})$ and showed

$$\mathsf{K}(\text{K-injectives}) \cong \mathsf{D}(\mathcal{A})$$

when $\mathcal{A}$ is a Grothendieck abelian category.

Example 7.9. (1) R associated ring $\Rightarrow \mathsf{D}(R) \coloneqq \mathsf{D}(\mathrm{L}\,\mathbf{Mod}_R)$ derived category of modules of R. Check: $\mathsf{D}(R)$ is generated by R as a triangulated category under coproducts. For example, $R = k$ is a field, then $\mathsf{D}(k) \cong \{\text{graded } k\text{-v.s.}\}$ where $V \mapsto \oplus_i \mathrm{H}^i(V)$. Note: in general, given $K \in \mathsf{D}(\mathcal{A})$, the objects K and $\oplus_i \mathrm{H}^i(K)[-i]$ are not isomorphic even though they have the same H^*. (ex: in $\mathsf{D}(\mathbb{Z}/4\mathbb{Z})$, take $K = (\mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z})$.)

(2) Given a ringed space $(X, \mathcal{O}_X)$, have $\mathsf{D}(X, \mathcal{O}_X) \coloneqq \mathsf{D}(\mathbf{Mod}(X, \mathcal{O}_X))$. Exercise: check that $\mathsf{D}(X, \mathcal{O}_X)$ is generated (as before) by $\{j_! \mathcal{O}_U\}$ where $j : U \subseteq X$ is an open subset.

(3) Say X scheme

$$\mathsf{D}_{\mathrm{qc}}(X) = \{K \in \mathsf{D}(X, \mathcal{O}_X) | \mathrm{H}^i(K) \in \mathbf{QCoh}_X \ \forall i\}$$

Claim: $\mathsf{D}_{\mathrm{qc}}(X)$ is a triangulated subcategory of $\mathsf{D}(X, \mathcal{O}_X)$ stable under all coproducts.

Remark 7.10. If X is quasicompact with affine diagonal, then $\mathsf{D}_{\mathrm{qc}}(X) \cong \mathsf{D}(\mathbf{QCoh}_X)$.

(4) X noetherian scheme

$$\mathsf{D}_{\mathbf{Coh}}(X) = \{K \in \mathsf{D}_{\mathrm{qc}}(X) | \mathrm{H}^i(K) \in \mathbf{Coh}_X \ \forall i\}$$

and similar statements for $\mathsf{D}^b_{\mathbf{Coh}}(X), \mathsf{D}^+_{\mathbf{Coh}}, \mathsf{D}^-_{\mathbf{Coh}}$,etc.

(5) X scheme. A quasicoherent sheaf of associated $\mathcal{O}_X$-algebras. Then

$$\mathsf{D}_{\mathrm{qc}}(X, \mathcal{A}) = \{K \in \mathsf{D}(\mathbf{Mod}(X, A)) | \mathrm{H}^i(K) \text{ is quasicoherent over } \mathcal{O}_X, \forall i\}$$

ex: $A = \mathscr{D}_X$ (X/k finite type scheme where k noetherian).

Remark 7.11. If k is a $\mathbb{Q}$-algebra and X/k is smooth, we have

$$\underbrace{\mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_X)}_{\text{important for our purpose}} \subseteq \underbrace{\mathsf{D}_{\mathbf{Coh}}(\mathscr{D}_X)}_{\text{coherent } \mathscr{D}_X} \subseteq \mathsf{D}_{\mathrm{qc}}(\mathscr{D}_X).$$

8. 09/28/2020

8.1. Derived category (continued).

Last time, for $\mathcal{A}$ abelian category, we have

- $\mathsf{K}(\mathcal{A})$ homotopy category of chain complexes (triangulated category)
- $\mathsf{D}(\mathcal{A})$ derived category $= \mathsf{K}(\mathcal{A})[\mathrm{qis}^{-1}]$ (triangulated category)

Example 8.1. (1) $\mathsf{D}(R)$ for associative ring R

(2) $\mathsf{D}(X, \mathcal{O}_X)$ for ringed space $(X, \mathcal{O}_X)$.

(3) $\mathsf{D}_{\mathrm{qc}}(X) \subseteq \mathsf{D}(X, \mathcal{O}_X)$ for scheme X

(4) $\mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_X) \subseteq \mathsf{D}_{\mathrm{qc}}(\mathscr{D}_X) \subseteq \mathsf{D}(X, \mathscr{D}_X)$ for smooth k-scheme X (where k is a field of char 0).

Generators (Thomason): Say X is a qcqs scheme. An object $K \in \mathsf{D}_{\mathrm{qc}}(X)$ is called *perfect* if it satisfies one of:

(1) locally on X, K can be represented by a finite complex of finite free $\mathcal{O}_X$-modules
(2) $\mathrm{Hom}(K, -)$ commutes with all coproducts

Theorem 8.2 (Thomason, Bondal-van der Begrgh, Hopkins, Neeman)**.**

$$\mathsf{D}_{\mathrm{perf}}(X) \coloneqq \{K \in \mathsf{D}_{\mathrm{qc}}(X) | K \textit{ perfect}\}$$

This generates $\mathsf{D}_{\mathrm{qc}}(X)$ *under coproducts as a triangulated category.*

8.2. Derived functors.

[Sta20, Tag 05S7]

Definition 8.3 (Deligne)**.** Say $F : \mathsf{K}(A) \to \mathsf{K}(B)$ is an exact functor and $X \in \mathsf{K}(A)$. We say that $\mathsf{R}F$ is defined on X if the ind-object $\{F(X')\}_{\substack{X \to X' \\ \text{qis in } \mathsf{K}(A)}}$ is essentially constant.

If this happens, then write $\mathsf{R}F(X) =$ constant value. (Dual notion of $\mathsf{L}F$).

Example 8.4 (Easy examples). (1) Any exact functor $F : A \to B$ induces a functor $F' : \mathsf{K}(A) \to \mathsf{K}(B)$. Then $\mathsf{R}F'$ is defined for all $X \in \mathsf{K}(A)$. The proof is that $F'(\text{qis})$ is invertible because F is exact.
(2) Say $F : A \to B$ is a left-exact functor. Assume A has enough injectives, then $F' : \mathsf{K}(A) \to \mathsf{K}(B)$.

Claim. *$\mathsf{R}F'$ is defined for all $X \in \mathsf{K}^+(A)$*

Proof. Assume $X \in A$ for simplicity. Then $\mathsf{R}F'(X) = F'(I^\bullet)$ for any injective resolution $X \to I^\bullet$. ☐

Properties of derived functors: Fix an exact functor $F : \mathsf{K}(A) \to \mathsf{K}(B)$.

(1) Consider $\mathscr{E} = \{X \in \mathsf{K}(A) | \mathsf{R}F \text{ defined of } X\} \subseteq \mathsf{K}(A)$. Then
  (a) $\mathscr{E} \subseteq \mathsf{K}(A)$ is a full triangulated subcategory and we have a functor $\mathsf{R}F : \mathscr{E} \to \mathsf{D}(\mathscr{E})$ sending X to $\mathsf{R}F(X)$. Moreover, $\mathsf{R}(\text{qis})$ is an isomorphism. and any qis in $\mathsf{K}(A)$ whose source/target lies in $\mathscr{E}$ itself lies in $\mathscr{E}$.
  (b) The map $\mathscr{D}' = \mathscr{E}[\text{qis}^{-1}] \to \mathsf{K}(A)[\text{qis}^{-1}] = \mathsf{D}(A)$ is fully faithful and realizes the source as a triangulated subcategory of target
  (c) The functor $\mathsf{R}F$ in (a) induces an exact functor $\mathsf{R}F : \mathscr{D}' \to \mathsf{D}(B)$.

  Note: $\mathscr{E}$ =preimage of $\mathscr{D}'$ under $\mathsf{K}(A) \to \mathsf{D}(A)$.

(2) Say A is a Grothendieck abelian category (ex: $\mathbf{Mod}(X, \mathcal{O}_X)$). Fix $F : \mathsf{K}(A) \to \mathsf{K}(B)$ an exact functor.
  Serpe-Spaltenstein: $\mathsf{R}F$ is defined on all $X \in \mathsf{K}(A)$. Explicitly, given $X \in \mathsf{K}(A)$, choose a "K-injective" representative $I^\bullet \in \mathrm{Ch}(A)$ of X, and set $\mathsf{R}F(X) = F(I^\bullet)$ [Sta20, Tag 070Y]
  (Dually, there exists a notion of "K-flat" complexes to compute $\mathsf{L}F$ for $F = - \otimes^{\mathsf{L}}_{\mathcal{O}_X} M$)

Example 8.5. Say $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ map of ringed spaces

(1) The pushforward $f_* : \mathbf{Mod}(X, \mathcal{O}_X) \to \mathbf{Mod}(Y, \mathcal{O}_Y)$ has a right derived functor $\mathsf{R}f_* : \mathsf{D}(X, \mathcal{O}_X) \to \mathsf{D}(Y, \mathcal{O}_Y)$. Note: if f is a qcqs map of schemes, then $\mathsf{R}f_*$ is a qcqs map of schemes, then $\mathsf{R}f_*$ preserves $\mathsf{D}_{\text{qc}}(-)$.
(2) The pullback $f^* : \mathbf{Mod}(Y, \mathcal{O}_Y) \to \mathbf{Mod}(X, \mathcal{O}_X)$ has a left derived functor $\mathsf{L}f^* : \mathsf{D}(Y, \mathcal{O}_Y) \to \mathsf{D}(X, \mathcal{O}_X)$ Note: For maps of schemes, $\mathsf{L}f^*$ preserves $\mathsf{D}_{\text{qc}}(-)$.

  Exercise 8.6. Show $\mathsf{L}f^*$ is left adjoint to $\mathsf{R}f_*$.

(3) For $K \in \mathrm{Ch}(X, \mathcal{O}_X)$, the functor $\mathrm{Hom}^\bullet_X(K, -) : \mathsf{K}(X, \mathcal{O}_X) \to \mathsf{D}(\mathbf{Ab})$ admits a right derived functor $\mathsf{R}\,\mathrm{Hom}_X(K, -) : \mathsf{D}(X, \mathcal{O}_X) \to \mathsf{D}(\mathbf{Ab})$. Dually, have $\mathsf{R}\,\mathrm{Hom}_X(-, K)$. Note that $\mathsf{R}\,\mathrm{Hom}_X(-, K)(L) = \mathsf{R}\,\mathrm{Hom}_X(L, -)(K)$.
(4) For $K \in \mathrm{Ch}(X, \mathcal{O}_X)$, the functor $\mathscr{H}om^\bullet_X(K, -)$ derives to $\mathsf{R}\mathscr{H}om_X(K, -) : \mathsf{D}(X, \mathcal{O}_X) \to \mathsf{D}(X, \mathcal{O}_X)$. (dual notion: $\mathsf{R}\mathscr{H}om_X(-, K)$.)
(5) For $K \in \mathrm{Ch}(X, \mathcal{O}_X)$, have $- \otimes_{\mathcal{O}_X} K : \mathsf{K}(X, \mathcal{O}_X) \to \mathsf{D}(X, \mathcal{O}_X)$ derives to $- \otimes^{\mathsf{L}}_{\mathcal{O}_X} K : \mathsf{D}(X, \mathcal{O}_X) \to \mathsf{D}(X, \mathcal{O}_X)$. FACT: $(- \otimes^{\mathsf{L}}_{\mathcal{O}_X} K)(M) = (M \otimes^{\mathsf{L}}_{\mathcal{O}_X} -)(K)$

  Exercise 8.7. $\mathsf{R}\mathscr{H}om_X(K, \mathsf{R}\mathscr{H}om_X(M, N)) \cong \mathsf{R}\mathscr{H}om_X(K \otimes^{\mathsf{L}}_{\mathcal{O}_X} M, N)$

(6) ([Nee96]) Say $f : X \to Y$ is a map of qcqs schemes. Then $\mathsf{R}f_* : \mathsf{D}_{\text{qc}}(X) \to \mathsf{D}_{\text{qc}}(Y)$ commutes with all coproducts. Then by adjoint functor theorem, $\mathsf{R}f_*$ has a right adjoint $f^\times$. One can show that if f is proper map of noetherian schemes, then $f^\times$ preserves coherence and we have natural isomorphisms

$$\mathsf{R}\,\mathrm{Hom}_X(F, f^\times G) \cong \mathsf{R}\,\mathrm{Hom}_Y(\mathsf{R}f_* F, G)$$

  and

$$\mathsf{R}f_* \mathsf{R}\mathscr{H}om_X(F, f^\times G) \cong \mathsf{R}\mathscr{H}om_Y(\mathsf{R}f_* F, G)$$

  (This is Grothendieck duality)

  Remark 8.8 (Notation). If f is proper, then write $f^!$ for $f^\times$.

Proposition 8.9. *Say f is a proper map of noetherian schemes. Assume that f has finite* Tor*-dimension (e.g., f is flat or Y is regular).*

(1) $\mathsf{R}f_*$ preserves $\mathsf{D}_{\text{perf}}(-)$, so $f^!(M) \cong \mathsf{L}f^*(M) \otimes f^! \mathcal{O}_Y$. (Formal part)

(2) If X and Y are smooth over a field k, then

$$\underbrace{f^!\mathcal{O}}_{\text{relative dualizing complex}} \cong \omega_X \otimes f^*\omega_Y^{-1}[\dim X - \dim Y]$$

(nonformal part)

8.3. **Neeman's Adjoint Functor theorem.** Fix qcqs schemes X, Y and $F : \mathsf{D}_{\mathrm{qc}}(X) \to \mathsf{D}_{\mathrm{qc}}(Y)$ exact functor. (In fact, one only needs much milder conditions)

Neeman:

(1) F has a right adjoint $\Leftrightarrow$ F perserves coproducts.
(2) Assume F has a right adjoint G, then G preserves coproducts $\Leftrightarrow$ F preserves $\mathsf{D}_{\mathrm{perf}}(-)$.

9. 09/30/2020

9.1. **Derived operations on $\mathscr{D}$-modules.** Say k a field of characteristic 0, $f : X \to Y$ map of smooth k-schemes.

We have "transfer $\mathscr{D}$-modules"

(1) $\mathscr{D}_{X\to Y} = \mathcal{O}_X \otimes^{\mathsf{L}}_{f^{-1}\mathcal{O}_Y} \mathscr{D}_Y$. Here we use that $\mathscr{D}_Y$ is flat over $\mathcal{O}_Y$. This is a $(\mathscr{D}_X, f^{-1}\mathscr{D}_Y)$-bimodule. Observation: Given $g : Y \to Z$ map of smooth k-schemes, we have a "transitivity isomorphism"

$$\mathscr{D}_{X\to Z} = \mathscr{D}_{X\to Y} \otimes^{\mathsf{L}}_{f^{-1}\mathscr{D}_Y} f^{-1}(\mathscr{D}_{Y\to Z})$$

Proof.

$$\begin{aligned}\text{RHS} &= \mathscr{D}_{X\to Y} \otimes^{\mathsf{L}}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{D}_{Y\to Z}\\ &= (\mathcal{O}_X \otimes^{\mathsf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{D}_Y) \otimes^{\mathsf{L}}_{f^{-1}\mathscr{D}_Y} f^{-1}(\mathcal{O}_Y \otimes_{g^{-1}\mathcal{O}_Z} g^{-1}\mathscr{D}_Z)\\ &= \mathcal{O}_X \otimes_{f^{-1}g^{-1}\mathcal{O}_Z} f^{-1}g^{-1}\mathscr{D}_Z = \text{LHS}\end{aligned}$$

□

(2) $\mathscr{D}_{Y\leftarrow X} = \omega_X \otimes^{\mathsf{L}}_{\mathcal{O}_X} \mathscr{D}_{X\to Y} \otimes^{\mathsf{L}}_{f^{-1}\mathcal{O}_Y} f^{-1}\omega_Y^{-1}$ because ω_X flat over $\mathcal{O}_X$ and ω_Y flat over $\mathcal{O}_Y$. This is a $(f^{-1}\mathscr{D}_Y, \mathscr{D}_X)$-bimodule. Also have for $g : Y \to Z$ as above, we have $\mathscr{D}_{Z\leftarrow X} = f^{-1}\mathscr{D}_{Z\leftarrow Y} \otimes^{\mathsf{L}}_{f^{-1}\mathscr{D}_Y} \mathscr{D}_{Y\leftarrow X}$ (exercise)

From now on, all functors are derived, e.g., $f_* = \mathsf{R}f_*$, $f^* = \mathsf{L}f^*$. And usual functor will be the zeroth homology of these derived ones.

9.1.1. *Pullback.* Have $f^* : \mathsf{D}(\mathscr{D}_Y) \to \mathsf{D}(\mathscr{D}_X)$ where $f^*(M) = \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}M$.

This has the following properties:

(1) On underlying $\mathcal{O}$-modules, have $f^* = f^*$ where the later is the $\mathcal{O}$-module pullback

$$\begin{aligned}f^*M &= \mathscr{D}_{X\to Y} \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}M\\ &= \mathcal{O}_X \otimes_{\mathcal{O}_Y} f^{-1}\mathscr{D}_Y \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}M\\ &= \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M = f^*M\end{aligned}$$

(2) f^* perserves $\mathsf{D}_{\mathrm{qc}}(\mathscr{D})$ use (1) + stability of $\mathsf{D}_{\mathrm{qc}}(\mathcal{O})$ under f^*
(3) Compositions: $(gf)^* = f^*g^*$ for $X \xrightarrow{f} Y \xrightarrow{g} Z$. The proof uses transitivity isomorphism.
(4) Cohomological dimension: f^* has bounded cohomological dimension. In fact, $f^*(\mathsf{D}^{[a,b]}(\mathscr{D}_Y)) \subseteq \mathsf{D}^{[a-\dim(Y),b]}(\mathscr{D}_X)$. The proof uses (1) + Y has global dimension $\leqslant \dim Y$ (so $\mathrm{Tor}_i^Y(M,N) = 0$ for any $i > \dim(Y)$ and $M, N \in \mathcal{O}_Y$-modules). (left as an exercise: use spectral sequence for Tor).

Remark 9.1. (a) If f is flat, then $f^* : \mathsf{D}(\mathscr{D}_Y) \to \mathsf{D}(\mathscr{D}_X)$ is t-exact (i.e., $f^*(\mathsf{D}^{[a,b]}) \subseteq \mathsf{D}^{[a,b]}$).

(b) Later, $\mathscr{D}_Y$ has global dimension $\leqslant 2\dim(Y)$.

(5) If f is smooth, then f^* preserves $\mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D})$: Say $f: X \to Y$ is smooth, first, reduce to X, Y being affine. Next, using a resolution over $\mathscr{D}_Y$, reduce to checking $f^*\mathscr{D}_Y$ is coherent over $\mathscr{D}_X$.
Using étale localization property of $\mathscr{D}$, reduce to the following:

$$\begin{array}{ccc} X = \mathbb{A}^{n+m} & \xrightarrow{\quad f \quad} & Y = \mathbb{A}^m \\ \big\downarrow = & & \big\downarrow = \\ \operatorname{Spec}(k[x_1, \dots, x_m, y_1, \dots, y_n]) & \longrightarrow & \operatorname{Spec}(k[x_1, \dots, x_m]) \end{array}$$

where

$$\mathscr{D}_Y = k[x_1, \dots, x_m, \partial_1, \dots, \partial_m]$$
$$f^*\mathscr{D}_Y = k[x_1, \dots, x_m, y_1, \dots, y_n, \partial_1, \dots, \partial_m]$$

Check $f^*\mathscr{D}_Y = \mathscr{D}_X / \mathscr{D}_X(\partial_{y_1}, \dots, \partial_{y_n})$. (exercise) Reference: [HTT08]

Remark 9.2. If f is not smooth, f^* need not preserve $\mathscr{D}$-coherence. The example is that $0 \overset{f}{\hookrightarrow} \mathbb{A}^1$, we show that $f^*\mathscr{D}_{\mathbb{A}^1} = k[x, \partial]/k[x, \partial] \cdot x$, which is not coherent over a point.

Example 9.3. For $i: \operatorname{Spec}(k) = 0 \hookrightarrow \operatorname{Spec}(k[x]) = \mathbb{A}^1$. Then $M = \frac{k[x, x^{-1}]}{k[x]} \in \mathrm{L}\,\mathbf{Mod}_{\mathbb{A}^1}$ and

$$i^* M = k \otimes^{\mathsf{L}}_{k[x]} \frac{k[x, x^{-1}]}{k[x]} = \Big(\frac{k[x, x^{-1}]}{k[x]} \xrightarrow{x} \underbrace{\frac{k[x, x^{-1}]}{k[x]}}_{\deg = 0}\Big) = k \cdot \frac{1}{x}[1] = k[1]$$

This motivates the renormalized pullback.

Definition 9.4. For $f: X \to Y$ map of smooth k-schemes, set $f^+ : \mathsf{D}(\mathscr{D}_Y) \to \mathsf{D}(\mathscr{D}_X)$ to be $f^+ = f^*[\dim(X) - \dim(Y)]$.

Remark 9.5.
(1) Under the left-right adjoint, the corresponding pullback $f^\dagger : \mathsf{D}(\mathscr{D}_Y^{\mathrm{op}}) \to \mathsf{D}(\mathscr{D}_X^{\mathrm{op}})$ concides with $f^!$ on underlying $\mathcal{O}$-modules.
(2) If f is a closed immersion, then f^+ is t-left exact, i.e., $f^+(\mathsf{D}^{\geqslant 0}) \subseteq \mathsf{D}^{\geqslant 0}$.
Proof.

$$\begin{aligned} f^+(M) &= \mathcal{O}_X \otimes_{\mathcal{O}_Y} M[\dim(X) - \dim(Y)] \\ &= \mathcal{O}_X \otimes_{\mathcal{O}_Y} M[-c] \end{aligned}$$

where $c = \operatorname{codim}(X \subseteq Y)$. Now use $\operatorname{Tor}\dim_{\mathcal{O}_Y}(\mathcal{O}_X) = c$ (because X, Y smooth) $\square$

Exercise 9.6. Show that the left $\mathscr{D}_X$-module $\mathscr{D}_{X \to Y}$ has

(1) $\operatorname{Tor}\dim 0$ if f is a closed immersion.
(2) $\operatorname{Tor}\dim \leqslant \dim(X) - \dim(Y)$ for f smooth.
(3) $\operatorname{Tor}\dim \leqslant \dim(X)$.

9.1.2. *Pushforward.* Let $f: X \to Y$ map of smooth qcqs k-schemes. Set $f_+ : \mathsf{D}(\mathscr{D}_X) \to \mathsf{D}(\mathscr{D}_Y)$ via

$$f_+(M) = f_*(\mathscr{D}_{Y \leftarrow X} \otimes_{\mathscr{D}_X} M) \in \mathsf{D}(\mathscr{D}_Y)$$

This has the following properties:

(1) f_+ perserves quasi-coherence D_{qc}: as f is qcqs, f_* preserves $\mathsf{D}_{\mathrm{qc}}(\mathcal{O})$. It sufficies to show that for $M \in \mathsf{D}_{\mathrm{qc}}(\mathscr{D}_X) \Rightarrow \mathscr{D}_{Y \to X} \otimes_{\mathscr{D}_X} M \in \mathsf{D}_{\mathrm{qc}}(\mathscr{D}_X)$. Check: $\mathscr{D}_{Y \leftarrow X} \in \mathscr{D}_{\mathrm{qc}}(\mathscr{D}_X^{\mathrm{op}})$
as derived tensor $- \otimes_{\mathcal{O}_X} -$ preserves $\mathsf{D}_{\mathrm{qc}}(\mathcal{O})$, use the "bar" resolution:

$$(\cdots \to \mathscr{D}_{Y \leftarrow X} \otimes_{\mathcal{O}_X} \mathscr{D}_X \otimes_{\mathcal{O}_X} M \to \mathscr{D}_{Y \leftarrow X} \otimes_{\mathcal{O}_X} M) \cong \mathscr{D}_{Y \leftarrow X} \otimes_{\mathscr{D}_X} M$$

(2) Transtivity: For $X \xrightarrow{f} Y \xrightarrow{g} Z$, have $g_+ f_+ \cong (gf)_+$ as functors $\mathsf{D}(\mathscr{D}_X) \to \mathsf{D}(\mathscr{D}_Y)$

Proof. Given $M \in \mathsf{D}(\mathscr{D}_X)$, we have

$$\begin{aligned} g_+(f_+(M)) &= g_+(f_*(\mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}_X} M)) \\ &= g_*(\mathscr{D}_{Z\leftarrow Y}\otimes_{\mathscr{D}_Y} f_*(\mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}_X} M)) \\ \text{(projection formula)} &= g_* f_*\left(f^{-1}\mathscr{D}_{Z\leftarrow Y}\otimes_{f^{-1}\mathscr{D}_Y}\mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}_X} M\right) \\ &= g_* f_*\left(\mathscr{D}_{Z\leftarrow X}\otimes_{\mathscr{D}_X} M\right) \\ &= (gf)_+(M) \end{aligned}$$

□

(3) if f is a closed immersion, then f_+ is t-exact. ($f_+(\mathsf{D}^{[a,b]})\subseteq \mathsf{D}^{[a,b]}$)
Proof. $f_+(M) = f_*(\mathscr{D}_{Y\leftarrow X}\otimes_{\mathscr{D}_X} M)$ where f_* is t-exact. Now use that $\mathscr{D}_{Y\leftarrow X}$ is flat over $\mathscr{D}_X$ (exercise). □

(4) f étale morphism $\Rightarrow f_+ = f_*$
Proof. $\mathscr{D}_{Y\leftarrow X}\cong\mathscr{D}_X$ for f étale. ($\Rightarrow f$ affine + étale $\Rightarrow f_+$ is t-exact). □

10. 10/05/2020

10.0.1. *Pushforward (continued).* Let $f : X \to Y$ map of smooth qcqs k-schemes.

(5) Chomological dimension: f_+ has bounded cohomological dimension. In fact, $f_+(\mathsf{D}^{[a,b]})\subseteq\mathsf{D}^{[a-\dim(X),b+\dim(X)]}$.
Proof. $f_+(M) = f_*(\mathscr{D}_{Y\leftarrow X}\otimes^{\mathsf{L}}_{\mathscr{D}_X} M)$. Use
- $\mathscr{D}_{Y\leftarrow X}$ has Tor dimension $\leqslant \dim(X)$ over $\mathscr{D}_X$.
- f_* has cohomological dimension $\leqslant\dim(X)$.

□

(6) $\mathscr{D}$-coherence for proper f: If f is proper, then $f_+(\mathsf{D}_{\mathbf{Coh}})\subseteq\mathsf{D}_{\mathbf{Coh}}$ (and not otherwise).
Proof for f a closed immersion. May assume $f : X \hookrightarrow Y$ where X, Y are both affine.
Devissage $\Rightarrow$ enough to show $f_+(\mathscr{D}_X)$ is $\mathscr{D}_Y$-coherent. In this case, $f_+(\mathscr{D}_X) = f_*(\mathscr{D}_{Y\leftarrow X}\otimes^{\mathsf{L}}_{\mathscr{D}_X}\mathscr{D}_X) = f_*(\mathscr{D}_{Y\leftarrow X})$.
Use induction on relative dimension to show that $\mathscr{D}_{Y\leftarrow X} = \mathscr{D}_Y/\mathscr{D}_Y\cdot I_{X\subseteq Y}$, where RHS is clearly $\mathscr{D}_Y$-coherent. □

Exercise 10.1. Say $f : X \to Y$ is a proper map. Fix $K\in\mathsf{D}^b_{\mathbf{Coh}}(X)\rightsquigarrow\mathscr{D}_X\otimes^{\mathsf{L}}_{\mathcal{O}_X} K\in\mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_X)$. Show that $f_+(K)$ is $\mathscr{D}$-coherent. (Using coherence of f_*K and projection formula.)

(7) Adjointness for proper maps: If $f : X\to Y$ is proper then $f_+ : \mathsf{D}_{\mathrm{qc}}(\mathscr{D}_X)\to\mathsf{D}_{\mathrm{qc}}(\mathscr{D}_Y)$ is left-adjoint to f^+.
Idea: Pass to right $\mathscr{D}$-modules, and use $\underbrace{f^\dagger}_{\text{right version of } f^+} = f^!$ on $\mathcal{O}$-modules

Exercise 10.2. Check this for closed immersions

Example 10.3 (Smooth Maps)**.** X smooth k-scheme, $f : X\to\operatorname{Spec}(k)$.

Claim. $f_+(\mathcal{O}_X)\cong\mathsf{R}\Gamma_{\mathrm{dR}}(X/k) := \mathsf{R}\Gamma(X,\Omega^\bullet_{X/k})$.

Proof. Since $\mathsf{L}\,\mathbf{Mod}_{\mathscr{D}_k}\cong\mathsf{R}\,\mathbf{Mod}_{\mathscr{D}_k}$ compatible with forgetful functor to $\mathbf{Vect}_k$. It sufficies to show that $f_\dagger(\omega_X) = \mathsf{R}\Gamma_{\mathrm{dR}}(X/k)$.

Recall: we have a de Rham resolution in $\mathsf{R}\,\mathbf{Mod}_{\mathscr{D}_X}$.

$$\mathrm{dR}(\mathscr{D}_X) := (\mathscr{D}_X\xrightarrow{\nabla}\Omega^1_X\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathscr{D}_X\to\cdots\to\omega_X\otimes^{\mathsf{L}}_{\mathcal{O}_X}\mathscr{D}_X)\xrightarrow{\eta}\omega_X$$

where η is the action map.

So

$$
\begin{aligned}
f_+(\omega_X) &= f_*(\omega_X \otimes^{\mathsf{L}}_{\mathscr{D}_X} \mathscr{D}_{X\to \mathrm{pt}}) \\
&= f_*(\omega_X \otimes^{\mathsf{L}}_{\mathscr{D}_X} \mathcal{O}_X) \\
&= f_*(\mathcal{O}_X \xrightarrow{\nabla} \Omega^1_X \to \cdots \to \omega_X) \quad (\nabla = \mathrm{d}_{\mathrm{dR}}) \\
&= \mathsf{R}\Gamma_{\mathrm{dR}}(X/k)[\dim(X)]
\end{aligned}
$$

□

Exercise 10.4. Say $f : X \to Y$ is a smooth map of smooth k-schemes. Then $f_+(\mathcal{O}_X) = f_*(\underbrace{\Omega^\bullet_{X/Y}}_{\text{relative de Rham complex}})$

Example 10.5. Say $\iota : X \hookrightarrow Y$ is a closed immersion of affine schemes such that there exists a fiber square

$$
\begin{array}{ccc}
X & \hookrightarrow & Y \\
\downarrow & & \downarrow \\
V(x) = \mathbb{A}^{d-1} & \hookrightarrow & \mathbb{A}^d
\end{array}
$$

with vertical maps étale. Let $x \in \mathcal{O}(Y)$ be the coordinates vanishing on x, and $\partial = \frac{\partial}{\partial x} \in \mathrm{T}_Y$ the corresponding vector field. Then $\mathscr{D}_{X\to Y} = \mathscr{D}_X[\partial]$ as a left $\mathscr{D}_X$-module. (exercise: use the explicit description of Weyl algebra)

For any $M \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X}$, then

$$\iota_+(M) = \iota_*(\mathscr{D}_{Y\leftarrow X} \otimes^{\mathsf{L}}_{\mathscr{D}_X} M) = k[\partial] \otimes_k M$$

Note $x \in \mathcal{O}(Y)$ acts nontrivially on $\iota_+(M)$, even though it kills M, e.g.

$$
\begin{aligned}
x\cdot(\partial^0 \otimes m) &= \partial^0 \otimes xm = 0 \qquad \text{because } x \cdot M = 0 \\
x \cdot (\partial \otimes m) &= (\partial x - 1)m = -m
\end{aligned}
$$

More generally, $\partial^n x = x\partial^n + n\partial^{n-1}$ tells us that

$$x(\partial^n m) = (\partial^n x - n\partial^{n-1})m = -n\partial^{n-1}m$$

So $\iota_+(M)$ has a filtration $\mathrm{F}_i = k[\partial]_{\leqslant i} \otimes M$ such that $x \cdot F_i \subseteq F_{i-1}$ and $x : \mathrm{gr}_i \cong \mathrm{gr}_{i-1}$ for $i - 1 \geqslant 0$.

Let us compute $\iota^+\iota_+(M)$:

$$
\begin{aligned}
\iota^+(k[\partial]\otimes_k M) &= \iota^*(k[\partial]\otimes_k M)[-1] \\
&= \left(k[\partial]\otimes_k M \underbrace{\xrightarrow{x}}_{\text{surjective by previous arguments}} \underbrace{k[\partial]\otimes_k M}_{\deg 0} \right)[-1] \\
&= ((k\cdot 1 \otimes_k M)[1])[-1] \\
&= M
\end{aligned}
$$

The upshot is that $\iota^+\iota_+(M) \cong M$, unlike $i^*i_*(M) = M \oplus M[1]$.

Theorem 10.6 (Kashiwara)**.** *Say $i : X \to Y$ closed immersion of smooth k-schemes.*

(1) $\mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{X,\mathscr{D}_Y} = \{M \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_Y} \,|\, M$ set-theoretically supported on $X\}$. i_+ and i^+ give naturally inverse equivalences $\mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X} \cong \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{X,\mathscr{D}_Y}$ ($\Rightarrow$ for $M \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{X,\mathscr{D}_Y}$, have $i^+(M) = \mathrm{H}^0(i^+(M))$)

(2) i_+ and i^+ give equivalences $\mathsf{D}^b_{\mathrm{qc}}(\mathscr{D}_X) \cong \mathsf{D}^b_{X,\mathrm{qc}}(\mathscr{D}_Y)$.

Proof. First, observe that $M \underbrace{\cong}_{\text{(unit of adjunction)}} i^+i_+(M)$ for any $M \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X}$ is an isomorphism by previous calculation.

So we get $i_+ : \mathsf{D}^b_{\mathrm{qc}}(\mathscr{D}_X) \to \mathsf{D}^b_{\mathrm{qc},X}(\mathscr{D}_Y)$ is fully faithful with left-inverse i^+ (category theory).

So if we know that i_+ is essentially surjective, it follows that i^+ is t-exact (i.e., for $M \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{X,\mathscr{D}_Y}$, $i^+(M) = \mathrm{H}^0(i^+(M))$).

So it sufficies to check that $i_+ : \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X} \to \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{X,\mathscr{D}_X}$ is essentially surjective.

May reduce to the case where Y is étale over $\mathbb{A}^d$ and $X = V(x) \subseteq Y$, x a coordinate on $\mathbb{A}^d$.

Let $\partial = \frac{\partial}{\partial x} \in \mathrm{T}_Y$ as before, fix $M \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{X,\mathscr{D}_Y}$.

Claim. *$M \cong k[\partial] \otimes_k M^{-1}$ (i.e., $i_+(M^{-1})$) for an $M^{-1} \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X}$.*

□

11. 10/07/2020

Proof. Consider $D = x\partial$ acts on M. Set $M^i = \{m \in M | Dm = im\}$ for each $i \in \mathbb{Z}$. We will show that $M^i = 0, \forall i \geqslant 0$ and $\oplus_{i \leqslant -1} M^i \cong M$.

We have the following properties:

(1) Left multiplication by x gives $x : M^i \to M^{i+1}$. Given $m \in M^i$, we have

$$\begin{aligned} D(xm) &= x\partial(xm) \\ &= x(\partial x m) \\ &= x(x\partial + 1)m \\ &= xDm + xm \\ &= xim + xm = (i+1)xm \\ \Rightarrow xm &\in M^{i+1} \end{aligned}$$

(2) ∂ gives $\partial : M^i \to M^{i+1}$ check this
(3) $x\partial : M^i \to M^i$ is an isomorphism unless $i = 0$: $x\partial = D$ acts by i on M^i.
(4) $\partial x : M^i \to M^i$ is an isomorphism unless $i = -1$: $\partial x = D + 1$ acts by $i + 1$ on M^i.
(5) $x : M^i \to M^{i+1}$ is bijective for $i \neq -1$: Use (3) and (4)
(6) $\partial : M^i \to M^{i-1}$ is bijective for $i \neq 0$: Use (3) and (4).

Note:

- Multiplication by x is locally nilpotent on M because M is supported on X
- Multiplication by ∂ is bijective on $\oplus_{i \geqslant 0} M^i$.

So we get an eigenspace decomposition.

$$\alpha : k[\partial] \otimes_k M^{-1} \underbrace{\cong}_{(6) + \oplus_{i \geqslant 0} M^i = 0} \oplus_i M^i \underbrace{\hookrightarrow}_{\text{injective as } D \text{ acts with distinct eigenvalues on the } M^i\text{s}} M$$

Claim. *α is surjective*

Proof. We show that given $m \in M$, if $x^k \cdot m = 0$, then m comes from $\oplus_{-k \leqslant i \leqslant -1} M^i \overset{\alpha}{\hookrightarrow} M$. For $k = 1$, we have $xm = 0$, want $m \in M^{-1} \Leftrightarrow Dm = -m$. But $Dm = x\partial m = (\partial x - 1)m = -m$.

For general $k \geqslant 1$, use similar (but tricky) algebra. □

To finish, need to show that $i_+i^+(M) \cong M$. Using $\oplus_{i\leqslant -1} M^i \cong M$, we have

$$\begin{aligned} i^+(M) = i^*M[-1] &= (R/(x) \otimes^{\mathsf{L}}_R M)[-1] \\ &= (M \xrightarrow{x} M)[-1] \\ &= (M^{-1}[1])[-1] = M^{-1} \end{aligned}$$

Finally, the decomposition

$$\alpha : k[\partial] \otimes_k M^{-1} \cong M$$

shows that

$$i_+i^+M = i_+M^{-1} \cong M$$

□

Remark 11.1. (1) Given $i : X \hookrightarrow Y$, the equivalence $i_+ : \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X} \cong \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{X,\mathscr{D}_Y}$. (Use that coherent $\mathscr{D}$-modules = noetherian objects of $\mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}}$)

(2) Say $i : X \hookrightarrow Y$ closed immersion of smooth k-schemes, $j : U := Y - X \hookrightarrow Y$ open complement. Then $j_* = j_+$ and $j^* = j^+$ (because $\mathscr{D}_Y|_U = \mathscr{D}_U$), and j_+ is right-adjoint to j^+. For any $M \in \mathsf{D}_{\mathrm{qc}}(\mathscr{D}_Y)$, the natural map $M \to j_*j^+M$ is a map of $\mathscr{D}_Y$-modules.

Set $\underline{\mathsf{R}\Gamma}_X(M) = \mathrm{Cone}(M \to j_*j^+M)[-1] \in \mathsf{D}_{\mathrm{qc}}(\mathscr{D}_Y)$. Then $i_+i^+M \cong \underline{\mathsf{R}\Gamma}_X(M)$. Because

$$\underline{\mathsf{R}\Gamma}_X(M) \to M \to j_+j^*M$$

Apply i^+ and use that $i^+j_* = 0 \Rightarrow i^+\underline{\mathsf{R}\Gamma}_X(M) \cong i^+M$.

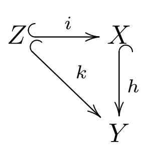
Apply i_+ and use Kashiwara: $\underline{\mathsf{R}\Gamma}_X(M) \cong i_+i^+M$.

11.1. **Applications of Kashiwara's Theorem.**

Corollary 11.2. *Let $i : Z \hookrightarrow X$ be a closed immersion of k-schemes with X smooth. Then $\mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{Z,\mathscr{D}_X}$ is independent of X.*

Proof. We give an argument for X affine. Fix a closed immersion $i : Z \hookrightarrow X$, $k : Z \hookrightarrow Y$ wit X, Y smooth affine.

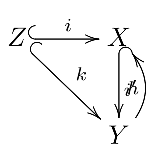
Step 1: Assume that there exists a closed immersion $h : X \hookrightarrow Y$ such that $h \circ i = k$.



Claim. $h_+ : \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{Z,\mathscr{D}_X} \cong \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{Z,\mathscr{D}_Y}$.

Proof. Use Kashiwara and supports matching up under h_+. □

Step 2: Assume $X = \mathbb{A}^n \times Y$, $h : X \to Y$ projection and $h \circ i = k$.



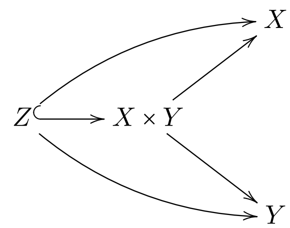
Claim. $h_+ : \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{Z,\mathscr{D}_X} \cong \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{Z,\mathscr{D}_Y}$.

Proof. This is local on Y, so may assume $Y = \mathrm{Spec}(R)$. We may choose $i' : Y \to X$ such that i' extends i and $h \circ i' = \mathrm{id}_Y$. Then $h_+i'_+ = (\mathrm{id}_Y)_+ = \mathrm{id}$. Kashiwara implies that $i'_+ : \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{Z,\mathscr{D}_Y} \cong \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{Z,\mathscr{D}_X}$. So left inverse h_+ is also an equivalence. □

Step 3: If there exists an $h : X \to Y$ such that $h \circ i = k$, then $h_+ : \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{Z,\mathscr{D}_X} \cong \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{Z,\mathscr{D}_Y}$.

Proof. Factor h as $X \hookrightarrow \mathbb{A}^n \times Y \to Y$ and use Step 2 ☐

Step 4: In general, use



and Step 3 for each projection. ☐

Corollary 11.3 (Beilinson-Bernstein)**.** *Let V be a finite dimensional k vector space, $X = \mathbb{P}(V)$. Then $\Gamma(X, -) : \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X} \to \mathbf{Vect}_k$ is exact, where Γ is the nonderived version.*

Proof. Let $U = V - \{0\} \overset{j}{\hookrightarrow} V$. Then $f : U \twoheadrightarrow X = \mathbb{P}(V)$ is a $\mathbb{G}_m$-tensor.

For any $M \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X}$,

$$\Gamma(U, f^*M) = \Gamma(V, (\mathsf{R}j_*)f^*M) = \oplus_{i \in \mathbb{Z}} \Gamma(X, M(i))$$

So it sufficies to show that $M \mapsto \Gamma(U, f^*M)_{\deg 0} = \Gamma(V, j_*f^*M)_{\deg 0}$ is exact.

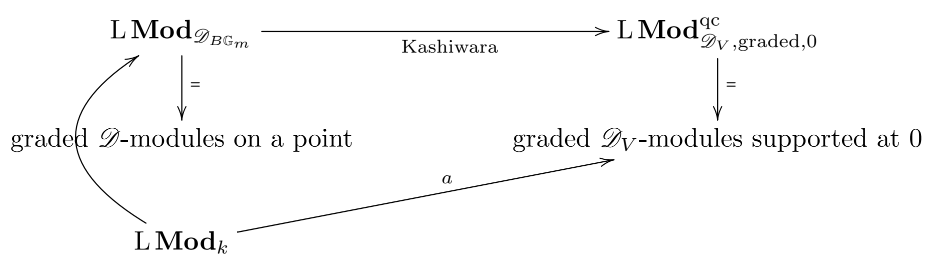
Given $0 \to K \to L \to M \to 0$ on X, we get

$$j_*f^*K \to j_*f^*L \to j_*f^*M$$

is a complex which is exact outside 0.

$\Rightarrow$ its cohomology groups are graded $\mathscr{D}_V$-modules suppored at 0.

So it sufficies to show that all graded $\mathscr{D}_V$-modules supported at 0 have no degree 0 summand. But this is relative easy by Kashiwara:



where the map a is sending k to $\mathrm{H}^d_0(\mathcal{O}_V) = \oplus_{i \in \mathbb{Z}} \mathrm{H}^{d-1}(\mathbb{P}(V), \mathcal{O}(i))$ where right-hand side vanishes in degree 0 or degree $\geqslant -d + 1$. [MZ12] ☐

12. 10/12/2020

12.1. **Application of Kashiwara's Theorem (continued).**

Corollary 12.1. *Let $X = \mathbb{P}(V)$ as above. Then every $M \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X}$ is generated by global sections.*

Proof. Use

$$\begin{array}{ccc} U = V - \{0\} & \overset{j}{\hookrightarrow} & V = \mathrm{Spec}(\mathrm{Sym}(V)) \\ \downarrow f & & \\ X = \mathbb{P}(V) = \mathrm{Proj}(\mathrm{Sym}(V)) & & \end{array}$$

Have $\mathrm{T}_V = \mathcal{O}_V \otimes_k V^\vee$. So we have a canonical element θ, which is the image of $1 \in k$ along

$$k \overset{\text{co-}e_V}{\to} V \otimes V^\vee \subseteq \mathcal{O}_{V \otimes V^\vee} = \mathrm{T}_V \subseteq \mathscr{D}_V.$$

So $\theta \in \Gamma(V, \mathscr{D}_V)$ (Euler operator).

(If $x_1, \dots, x_n$ basis for V, then $\theta = \sum x_i \cdot \frac{\partial}{\partial x_i} \in \mathscr{D}_V$.)

Note that the Euler sequence on $\mathbb{P}(V)$ pulls back (along f) to the sequence

$$0 \to \mathcal{O}_U \overset{\theta}{\to} \mathrm{T}_U \overset{\text{can}}{\to} f^*\mathrm{T}_X \to 0$$

To prove the corollary, it sufficies to show that if $M \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X}$ is nonzero, then $\Gamma(X, M) \neq 0$. (Using extension of $\Gamma(X, -)$).

Using $f : U \to X$, we have $\Gamma(U, f^*M) \cong \oplus_{n \in \mathbb{Z}} \Gamma(X, M(n))$.

This has a $\mathscr{D}_V$-action. Concretely:

- $\omega \in V^\vee$ acts by lowering degree by 1
- θ acts on degree n piece by n.

Observation: $\Gamma(X, M(n)) \neq 0$ for $n \gg 0$ (by projective geometry)

Want to show $\Gamma(X, M) \neq 0$ if $M \neq 0$.

Choose $n \geqslant 0$ such that $\Gamma(X, M(n)) \neq 0$.

If $n = 0$, done

If $n > 0$, then each $w \in V^\vee$ kills $\Gamma(X, M(n))$ by minimality of n. But $\theta \in \mathscr{D}_X \cdot V^\vee$. So θ kills $\Gamma(X, M(n))$. But θ acts by $n \neq 0$ on $\Gamma(X, M(n))$, which is a contradiction! □

Remark 12.2.
(1) Beilinson-Bernstein upgraded this to show
  (a) (THM) $\mathrm{L}\,\mathbf{Mod}^{\mathrm{qc}}_{\mathscr{D}_X} \cong \Gamma(X, \mathscr{D}_X)$-modules, and explicit Lie theoretic description of $\Gamma(X, \mathscr{D}_X) = U(\mathfrak{gl}(V))$/explicit ideal.
  (b) Variant for any $X = G/P$ where G an algebraic group and P parabolic
  (c) Variant with twisted differential operators $\mathscr{D}_{X,\lambda}$ for any chacater $\lambda : T \to \mathbb{G}_m$.
(2) It's conjectured (in characteristic 0) that $\mathscr{D}$-affine smooth projective varieties $\Leftrightarrow G/P$.
(3) Characteristic p: mystery (Larger's 200 paper)

Corollary 12.3. *For X smooth variety over k, $x \in X$ closed point $k = \overline{k}$. Then by Kashiwara, we have*

$$\{\mathscr{D}\text{-modules on } X \text{ support at } x\} \cong \mathrm{L}\,\mathbf{Mod}_k$$
$$i_+(k) = \mathrm{H}^{\dim(X)}_{\{x\}}(\mathcal{O}_X) \leftarrow k$$

Proof. To prove this, we want to show that $i^*(\mathrm{H}^{\dim(X)}_{\{x\}}(\mathcal{O}_X))[-\dim(X)]$ is 1-dimensional (which can be shown explicitly).

$$i^*(\mathrm{H}^{\dim(X)}_{\{x\}}(\mathcal{O}_X))[-\dim(X)] = \mathrm{H}^{\dim(X)}_{\{x\}}(\mathcal{O}_X)[-\dim(X)] \otimes^{\mathsf{L}}_{\mathcal{O}_X} k$$
$$(X \text{ is CM}) = \mathsf{R}\Gamma_{\{x\}}(\mathcal{O}_X) \otimes^{\mathsf{L}}_{\mathcal{O}_X} k$$
$$(\mathrm{Cone}(\mathsf{R}\Gamma_{\{x\}}(\dim X) \to \mathcal{O}_X) = \mathsf{R}i_* \mathcal{O}_{X - \{x\}}) = \mathcal{O}_X \otimes^{\mathsf{L}}_{\mathcal{O}_X} k = k$$

□

Proposition 12.4. *$\mathcal{O}$-coherent $\mathscr{D}_X$-modules M are locally free*

Proof. Since X is reduced, it sufficies to show that $\mathrm{H}^0(i_x^* M)$ has constant rank for any $x \in X^{\mathrm{closed}}$. So we may assume that X is a smooth curve.

By structure theorem of coherent sheaves on X, it sufficies to show that $M_{\mathrm{tors}} = 0$.

Since M_{tors} is supported at finitely many points, we may localize to assume that M_{tors} supported at $\{x\}$. Then $M_{\text{tors}} = \Gamma_x(M) \subseteq M$. So M_{tors} is a $\mathscr{D}_X$-module supported at x and M_{tors} is coherent. By Proposition 12.4, $M_{\text{tors}} = 0$ (because $\mathrm{H}^d_{\{x\}}(\mathcal{O}_X)$ is not finitely generated). □

12.2. **Homological properties of $\mathscr{D}$-modules.**

Theorem 12.5. *Let $(A, \{\mathrm{F}_i(A)\}_{i\geqslant 0})$ be a filtered associated ring which is almost commutative (i.e., $\mathrm{gr}(A)$ is commutative). Assume that $\mathrm{gr}(A)$ is a regular ring of dimension d.*

(1) A is left/right notherian.
(2) Each $M \in \mathrm{L}\,\mathbf{Mod}_A$ has global dimension $\leqslant d$.
(3) $\mathbb{D}(-) := \mathsf{R}\,\mathrm{Hom}_A(-, A) : \mathsf{D}^b_{\mathbf{Coh}}(A) \to \mathsf{D}^b_{\mathbf{Coh}}(A^{\mathrm{op}})$ is an equivalence.

12.2.1. *Good filtrations.*

Definition 12.6. A *filtered A-module M* is a pair $(M, \mathbb{F})$ where $M \in \mathrm{L}\,\mathbf{Mod}_A$ and $F = \{F_i(M)\}_{i\in\mathbb{Z}}$ is a filtration such that

(1) $\mathrm{F}_i M = 0$ for any $i \ll 0$ and $\varinjlim_i \mathrm{F}_i M = M$.
(2) $\mathrm{F}_i(A)\cdot \mathrm{F}_i(M) \subseteq \mathrm{F}_{i+j}(M) \Rightarrow \mathrm{gr}(M)$ is a $\mathrm{gr}(A)$-module.

We call (M, F) a *good filtration* if $\mathrm{gr}(M)$ is finitely generated over $\mathrm{gr}(A)$.

Example 12.7. (1) $A = \mathscr{D}_{\mathbb{A}^1} = k[x, \partial]$ with order filtration $\Rightarrow \mathrm{gr}(A) = k[x, p]$ where p is the class of ∂. $M = k[x, x^{-1}]/k[x] \in \mathrm{L}\,\mathbf{Mod}_A$. Set $\mathrm{F}_n(M) = \frac{1/x^n k[x]}{k[x]} \subseteq M$.

Claim. F *gives a good filtration on M.*

Proof.
- $\mathrm{F}_n(M) = 0$ for any $n < 0$.
- $\cup_n \mathrm{F}_n(M) = M$
- $\partial(\frac{1}{x^n}) = \frac{-n}{x^{n+1}} \Rightarrow \partial\mathrm{F}_n \subseteq \mathrm{F}_{n+1} \Rightarrow (M, \mathrm{F})$ is a filtered A-module.

Check that $\mathrm{gr}(M) = \mathrm{gr}(A)/(x)$ where the map is given by $P\cdot(\frac{1}{x}) \leftarrow\!\shortmid P$. So (M, F) is a good filtration. □

(2) $M = k[x, x^{-1}]$ and $\mathrm{F}_n(M) = \frac{1}{x^n}k[x]$ for any $n \geqslant 0$ and $= 0$ for $n < 0$. Check that F is a good filtration and $\mathrm{gr}(M) = k[x, y]/x\cdot y$.

Proposition 12.8. *Fix an A-module M.*

(1) M is finitely generated $\Leftrightarrow$ M admits a good filtration.
(2) If F and G are good filtrations on M, then there exists $c > 0$ such that $\mathrm{F}_{i-c} \subseteq \mathrm{G}_i \subseteq \mathrm{F}_{i+c}$ for any i.

Proof. (1) Assume that M is finitely generated. Fix generators $m_1, \dots, m_k \in M$ and $a_1, \dots, a_k \in \mathbb{Z}$. Set $\mathrm{F}_n(M) = \sum_{i=1}^k \mathrm{F}_{n-a_i}(A)\cdot M_i \subseteq M$. It is easy to see that $\mathrm{F}_n(M) = 0$ for $n \ll 0$ and $\mathrm{F}_i(A)\cdot\mathrm{F}_j(M) \subseteq \mathrm{F}_{i+j}(M)$. So (M, F) is a filtered A-module. Also, $\mathrm{gr}(M)$ is generated over $\mathrm{gr}(A)$ by the classes of $\overline{m}_i \in \mathrm{gr}_{a_i}(M)$. So F is a good filtration on M.

Conversely, any good filtration (M, F) has this form: Choose $m_1, \dots, m_k \in M$ such that $\{\overline{m}_i \in \mathrm{gr}_{a_i}(M)\}$ is a generating set for $\mathrm{gr}(M)$ over $\mathrm{gr}(A)$. Check that $\mathrm{F}_n(M) = \sum_{i=1}^k \mathrm{F}_{n-a_i}(A)\cdot m_i$ (by induction on n).

(2) left as an exercise. □

So $\mathsf{R}\,\mathrm{Hom}_R(M, \omega_R) \cong \mathsf{R}\,\mathrm{Hom}_R(\mathsf{R}\Gamma_M(M), E)$ for any $M \in \mathsf{D}(R)$.

13.1. **Good filtrations (continued).** Let $(A, \{\mathrm{F}_i(A)\}_{i\geqslant 0}$ be a filtered almost commutative ring, $\mathrm{gr}(A)$ regular of dimension d.

Corollary 13.1. *Let $M \in \mathrm{L}\,\mathbf{Mod}_A^{\mathrm{f.g.}}$. For any good filtration F on M, the set $\mathrm{SS}(M) \coloneqq \mathrm{Supp}(\mathrm{gr}(M))_{\mathrm{red}} \subseteq \mathrm{Spec}(\mathrm{gr}(A))$ is closed, conical (=$\mathbb{G}_m$-equivariant) and depends only on M.*

Proof. Say (M, G) is another good filtration on M. Choose $c > 0$ such that $\mathrm{F}_{i-c} \subseteq \mathrm{G}_i \subseteq \mathrm{F}_{i+c}$ for any i. It sufficies to show that $\mathrm{Supp}(\mathrm{gr}^{\mathrm{F}}(M))_{\mathrm{red}} \supseteq \mathrm{Supp}(\mathrm{gr}^{\mathrm{G}}(M))_{\mathrm{red}}$, which is equivalent to

$$\mathrm{Ann}(\mathrm{gr}^{\mathrm{F}}(M)) \subseteq \sqrt{\mathrm{Ann}(\mathrm{gr}^{\mathrm{G}}(M))}$$

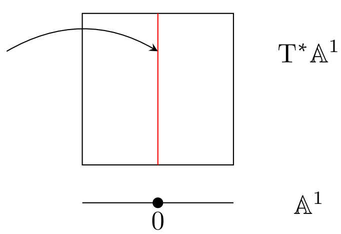
Pick $f \in \mathrm{Ann}_{\mathrm{gr}(A)}(\mathrm{gr}^{\mathrm{F}}(M)) \Rightarrow$ we can lift $\widetilde{f} \in A$ satisfies $\widetilde{f} \cdot \mathrm{F}_i(M) \subseteq \mathrm{F}_{i+1}(M)$ for any i.

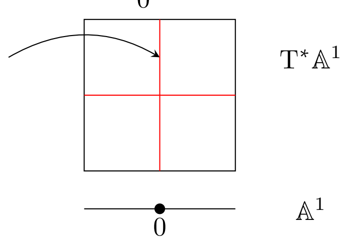
This implies that $\widetilde{f}^{2c+1} \cdot \mathrm{F}_{i+c}(M) \subseteq \mathrm{F}_{i-c-1}(M)$ for any i.

But $\mathrm{F}_{i-c-1}(M) \subseteq \mathrm{G}_{i-1}(M) \subseteq \mathrm{G}_i(M) \subseteq \mathrm{F}_{i+c}(M)$. So we have $\widetilde{f}^{2c+1} \cdot \frac{\mathrm{G}_i(M)}{\mathrm{G}_{i+1}(M)} = 0$ for any i. So $f^{2c+1} \in \mathrm{Ann}(\mathrm{gr}^{\mathrm{F}}(M))$ which implies that $f \in \sqrt{\mathrm{Ann}(\mathrm{gr}^{\mathrm{G}}(M))}$.

As in the lecture, say A is an almost commutative filtered ring with $\mathrm{gr}(A)$ regular of dimension d. Let M be a finitely generated A-module. Fix two good filtrations F and G on M. Our goal is to show that $\mathrm{Supp}(\mathrm{gr}^{\mathrm{F}}(M)) = \mathrm{Supp}(\mathrm{gr}^{\mathrm{G}}(M))$ as Zariski closed subsets of $\mathrm{Spec}(\mathrm{gr}(A))$; equivalently, we must show that the annihilators of $\mathrm{gr}^{\mathrm{F}}(M)$ and $\mathrm{gr}^{\mathrm{G}}(M)$ agree up to radicals in $\mathrm{gr}(A)$. By symmetry, it is enough to show that if $f \in \mathrm{gr}(A)$ annihilates $\mathrm{gr}^{\mathrm{F}}(M)$, then some power of f annihilates $\mathrm{gr}^{\mathrm{G}}(M)$. Moreover, as everything is graded, it is enough to check this for f homogenous, say of degree j; choose a lift $g \in \mathrm{F}_j(A)$ of f. Also, choose $c > 0$ such that $\mathrm{F}_{i-c} \subseteq \mathrm{G}_i \subseteq \mathrm{F}_{i+c}$ as submodules of M. The assumption on f implies that $g * \mathrm{F}_i$ is contained F_{i+j-1} for all i. By induction on k, we have $g^k * \mathrm{F}_i$ is contained F_{i+kj-k} for all $k > 0$ and all i. Using the containment $\mathrm{G}_i \subseteq \mathrm{F}_{i+c}$, this implies $g^k \mathrm{G}_i$ is contained in $\mathrm{F}_{i+c+kj-k}$ for all $k > 0$ and all i. Using the containment that $\mathrm{F}_{i-c} \subseteq \mathrm{G}_i$ for all i, this then implies that $g^k \mathrm{G}_i$ is contained in $\mathrm{G}_{i+2c+kj-k}$ for all $k > 0$ and all i. For $k > 2c$, this gives that $g^k \mathrm{G}_i$ is contained in G_{i+kj-i} for all i. But this implies by passing to the associated graded that f^k (which is homogeneous of degree kj) annihilates $\mathrm{gr}^{\mathrm{G}}(M)$ for $k > 2c$, as wanted. □

Example 13.2. Take $A = \mathscr{D}_{\mathbb{A}^1}$.

- $M = \frac{k[x,x^{-1}]}{k[x]} \rightsquigarrow \mathrm{SS}(M) =$ 

- $M = k[x, x^{-1}] \rightsquigarrow \mathrm{SS}(M) =$ 

- $M = \mathscr{D}_{\mathbb{A}^1}$. Then we can use the order filtration to get $\mathrm{gr}(M) = \mathrm{gr}(A) = \mathrm{T}^*\mathbb{A}^1$.

Remark 13.3 (Characteristic cycles). Say M is a finitely generated A-module. Set

$$\mathrm{Supp}_0(M) = \{\text{minimal points of } \mathrm{SS}(M)\}.$$

We will define $\ell_P(M) \in \mathbb{N}_{\geqslant 0}$ for any $P \in \mathrm{Supp}_0(M)$. We get

$$\mathrm{CC}(M) = \sum_{P \in \mathrm{Supp}_0(M)} \ell_P(M) \cdot [\overline{\{P\}}]$$

where $[\overline{\{P\}}]$ are cycles on $\mathrm{Spec}(\mathrm{gr}(A))$.

To construct it, choose a good filtration F on M and set

$$\ell_P(M) = \mathrm{length}_{\mathrm{gr}(A)_P}(\mathrm{gr}^{\mathrm{F}}(M)_P) \in \mathbb{N}$$

We need to check the well-definedness. Let G be another good filtration. Assume that $\mathrm{F}_i \subseteq \mathrm{G}_i \subseteq \mathrm{F}_{i+1}$ for any i. (Slogan: F and G are "neighbours".)

Consider $\mathrm{F}_i/\mathrm{F}_{i-1} \xrightarrow{\phi_i} \mathrm{G}_i/\mathrm{G}_{i-1}$. Then $\mathrm{Ker}(\phi_i) = \frac{\mathrm{G}_{i-1}}{\mathrm{F}_{i-1}}$ and $\mathrm{Coker}(\phi_i) = \frac{\mathrm{G}_i}{\mathrm{F}_i}$. So we get an exact sequence

$$0 \to \mathrm{Ker}(\phi) \to \mathrm{gr}^{\mathrm{F}}(M) \xrightarrow{\phi} \mathrm{gr}^{\mathrm{G}}(M) \to \mathrm{Coker}(\phi) \to 0$$

and $\mathrm{Ker}(\phi) \cong \mathrm{Coker}(\phi)$ as $\mathrm{gr}(A)$-modules.

Now localize at $P \in \mathrm{Supp}_0(M)$ to get

$$\ell_P(\mathrm{gr}^{\mathrm{F}} M) - \ell_P(\mathrm{gr}^{\mathrm{G}} M) = -\ell_P(\mathrm{Ker}(\phi)) + \ell_P(\mathrm{Coker}(\phi)) = 0.$$

In general, for any $k \in \mathbb{Z}$, define a good filtration $I(k)$ via $I(k)_i = \mathrm{F}_i + \mathrm{G}_{i+k}$ for any i. The we have following observation

- $I(k) = \mathrm{F}$, $\forall k \ll 0$
- $I(k) = \mathrm{G}_{*+k}$, $\forall k \gg 0$
- $I(k)$ and $I(k+1)$ are neighbours for any k.

Remark 13.4. If $0 \to K \to L \to M$ is a SES of finitely generated A-modules, then

$$\mathrm{SS}(L) = \mathrm{SS}(K) \cup \mathrm{SS}(M)$$

Proof. Choose a good filtration F_L on L. It induces good filtrations on M and K as follows

$$\mathrm{F}_{M,i} := \mathrm{Im}(\mathrm{F}_{L,i} \to M) \subseteq M$$
$$\mathrm{F}_{K,i} := \mathrm{F}_{L,i} \cap K \subseteq K$$

Then $0 \to \mathrm{gr}(K) \to \mathrm{gr}(L) \to \mathrm{gr}(M) \to 0$ is a SES. So we have $\mathrm{Supp}(\mathrm{gr}(L)) \subseteq \mathrm{Supp}(\mathrm{gr}(K)) \cup \mathrm{Supp}(\mathrm{gr}(M))$. ☐

This implies that for $M \in \mathsf{D}^b_{\mathbf{Coh}}(A)$, setting $\mathrm{SS}(M) = \cup_i \mathrm{SS}(\mathrm{H}^i(M))$ gives a reasonable notion of support. E.g., if $K \to L \to M \to K[1]$ is an exact triangle, then $\mathrm{SS}(L) \subseteq \mathrm{SS}(K) \cup \mathrm{SS}(M)$ by SES of Coh.

13.2. **Filtered resolutions, global dimension, duality.** Notation: For a filtered A-module N and $j \in \mathbb{Z}$, set

(1) $N(j)$ =filtered A-module defined by $\mathrm{F}_i(N(j)) = \mathrm{F}_{i+j}(N)$. So $A(-j)$ is generated by the class of $1 \in \mathrm{F}_j(A(-j))$.
(2) A map $(M, F) \to N, G)$ of filtered A-modules is *strict* if $\mathrm{Im}(f) \cap \mathrm{G}_i(N) = f(\mathrm{F}_i(M))$ for any i. ($\Leftrightarrow$ the filtration on $\mathrm{Im}(f)$ coming from $M \twoheadrightarrow \mathrm{Im}(f)$ and $\mathrm{Im}(f) \subseteq N$ concide.)

Proposition 13.5. *Say (M, F) is an A-module with a good filtration. Then there exists a strict complex $P^\bullet$ of filtered finite free A-modules in degree $\leqslant 0$ and a strict augmentation $\epsilon : P^\bullet \to M$ such that $\mathrm{gr}(\epsilon) : \mathrm{gr}(P^\bullet) \to \mathrm{gr}(M)$ is a resolution.*

Proof. By strictness, it sufficies to construct a strict filtered finite free resolution $\epsilon : P^\bullet \to (M, \mathrm{F})$.

To construct $\oplus_{j=1}^k A(-a_j) = P^0 \xrightarrow{\epsilon} M$, pick generators $\{m_i \in \mathrm{F}_{a_i}(M)\}_{i=1,\ldots,k}$ as in proof of existence of good filtrations, and define ϵ by $\epsilon(e_i) = m_i$. Since $\mathrm{F}_n(M) = \sum_{i=1}^k \mathrm{F}_{n-a_i}(A) \cdot M_i$, the map ϵ is strict.

The rest of the resolution is constructed inductively by giving $\mathrm{Ker}(\epsilon)$ the indeuced good filtration from P^0. ☐

Theorem 13.6. *Any A-module M has global dimension $\leqslant d$ ($= \dim(\mathrm{gr}(A))$) ($\Leftrightarrow \mathrm{Ext}^i_A(M,-) = 0$ for any $i > d$).*

Proof. Reduce to M being finitely generated [Sta20, Tag 065T].

Say M is a finitely generated A-module. Choose a good filtration F on M and a filtered free resolution $P^\bullet \xrightarrow{\epsilon} (M, \mathrm{F})$ as before. Then $\mathsf{R}\mathrm{Hom}_A(M,N) = \mathsf{R}\mathrm{Hom}_A(P^\bullet, N)$.

Since each P^i is finitely generated, $\mathrm{Hom}_A(P^i,-)$ commutes with filtered colimits, so we may assume N is finitely generated as well.

So it suffices to show that for any finitely generated A-module N, $\mathrm{H}^i(\mathrm{Hom}_A(P^\bullet,N)) = 0$ for any $i > d$.

Since N is finitely genearted, may choose a good filtration G on N, For each i, define a filtration on $\mathrm{Hom}_A(P^i,N)$ by

$$\mathrm{F}_j \mathrm{Hom}_A(P^i,N) = \{f : P^i \to N \mid f(\mathrm{F}_*) \subseteq \mathrm{G}_{*+j}\}$$

Check this gives a filtration $\{\mathrm{F}_i\}$ on $\mathrm{Hom}_A(P^\bullet,N)$. Using that each P^i is filtered free, have

(1) $\mathrm{F}_j \mathrm{Hom}_A(P^i,N) = 0$ for any $j \ll 0$
(2) $\cup_j \mathrm{F}_j \mathrm{Hom}_A(P^\bullet,N) = \mathrm{Hom}_A(P^\bullet,N)$
(3) $\mathrm{gr}^{\mathrm{F}} \mathrm{Hom}_A(P^\bullet,N) \cong \mathrm{Hom}_{\mathrm{gr}(A)}(\mathrm{gr}(P^\bullet),\mathrm{gr}(N))$: an element $f \in \mathrm{F}_j \mathrm{Hom}_A(P^\bullet,N)$ gives a degree j map $\mathrm{gr}^{\mathrm{F}}(P^\bullet) \to \mathrm{gr}(N)$ over $\mathrm{gr}(A)$. So we get a map $\mathrm{gr}^{\mathrm{F}} \mathrm{Hom}_A(P^\bullet,N) \to \mathrm{Hom}_{\mathrm{gr}(A)}(\mathrm{gr}(P^\bullet),\mathrm{gr}(N))$. Check that since $P^\bullet$ is finite free, this map is isomorphism of complexes.

Upshot: have a bounded below, exhaustive filtration on $\mathrm{Hom}_A(P^\bullet,M)$ with gr^{F} given by

$$\mathrm{Hom}_{\mathrm{gr}(A)}(\mathrm{gr}(P^\bullet),\mathrm{gr}(N)) = \mathsf{R}\mathrm{Hom}_{\mathrm{gr}(A)}(\mathrm{gr}(M),\mathrm{gr}(N)) \underbrace{\in}_{\mathrm{gr}(A) \text{ is comm. regular of dim } d} \mathsf{D}^{[0,d]}$$

There exists a filtration on $\mathrm{Ext}^i_A(M,N)$ whose associated graded is a subquotient of $\mathrm{Ext}^i_{\mathrm{gr}(A)}(\mathrm{gr}(M),\mathrm{gr}(N))$, which implies that $\mathrm{Ext}^i_A(M,N) = 0$ for any $i > d$ because the same is true on $\mathrm{gr}(A)$. ☐

14. 10/19/2020

14.1. The codimension filtration.

(A,F) almost commutative filtered, $\mathrm{gr}(A)$ regular of dimension d ($+\varepsilon$).

Definition 14.1. Say M is a finitely generated A. Define $d(M), c(M)$ as follows:

$$d(M) = \dim(\mathrm{SS}(M)) \in [0,d]$$
$$c(M) = \min\{i | \mathrm{Ext}^i_A(M,A) \neq 0\} \in [0,d] \text{ if } M \neq 0$$

Theorem 14.2 (Duality)**.** $\mathbb{D}(-) = \mathsf{R}\mathrm{Hom}_A(-,A) : \mathsf{D}^b_{\mathbf{Coh}}(A) \to \mathsf{D}^b_{\mathbf{Coh}}(A^{\mathrm{op}})$ *is an equivalence with inverse* $\mathbb{D}'(-) = \mathsf{R}\mathrm{Hom}_{A^{\mathrm{op}}}(-,A^{\mathrm{op}})$.

Proof. Have a natural map: for $M \in \mathsf{D}^b_{\mathbf{Coh}}(A)$, we have $\eta_M : M \to \mathbb{D}'(\mathbb{D}(M))$. Let

$$\mathcal{C} = \{M \in \mathsf{D}^b_{\mathbf{Coh}}(A) | \eta_M \text{ is an isomorphism}\} \subseteq \mathsf{D}^b_{\mathbf{Coh}}(A).$$

Then

(1) $\mathcal{C}$ is a triangulated subcategory of $\mathsf{D}^b_{\mathbf{Coh}}(A)$.
(2) $\mathcal{C}$ is closed under direct summands.
(3) $A \in \mathcal{C}$.

This implies $\mathcal{C} = \mathsf{D}^b_{\mathbf{Coh}}(A)$: As $\mathcal{C} \subseteq \mathsf{D}^b_{\mathbf{Coh}}(A)$ is full, triangulated subcategory, it suffices to show that $M \in \mathcal{C}$ for any $M \in \mathrm{L}\,\mathbf{Mod}_A^{\mathrm{f.g.}}$.

As any such M has a finite resolution by finite projective A-modules, we may assume M is finite projective. By stability under direct summands, may assume M is finite free. But $A \in \mathcal{C} \Rightarrow M \in \mathcal{C}$. □

Corollary 14.3. *If $M \in \mathsf{D}^b_{\mathbf{Coh}}(A)$, then $\mathrm{SS}(M) = \mathrm{SS}(\mathbb{D}(M))$.*

Proof. By symmetry + duality, it suffices to show that $\mathrm{SS}(\mathbb{D}(M)) \subseteq \mathrm{SS}(M)$. Assume for simplicity that M is concentrated in degree 0.

Then $\mathrm{SS}(\mathbb{D}M) = \cup_i \mathrm{SS}(\mathrm{Ext}^i_A(M,A))$. But each $\mathrm{Ext}^i_A(M,A)$ has a good filtration whose associated graded is a subquotient of $\mathrm{Ext}^i_{\mathrm{gr}(A)}(\mathrm{gr}(M),\mathrm{gr}(A))$. Thus

$$\mathrm{SS}(\mathrm{Ext}^i_A(M,A)) \subseteq \mathrm{Supp}\left(\mathrm{Ext}^i_{\mathrm{gr}(A)}(\mathrm{gr}(M),\mathrm{gr}(A))\right) \subseteq \mathrm{Supp}(\mathrm{gr}(M)) =: \mathrm{SS}(M).$$

□

Theorem 14.4 (Gabber)**.** *Say M is a finitely generated A-module. For $0 \leqslant s \leqslant d$, write $C^s(M) = \sum_{N \subseteq M, c(N) \geqslant s} N \subseteq M$, we have*

$$M = C^0(M) \supseteq C^1(M) \supseteq \cdots \supseteq C^d(M) \supseteq C^{d+1}(M) = 0$$

Then $\mathrm{SS}(C^s(M)/C^{s+1}(M))$ is equidimensional of dimension $d - s$.

Lemma 14.5. *Say M is a finitely generated A-module, $M \neq 0$. Then*

(1) $d(M) + c(M) = d$.
(2) $d(\mathrm{Ext}^i_A(M,A)) \leqslant d - i$.
(3) $d(\mathrm{Ext}^{c(M)}_A(M,A)) = d(M)$.

In particular, $c(\mathrm{Ext}^i_A(M,A)) \geqslant i, \forall i$.

Proof for A commutative. First, prove (2): Fix $x \in \mathrm{Supp}(\mathrm{Ext}^i_A(M,A))$ generic. Then $\mathrm{Ext}^i_A(M,A)_x \neq 0 \Rightarrow \mathrm{Ext}^i_{A_x}(M_x, A_x) \neq 0$.

$$i \underbrace{\leqslant}_{\mathrm{gl}\dim(A_x) \leqslant \dim(A_x)} \dim(A_x) = d - \dim(\overline{\{x\}})$$

which implies that $\dim(\overline{\{x\}}) \leqslant d - i \Rightarrow \dim(\mathrm{Supp}(\mathrm{Ext}^i_A(M,A))) \leqslant d - i$, which is $d(\mathrm{Ext}^i_A(M,A)) \leqslant d - i$.

Next we show that $d(\mathrm{Ext}^{c(M)}_A(M,A) = d - c(M)$.

- $\leqslant$ comes from (2)
- $\geqslant$ $\mathrm{Ext}^{c(M)}_A(M,A)$ is the smallest nonzero Cohomology group of $K = \mathsf{R}\mathrm{Hom}_A(M,A)$, and $\mathrm{Tor\,Amp}(K) \in [0,d]$.

Assuming Lemma 14.6, we learn for all associated primes P of $\mathrm{Ext}^{c(M)}_A(M,A)$, we have $\dim(A/p) \geqslant d - c(M)$. In particular, we learn $d(\mathrm{Ext}^{c(M)}_A(M,A)) \geqslant d \cdot c(M)$. Combining this with (2) + definition of $c(-)$, we get

$$d(M) = d(\mathbb{D}(M)) = \max_i d(\mathrm{Ext}^i_A(M,A)) = d - c(M).$$

So $d(M) = d - c(M)$, giving (1) and then also (3). □

Lemma 14.6. *Let R be a regular ring, $K \in \mathsf{D}^b_{\mathbf{Coh}}(R)$ with $\mathrm{Tor\,Amp}(K) \in (0,\infty)$. Let $j = \min\{i | \mathrm{H}^i(K) \neq 0\}$. Then all associated primes P of $\mathrm{H}^j(K)$ satisfy $\dim(R/P) \geqslant d \cdot j$.*

Proof. Say $M \in \mathrm{L}\,\mathbf{Mod}_A^{\mathrm{f.g.}}$ with good filtration F. Then $d(M) = \dim(\mathrm{SS}(M)) = d(\mathrm{gr}(M))$. Let us prove the same for $c(-)$.

Since $\mathrm{Ext}^i_A(M,A)$ has a good filtration with associated graded being a subquotient of $\mathrm{Ext}^i_{\mathrm{gr}(A)}(\mathrm{gr}(M),\mathrm{gr}(A))$, get $c(M) \geqslant c(\mathrm{gr}(M))$.

For the other direction, assume towards contradiction that $c(M) > c(\mathrm{gr}(M))$. Then

$$d(M) = d(\mathbb{D}M) = \max_i d(\mathrm{Ext}^i_A(M,A)) = \max_{i > c(\mathrm{gr}(M))} d(\mathrm{Ext}^i_A(M,A)) < d - c(\mathrm{gr}(M)).$$

But $d(M) = d(\mathrm{gr}(M)) = d - c(\mathrm{gr}(M))$. So we get a contradiction, whence $c(M) = c(\mathrm{gr}(M))$. □

Remark 14.7. For a short exact sequence $0 \to M \to N \to K \to 0$ of finitely generated A-modules, we have

$$d(N) = \max(d(M), d(K)) \text{ and}$$
$$c(N) = \min(c(M), c(K))$$

In particular, $C^s(M) = \sum_{N \subseteq M, c(N) \geqslant s} N \subseteq M$ itself has $c(C^s(M)) \geqslant s$.

14.2. **Truncations.** Let R be an associated ring $K \in \mathsf{D}(R)$, $n \in \mathbb{Z}$. Then there exists a universal map $\tau^{\leqslant n}K \xrightarrow{\alpha} K$ such that

(1) $\mathrm{H}^i(\tau^{\leqslant n}K) = 0$ for any $i > n$
(2) $\mathrm{H}^i(\alpha)$ isomorphism for any $i \leqslant n$.

Construction: If K is represented by $K^\bullet$, then $\tau^{\leqslant n}K$ is given by

$$\begin{array}{ccccccccccccc}
(\tau^{\leqslant n}K)^\bullet & = & (\cdots & \to & K^{n+2} & \to & K^{n+1} & \to & \mathrm{Ker}(\mathrm{d}: K^n \to K^{n+1}) & \to & 0 & \to & 0 \to \cdots) \\
 & & & & \downarrow\subseteq & & \downarrow\subseteq & & \downarrow\subseteq & & \downarrow\subseteq & & \downarrow\subseteq \\
K^\bullet & = & (\cdots & \to & K^{n+2} & \to & K^{n+1} & \to & K^n & \to & 0 & \to & 0 \to \cdots)
\end{array}$$

As n varies, we get a "filtration" of $K : \{\cdots \to \tau^{\leqslant n}K \to \tau^{\leqslant n+1}K \to \tau^{\leqslant n+2}K \to \cdots\}$.

14.3. **Sato-Kashiwara filtration.** For a finitely generated A-module M, the dual $\mathbb{D}(M)$ carries a truncation filtration $\{\tau^{\geqslant s}\mathbb{D}(M)\}_{s \in \mathbb{Z}}$, which induces a filtration on $\mathbb{D}'(\mathbb{D}(M)) = M$. So we get an actual filtration on M.

Explicitly,

$$S^s(M) = \mathrm{Im}(\mathrm{Ext}^0_{A^{\mathrm{op}}}(\tau^{\geqslant s}\mathbb{D}(M), A^{\mathrm{op}}) \to \mathrm{Ext}^0_{A^{\mathrm{op}}}(\mathbb{D}M, A^{\mathrm{op}}) = M) \subseteq M.$$

Since $\mathbb{D}(M) \in \mathsf{D}^{[0,d]}$, we have

$$M = S^0(M) \supseteq S^1(M) \supseteq \cdots \supseteq S^d(M) \supseteq S^{d+1}(M) = 0$$

Proposition 14.8. *$S^s(M) = C^s(M)$ for any s. In particular, $C^s(M)/C^{s+1}(M) \subseteq \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M,A), A^{\mathrm{op}})$.*

15. 10/21/2020

Proof of Proposition. Step 1: The map $\mathrm{Ext}^0_{A^{\mathrm{op}}}(\tau^{\geqslant s+1}\mathbb{D}M, A^{\mathrm{op}}) \to \mathrm{Ext}^0_{A^{\mathrm{op}}}(\tau^{\geqslant s}\mathbb{D}M, A^{\mathrm{op}})$ is injective for any s, which implies that $\mathrm{Ext}^0_{A^{\mathrm{op}}}(\tau^{\geqslant s}\mathbb{D}M, A^{\mathrm{op}}) = S^s(M)$.

Proof. Have a triangle

$$\mathrm{Ext}^s_A(M,A)[-s] \to \tau^{\geqslant s}\mathbb{D}M \to \tau^{\geqslant s+1}\mathbb{D}M \to \mathrm{Ext}^S_A(M,A)[-s+1]$$

So it sufficies to show that $\mathrm{Ext}^0_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M,A)[-s+1], A^{\mathrm{op}}) = 0$. The left-hand side is $\mathrm{Ext}^{s-1}_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M,A), A^{\mathrm{op}})$. But $c(\mathrm{Ext}^s_A(M,A)) \geqslant s$ by codim lemma. So LHS=0. □

Step 2: Same analysis shows that there exists a short exact sequence

$$0 \to S^{s+1}(M) \to S^s(M) \to \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A)$$

By desceding induction on s and $c(\mathrm{Ext}^s_{A^{\mathrm{op}}}(-, A)) \geqslant s$, we learn that $c(S^s(M)) \geqslant s \Rightarrow S^s(M) \subseteq C^s(M)$.

Step 3: Want to show $C^s(M) \subseteq S^s(M)$.

Known:

$$\begin{aligned} & c(C^s(M)) \geqslant s \\ \Rightarrow & \mathbb{D}(C^s(M)) \cong \tau^{\geqslant s}\mathbb{D}(C^s(M)) \\ \Rightarrow & S^s(C^s(M)) = C^s(M) \\ \overset{\text{functoriality}}{\Rightarrow} & \underbrace{S^s(C^s(M))}_{=C^s(M)} \subseteq S^s(M) \end{aligned}$$

□

Theorem 15.1 (Gabber)**.** *Say M is a finitely generated A-module. Then $\mathrm{SS}(C^s(M)/C^{s+1}(M))$ is equidimensional of dimension $d - s$ and has no associated points.*

Proof. Step 1: Each $C^s(M)/C^{s+1}(M) =: N$ has pure codimension s, i.e., $C^k(N)/C^{k+1}(N) = 0$ if $k \neq s$ and N if $k = s$. (exercise).

Step 2: Any M that has pure codimension s satisfies $\mathrm{SS}(M)$ is equidimension of dim $d-s$, without assocaited points.

Proof. Say $N = \mathrm{Ext}^s_A(M, A) \Rightarrow \mathrm{SS}(N) \subseteq \mathrm{SS}(M)$. On the other hand, we have

$$M = \frac{S^s(M)}{S^{s+1}(M)} \subseteq \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A^{\mathrm{op}}) = \mathrm{Ext}^s_{A^{\mathrm{op}}}(N, A^{\mathrm{op}}).$$

So $\mathrm{SS}(M) \subseteq \mathrm{SS}(N)$, so $\mathrm{SS}(M) = \mathrm{SS}(N)$. □

Step 3: Choose a good filtration on M, then

$$\mathrm{SS}(N) \subseteq \mathrm{Supp}(\mathrm{Ext}^s_{\mathrm{gr}(A)}(\mathrm{gr}(M), \mathrm{gr}(A))) \subseteq \mathrm{Supp}(\mathrm{gr}(M)) = \mathrm{SS}(M)$$

So $\mathrm{SS}(N) = \mathrm{Supp}(\mathrm{Ext}^s_{\mathrm{gr}(A)}(\mathrm{gr}(M), \mathrm{gr}(A))$ with $s = c(M) = c(\mathrm{gr}(M))$.

Now conclude via the general fact: For R regular ring, $K \in \mathrm{L}\,\mathbf{Mod}^{\mathrm{f.g.}}_R$, $c = c(K)$. Then $\mathrm{Supp}(\mathrm{Ext}^c_R(K, R))$ is equidimensional of dimension $\dim(R) - c$ and has no associated points. □

15.1. **Holonomic $\mathscr{D}$-modules.** Let k be a characteristic 0 field. For X/k smooth variety, $\mathscr{D}_X = \mathscr{D}_{X/k}$, we have $f : \mathrm{T}^*X \to X$ structure map where $\mathrm{T}^*(X) = \underline{\mathrm{Spec}}_X(\mathrm{Sym}(\mathrm{T}_X))$.

Notation: For $Z \subseteq X$ locally closed, $\mathrm{T}^*_Z X = (\mathrm{T}^*X)|_Z$. (might change)

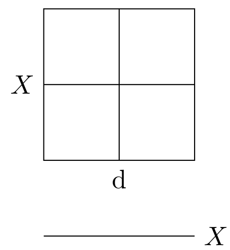
Lemma 15.2. *For $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$. Then M admits a good filtration F and any two such filtrations are commensurate: $\mathrm{SS}(M) \coloneqq \mathrm{Supp}(\mathrm{gr}^{\mathrm{F}}(M))_{\mathrm{red}} \subseteq \mathrm{T}^*X$ is well-defined.*

Proof. Choose $N \subseteq M$ such that N is $\mathcal{O}$-coherent and $\mathscr{D}_X \cdot N = M$ (ex: $M = \cup_i N_i$, N_i coherent over $\mathcal{O}$ and take $N = N_i$ for $i \gg 0$). Then $\mathrm{F}_n(M) \coloneqq \mathrm{F}_n(\mathscr{D}_X) \cdot N \subseteq M$. Check: thsi is a good filtration. □

Remark 15.3. If (M, F) is a good filtration, then $\mathrm{gr}^{\mathrm{F}}(M) \in \mathbf{Coh}(\mathrm{T}^*X)$, i.e., (Higgs sheaf (depends on F)) $\mathrm{gr}^{\mathrm{F}}(M) \in \mathbf{QCoh}(X)$ + $\mathrm{gr}^{\mathrm{F}}(M) \xrightarrow{\theta} \mathrm{gr}^{\mathrm{F}}(M) \otimes \Omega^1_X$ is $\mathcal{O}_X$-linear satisfying $\theta \wedge \theta = 0 : \mathrm{gr}^{\mathrm{F}}(M) \to \mathrm{gr}^{\mathrm{F}}(M) \otimes \Omega^2_X$.

15.2. **Topological properties of** SS**.** For $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$,

(1) $\mathrm{SS}(M) \subseteq \mathrm{T}^*X$ is conical closed (closed under scaling on T^*X)



Claim. *$f : \mathrm{T}^*X \to X$, $f(\mathrm{SS}(M))$ is closed and in fact $\mathrm{SS}(M) \cap X = f(\mathrm{SS}(M))$ where $X \subseteq \mathrm{T}^*X$ via 0-section.*

Proof. $\subseteq$ clear

$\supseteq$ Fix $y \in f(\mathrm{SS}(M))$, so there exists $x \in \mathrm{SS}(M) \cap f^{-1}(y)$. Now scale x to obtain $y = \varinjlim_{t\to\infty} tx \in \overline{\mathrm{SS}(M)} = \mathrm{SS}(\mathscr{D})$. □

(2) For $U \subseteq X$ open, $\mathrm{SS}(M)|_{(\mathrm{T}^*X)_U} = \mathrm{SS}(M|_U)$.

(3) $f(\mathrm{SS}(M)) = \{x \in X | M_x \neq 0\}$ (So RHS is closed).

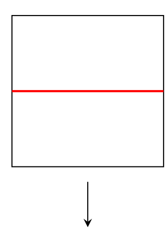
Example 15.4. (1) $M = \mathscr{D}_X \Rightarrow \mathrm{SS}(M) = \mathrm{T}^*X$

(2)

Claim. $\mathrm{SS}(M) = X \Leftrightarrow M$ *is $\mathcal{O}$-coherent.*

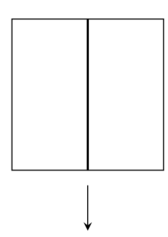
Proof. $\Leftarrow$ M is locally free over $\mathcal{O}$. Use $\mathrm{F}_0 = M, \mathrm{F}_{-1} = 0$ as a good filtration on $M \rightsquigarrow \mathrm{SS}(M) = \mathrm{Supp}(M) = X$.

So say $\mathrm{SS}(M) = X$. So if F is a good filtration then $\mathrm{gr}(M) := \oplus_i \mathrm{gr}_i(M) \in \mathbf{Coh}(\mathrm{T}^*X)$ is set-theoretically supported on $X \subseteq \mathbf{Coh}(\mathrm{T}^*X)$. $\mathrm{gr}(M)$ is coherent over an infinitestimal neighbourhood of $X \subseteq \mathrm{T}^*X$. So $f_*(\mathrm{gr}(M))$ is coherent over X. So $\mathrm{gr}_i(M) = 0$ for any $i \gg 0$.

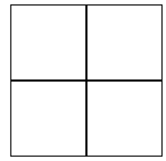


But each is $\mathrm{F}_i(M)$ is $\mathcal{O}$-coherent, so M is also $\mathcal{O}$-coherent. □

(3) $X = \mathbb{A}^1$, $M = \frac{k[x,x^{-1}]}{k[x]}$ gives $\mathrm{SS}(M) = (\mathrm{T}^*X) \times_X \{0\}$



(4) For $X = \mathbb{A}^1$, $j : X \smallsetminus \{0\} = U \hookrightarrow X$, $M = j_* \mathcal{O}_U$. Then

$\mathrm{SS}(M)$=(x-axis)∪(y-axis)⊆ 

Exercise: $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_{\mathbb{A}^1}}$ such that $M|_U$ is a vector bundle. $\mathrm{SS}(M) = X \cup (\mathrm{T}^*X)_0$ unless M is $\mathcal{O}$-coherent.

Theorem 15.5 (Bernstein)**.** *If $0 \neq M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$, then every irreducible component of $\mathrm{SS}(M)$ has dimension $\geqslant \dim(X)$ $(= \frac{1}{2}\dim \mathrm{T}^*X)$.*

Definition 15.6. M is *holonomic* if $\dim(\mathrm{SS}(M)) = \dim(X)$.

Lemma 15.7. *Let $i : Z \hookrightarrow X$ is a closed immersion of smooth varieties. Let $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_Z} \rightsquigarrow i_+(M) \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$.*

Consider the diagram

$$\begin{array}{ccccc} \mathrm{T}^*Z & \xleftarrow{a} & (\mathrm{T}^*X)|_Z & \xhookrightarrow{b} & \mathrm{T}^*X \\ \downarrow & & \downarrow & & \downarrow \\ Z & = & Z & \hookrightarrow & X \end{array}$$

where Z is $\mathrm{SS}(i_+(M)) = ba^{-1}(\mathrm{SS}(M))$, a gives vector bundle of rank $\dim X - \dim Z$, and b is a closed immersion.

16. 10/26/2020

16.1. Bernstein's inequality and holonomic $\mathscr{D}$-modules.

Lemma 16.1. *For $i : Z \hookrightarrow X$ closed immersion of smooth varieties. Have*

$$\begin{array}{ccccc} \mathrm{T}^*Z & \xleftarrow{a} & (\mathrm{T}^*X)|_Z & \xhookrightarrow{b} & \mathrm{T}^*X \\ \downarrow & & \downarrow & & \downarrow \\ Z & = & Z & \hookrightarrow & X \end{array}$$

For $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_Z}$, $\mathrm{SS}(i_+M) = b(a^{-1}(\mathrm{SS}(M)))$.

Proof. It sufficies to show that $M \in \mathrm{R}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_Z}$, then $\mathrm{SS}(i_+(M)) = b(a^{-1}(\mathrm{SS}(M)))$.

Recall: $i_+(M) = M \otimes^{\mathsf{L}}_{\mathscr{D}_Z} \mathscr{D}_Z \to X$ where $\mathscr{D}_{Z\to X} = i^*(\mathscr{D}_X) = \mathscr{D}_X / I_Z \cdot \mathscr{D}_X$.

So we have a surjection $\mathscr{D}_X \to \mathscr{D}_{Z\to X} = \mathscr{D}_X / I_Z \mathscr{D}_X$ of right $\mathscr{D}_X$-modules.

Choose a good filtration G on M. Use the filtration $\mathrm{F}_i(\mathscr{D}_{Z\to X}) \coloneqq \mathrm{Im}(\mathrm{F}_i \mathscr{D}_X \to \mathscr{D}_{Z\to X})$ as a good filtration of the right $\mathscr{D}_X$-module $\mathscr{D}_{Z\to X}$.

Then we have

(1) $\mathrm{F}_\bullet$ gives a good filtration of $\mathscr{D}_{Z\to X} \in \mathrm{R}\,\mathbf{Mod}_{\mathscr{D}_X}$.
(2) $\mathrm{F}_\bullet$ gives a filration of $\mathscr{D}_{Z\to X} \in \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_X}$: this follows by transfering the "order filtration of $\mathscr{D}_X$ is a good filtration in $\mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_X}$" across the surjection $\mathscr{D}_X \to \mathscr{D}_{Z\to X}$.
(3) $\mathrm{gr}^{\mathrm{F}}(\mathscr{D}_{Z\to X}) = \frac{\mathrm{gr}(\mathscr{D}_X)}{I_Z \cdot \mathrm{gr}(\mathscr{D}_X)} = \frac{\mathrm{Sym}^*_{\mathcal{O}_X}(\mathrm{T}_X)}{I_Z \cdot \mathrm{Sym}^*_{\mathcal{O}_X}(\mathrm{T}_X)} = \mathrm{Sym}^*_{\mathcal{O}_Z}(\mathrm{T}_X|_Z)$.

Now, using the chosen good filtration G on M, get a $\otimes$-product filtration on $i_+(M) = M \otimes^{\mathsf{L}}_{\mathscr{D}_Z} \mathscr{D}_{Z\to X}$.

As passing to gr is a $\otimes$-functor, we get

$$\mathrm{gr}(i_+(M)) = \mathrm{gr}(M) \otimes^{\mathsf{L}}_{\mathrm{gr}(\mathscr{D}_Z)} \mathrm{gr}(\mathscr{D}_{Z\to X}) = \mathrm{gr}(M) \otimes^{\mathsf{L}}_{\mathrm{Sym}^*_{\mathscr{O}_Z}(\mathrm{T}_Z)} \mathrm{Sym}^*_{\mathscr{O}_Z}(\mathrm{T}_X|_Z) = b_* a^* \,\mathrm{gr}(M)$$

where $\mathrm{T}^*Z \xleftarrow{a} (\mathrm{T}^*X)|_Z \xrightarrow{b} \mathrm{T}^*X$.

$$\Rightarrow \mathrm{SS}(i_+M) = \mathrm{Supp}(\mathrm{gr}(i_+M)) = \mathrm{Supp}(b_* a^* \,\mathrm{gr}(M)) = ba^{-1}\mathrm{SS}(M)$$

□

Theorem 16.2 (Bernstein's inequality)**.** *For X smooth over k, $0 \neq M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}} \Rightarrow \dim(\mathrm{SS}(M)) \geqslant \dim(X)$. (In fact, each irreducible component of $\mathrm{SS}(M)$ has $\dim \geqslant \dim(X)$.)*

Proof. Using pure dimensionality of $\mathrm{SS}(\mathrm{gr}^\bullet_{\mathrm{C}}(M))$ from last time where C is the codimension filtration.

It is enough to show that $\dim(\mathrm{SS}(M)) \geqslant \dim(X)$, we proceed by induction on $\dim(X)$ for X smooth.

- $\dim(X) = 0$: easy
- Assume the statement for all smaller dimensional varieties. If $f(\mathrm{SS}(M)) = X$, the claim is clear. So we may assume that $Z = f(\mathrm{SS}(M)) \subseteq X$ has $\dim(Z) < \dim(X)$.
  Now $M|_{X-Z} = 0$: $f(\mathrm{SS}(M)) = \{x \in X | M_x \neq 0\}$.
  Replace X with a small open meeting Z in Z_{sm} densely, may assume Z smooth (using that $\mathrm{SS}(M) \twoheadrightarrow Z$).
  Kashiwara$\Rightarrow$ $M = i_+(N)$, where $N = i^+(M), i : Z \hookrightarrow X$. M coherent over $\mathscr{D}_X \Rightarrow N$ is coherent over $\mathscr{D}_Z$.
  By induction on dimension, $\dim(\mathrm{SS}(M)) = \dim(\mathrm{SS}(N)) + \dim(X) - \dim(Z) \geqslant \dim(X)$.

□

Corollary 16.3. *For $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$, we have $\mathbb{D}(M) \in \mathsf{D}^{[0,\dim(X)]}$ where $\mathbb{D}(-) = \underline{\mathsf{R}\,\mathrm{Hom}}_{\mathscr{D}_X}(-, \mathscr{D}_X)$.*

Proof. Want to show that $\mathrm{Ext}^i_{\mathscr{D}_X}(M, \mathscr{D}_X) = 0$ for any $i > \dim(X)$.

Write N for this Ext^i. Assume for contradiction that $N \neq 0$. Since N is a coherent $\mathscr{D}_X$-module.

$$\text{Bernstein} \Rightarrow \dim(N) \geqslant \dim(X)$$
$$\text{Codimension lemmas} \Rightarrow \dim(N) \leqslant 2\dim(X) - i < \dim(X)$$

So this is a contradiction! □

Remark 16.4 (Gabber's inductivity theorem - Ginzburgs' notes)*.* Say X/k smooth, $Y = \mathrm{T}^*X$. Note/recall: Y carries a canonical non-degenerate symplectic 2-form $\omega_{\mathrm{can}} \in \bigwedge^2 \Omega^1_Y$: If $x_1, \ldots, x_n$ are étale coordinates on X, then $\omega = \sum_{i=1}^n \mathrm{d}(x_i) \wedge \mathrm{d}(\frac{\partial}{\partial x_i})$. So for any $y \in Y$, the space $\mathrm{T}_{Y,y}$ is a symplectic vector space.

Theorem 16.5 (Gabber)**.** *Say $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$. Choose $y \in \mathrm{SS}(M)_{\mathrm{sm}}$. Then $\mathrm{T}_{\mathrm{SS}(M),y} \subseteq \mathrm{T}_{Y,y}$ is involutive (i.e., it contains $(\mathrm{T}_{\mathrm{SS}(M),y})^\perp$).*

So $\dim(\mathrm{T}_{\mathrm{SS}(M),y}) \geqslant \dim(\mathrm{T}_{Y,y}) - \dim(\mathrm{T}_{\mathrm{SS}(M),y})$, which implies that $2\dim(\mathrm{T}_{\mathrm{SS}(M),y}) \geqslant 2\dim(X)$. So we get Bernstein.

16.2. **Holonomic $\mathscr{D}$-modules.**

Definition 16.6. $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$ is *holonomic* if $\dim(\mathrm{SS}(M)) = \dim(X)$ or $M = 0$.

1. We have $\mathrm{Hol}_X \subseteq \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$.
2. $M \in \mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_X)$ is *holonomic complex* if each $\mathrm{H}^i(M) \in \mathrm{Hol}_X$. So $\mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_X) \subseteq \mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_X)$.

Example 16.7. For $i: Z \hookrightarrow X$ closed immersionof smooth varieties. For $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_Z}$. Then M holonomic $\Leftrightarrow i_+(M)$ is holonomic (first lemma today).

For X smooth variety, $x \in X$ closed point.

$$i_+(k(x)) = \mathrm{H}^{\dim(X)}_{\{x\}}(\mathcal{O}_X) \in \mathrm{Hol}_X$$

Proposition 16.8. *(1) Subquotients and extensions of holonomic $\mathscr{D}_X$-moduels are holonomic. So $\mathrm{Hol}_X \subseteq \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$ is an abelian Serre subcategory, and $\mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_X)$ is a triangulated subcategory of $\mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_X)$.*

(2) If $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$, then $\mathrm{H}^{\dim(X)}\mathbb{D}(M) = \underline{\mathrm{Ext}}^{\dim(X)}_{\mathscr{D}_X}(M, \mathscr{D}_X) \in \mathrm{Hol}_X$.

(3) If M is holonomic, then $c(M) = \dim(X)$ and $\mathbb{D}(M) = \underline{\mathrm{Ext}}^{\dim}_{\mathscr{D}_X}(M, \mathscr{D}_X)[-\dim X]$ is concentrated in a single degree.

(4) M holonomic $\Leftrightarrow \mathbb{D}(M)$ concentrated in cohomological degree $\dim(X)$.

(5) $M \mapsto \mathbb{D}(M)[\dim X]$ gives an autoduality of Hol_X.

(6) Hol_X is noetherian and artinian. So every object has finite length.

Proof. (1) This follows from properties of $d(-)$ under exact sequences (because holonomic $\Leftrightarrow d(-)$ is minimal).

(2) We know $d(\underline{\mathrm{Ext}}^{\dim(X)}_{\mathscr{D}_X}(M, \mathscr{D}_X)) \leqslant 2\dim(X) - \dim(X) = \dim(X)$. So by Bernstein, $d(\underline{\mathrm{Ext}}^{\dim(X)}_{\mathscr{D}_X}(M, \mathscr{D}_X)) = \dim(X)$. Thus $\underline{\mathrm{Ext}}^{\dim(X)}_{\mathscr{D}_X}(M, \mathscr{D}_X)$ is holonomic.

(3) M holonomic. Want to show that $\mathbb{D}(M)$ is canonical in degree $\dim(X)$. We know that $\mathbb{D}(M) \in \mathsf{D}^{[0,\dim(X)]}$ (corollary of Bernstein). So it sufficies to show

$$\begin{aligned} & c(M) \geqslant \dim(X) \\ \Leftrightarrow\ & d(M) \underbrace{\leqslant}_{\text{true by holonomicity}} 2\dim(X) - \dim(X) = \dim(X) \end{aligned}$$

(4) M holonomic $\Leftrightarrow \mathbb{D}(M)$ is concentrated in degree $\dim(X)$. "$\Rightarrow$" by (3). "$\Leftarrow$" use Sato-Kashiwara description of the codimension filtration (or use (2)).

(5) Clear: use (3)+(4)

(6) noetheriness clear (as $\mathscr{D}_X$ is noetherian). Artininess follows from duality.

$\square$

17. 10/28/2020

17.1. Holonomicity (contd).

Example 17.1. Let X be a smooth variety of dimension d over k.

(1) $M \in \mathbf{Vect}^{\nabla} = \{$vector bundle with flat connections over $X\}$
- M is holonomic (because $\mathrm{SS}(M) = X \overset{\circ}{\subseteq} \mathrm{T}^*X$).
- $\ell(M) \leqslant \mathrm{rank}(M)$: any subquotient N of M is a verctor bundle with flat connection (because $\mathrm{SS}(N) \subseteq \mathrm{SS}(M) = X$, $\dim(\mathrm{SS}(N)) \geqslant \dim(X) \Rightarrow \mathrm{SS}(N) = X$). So $\ell(M) \leqslant \mathrm{rank}(M)$. Exercise: M as above, $\mathbb{D}(M)[\dim(X)] = M^{\vee}$.

(2) Say $x \in X$ closed point $\rightsquigarrow \mathrm{H}^d_x(\mathcal{O}_X)(= \mathsf{R}\Gamma_{\{x\}}(\mathcal{O}_X)[d])$ is a holonomic $\mathscr{D}_X$-module of length 1:

$$i_+ : \mathrm{Hol}_{\mathrm{pt}} \cong \mathrm{Hol}_{\mathrm{pt}\subseteq X} = \{M \in \mathrm{Hol}_X | M \text{ support at point}\}$$

preserves length, so it sufficies to observe

$$i^+(\mathrm{H}^d_x(\mathcal{O}_X)) = i^+ i_+ k(x) = k(x)$$

which is length 1 in $\mathrm{Hol}_{\mathrm{pt}}$.

(3) Let $X = \mathbb{A}^1$, $M = k[x, x^{-1}]$ is holonomic of length 2: have a short exact sequence of $\mathscr{D}$-modules:

$$0 \to k[x] \to k[x, x^{-1}] \to k[x, x^{-1}]/k[x] \to 0$$

and both outer terms have length 1 by exercise 1+2

Remark 17.2. Say $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$. By Bernstein's inequality, the codimension filtration looks like
$$0 = \mathrm{C}^{\dim(X)+1}(M) \subseteq \mathrm{C}^{\dim(X)}(X) \subseteq \cdots \subseteq \mathrm{C}^0(M) = M$$
So $\mathrm{C}^{\dim(X)}(X) \subseteq M$ is the maximal holonomic submodule of M by functoriality of codimension filtration.

Also, $\mathrm{C}^{\bullet}(M)$ is compatible with passing to open subsets. So
$$\big(\mathrm{C}^{\dim(X)}(M) \subseteq M\big)|_U = \mathrm{C}^{\dim(U)}(M|_U) \subseteq M|_U$$
for all $\varnothing \neq U \subseteq X$.

Corollary 17.3. *For $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$, $U \subseteq X$ open, $N \subseteq M|_U$ holonomic submodule. Then there exists a holonomic submodule $N' \subseteq M$ such that $N'|_U = N$ inside $M|_U$.*

Proof. By previous approach, may replace M with its maximal holonomic submodule to assume M is holonomic.

Now it is enough to lift N to any $\mathscr{D}$-submodule $N' \subseteq M$. Write $N = \mathscr{D}_U \cdot N_0$ where $N_0 \subseteq M|_U$ is $\mathcal{O}$-coherent. Lift N_0 to an $\mathcal{O}$-coherent subsheaf $N_0' \subseteq M$. Set $N' = \mathscr{D}_X \cdot N_0'$. □

17.2. **Stability under pushforward/pullback.** Say $f : X \to Y$ map of smooth varieties over k. We have
$$f_+ : \mathsf{D}(\mathscr{D}_X) \to \mathsf{D}(\mathscr{D}_Y)$$
$$f^* : \mathsf{D}(\mathscr{D}_Y) \to \mathsf{D}(\mathscr{D}_X)$$
and $f^+ = f^*[dim(X) - \dim(Y)]$.

These have the following properties:

(1) f proper $\Rightarrow$ f_+ preserves $\mathscr{D}$-coherence and is adjoint to f^*
(2) f étale $\Rightarrow$ f^* preserves $\mathscr{D}$-coherence

In general, f_+ and f^* do not preserve $\mathscr{D}$-coherence.

Theorem 17.4. *f_+ and f^* (and thus f^+) preserve $\mathsf{D}^b_{\mathrm{Hol}}(-)$.*

Lemma 17.5. *Say $j : U \to X$ open immersion. Then $j_+ (= j_*)$ and j^* preserves $\mathsf{D}^b_{\mathrm{Hol}}(-)$.*

Proof. Clear for j^*; rest comes later. □

Lemma 17.6. *Say $i : Z \hookrightarrow X$ closed immersion. Then i_+ and i^* preserve $\mathsf{D}^b_{\mathrm{Hol}}(-)$.*

Proof. i_+ clear from previous discussion (because $\mathrm{SS}(i_+, M)$ vector bundle of rank $\dim X - \dim Z$ over $\mathrm{SS}(M)$).

For i^* (or equivalently i^+), we use Lemma 17.5: Given $M \in \mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_X)$, have an exact triangle
$$\begin{array}{ccccc} i_+i^+(M) & \longrightarrow & M & \longrightarrow & j_+j^+M \\ \| & & \| & & \downarrow \\ \underline{\mathsf{R}\Gamma}_Z(M) & \longrightarrow & M & \longrightarrow & j_*j^*M \end{array}$$
Since M is holonomic by assumption and j_*j^*M holonomic by Lemma 17.5, then $i_+i^+(M)$ holonomic.

As $\mathrm{SS}(i_+N)$ = vector bundle of rank $\dim X - \dim Z$ over $\mathrm{SS}(N)$, it follows that i^+M is holonomic. □

Lemma 17.7. *Say $f : X \to Y$ is smooth and M a coherent $\mathscr{D}_Y$-module. Then f^*M is $\mathscr{D}$-coherent and $\mathrm{SS}(f^*M) = ab^{-1}(\mathrm{SS}(M))$ where a and b are defined via $\mathrm{T}^*Y \xleftarrow{b} \mathrm{T}^*Y \times_Y X \xrightarrow{a} \mathrm{T}^*X$, where*

- *b smooth map of relative dimension $\dim X - \dim Y$.*
- *a closed immersion.*

So if M is holonomic, then f^*M is also holonomic.

Proof. Choose a good filtration F on M. Note: $f^*M = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M$. Endow $\mathcal{O}_X$ and $f^{-1}\mathcal{O}_Y$ with trivial filtration (so $\mathrm{gr}_i = 0$ for any $i \neq 0$). So we get a filtration on f^*M.

Thinking through the action of $\mathscr{D}_X$ on f^*M, we learn that this construction makes f^*M a filtered $\mathscr{D}_X$-module.

Also,

$$\begin{aligned}
\mathrm{gr}(f^*M) &= \mathrm{gr}(\mathcal{O}_X) \otimes_{\mathrm{gr}(f^{-1}\mathcal{O}_Y)} \mathrm{gr}(f^{-1}M) \\
&= \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\,\mathrm{gr}(M) \\
&= (\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathrm{Sym}^*_{\mathcal{O}_Y}(\mathrm{T}_Y)) \otimes_{\mathrm{Sym}^*_{\mathcal{O}_Y}(\mathrm{T}_Y)} \mathrm{gr}(M) \\
&= a_* b^*\,\mathrm{gr}(M) \\
&= \mathrm{Supp}(\mathrm{gr}(f^*M)) \\
&= ab^{-1}\,\mathrm{Supp}(\mathrm{gr}(M))
\end{aligned}$$

□

Corollary 17.8. *For any $f : X \to Y$, f^* preserves $\mathsf{D}^b_{\mathrm{Hol}}(-)$.*

Proof. Reduce to X, Y being affine.

Factor f as $X \overset{i}{\hookrightarrow} \mathbb{A}^n \times Y \overset{\pi}{\to} Y$. Then

- i^* preserves $\mathsf{D}^b_{\mathrm{Hol}}$ (Lemma 17.6)
- π^* preserves $\mathsf{D}^b_{\mathrm{Hol}}$ (Lemma 17.7)

So f^* preserves $\mathsf{D}^b_{\mathrm{Hol}}$. □

Lemma 17.9. *Say $M \in \mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_X)$. M is holonomic $\Leftrightarrow$ for any $x \in X$ closed, $i_x^* M \in \mathsf{D}^b(\mathscr{D}_{\mathrm{pt}})$ is holonomic (= has finite dimension total coh)*

Proof. Assume M is holonomic, $x \in X$ closed point, $i_x : \{x\} \hookrightarrow X$.

$\Rightarrow$: As i_x is a closed immersion, i_x^* preserves holonomicity Lemma 17.6. So $i_x^* M$ has finite dimen total coh.

$\Leftarrow$: Assume each $i_x^* M$ is holonomic. Let $\mathrm{S}(M) = \mathrm{Supp}(M) \overset{g}{\subseteq} X$ which is a closed subset. We prove the claim by induction on $\dim(\mathrm{S}(M))$. If $\dim(\mathrm{S}(M)) = 0$, then $\mathrm{S}(M)$ is a finite collection of closed points, so statement true by assumption + stability of $\mathsf{D}^b_{\mathrm{Hol}}(-)$ under $i_{x,+}$. □

In general, we will use:

Lemma 17.10. *Let $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_X}$. Then there exists $U \subseteq X$ dense open such that $M|_U$ is a projective $\mathcal{O}_U$-module (but not necessarily finite projective).*

Assuming this, (apply Lemma 17.10 to $g^+ M$ where $g : \mathrm{S}(M)_{\mathrm{sm}} \hookrightarrow X$), may choose a dense open $U \subseteq X$ such that

- $\mathrm{S}(M)_U = U \cap \mathrm{S}(M) \subseteq \mathrm{S}(M)$ is a smooth dense open subset.
- $g^+ M|_U$ has projective cohomology sheaves over $\mathcal{O}_U$.

Applying the assumption $i_x^*(M) (= i_x^+ M[\text{shift}])$ has finite dimension total Coh to $x \in \mathrm{S}(M)_U$, get $g^+ M|_U$ is a finite projective $\mathcal{O}_U$-module.

Goal: $i_x^*(M)$ perfect for any $x \in X^{\mathrm{cl}} \Rightarrow M$ holonomic.

Write $\mathrm{S}(M) = \mathrm{Supp}(M)$, prove by induction on $\dim(\mathrm{S}(M))$.

Dim 0: clear

Observation: If $Z \overset{i}{\hookrightarrow} X$ closed with complement $j : U \hookrightarrow X$, then for any $N \in \mathsf{D}_{\mathrm{qc}}(\mathscr{D}_X)$, have exact triangle

$$\underline{\mathsf{R}\Gamma}_Z(N) \to N \to j_*(N|_U)$$

N is fiberwise perfect $\Leftrightarrow \underline{\mathsf{R}\Gamma}_Z(N), j_*(N|_U)$ are fiberwise perfect.

Step 1: Reduce to the case $\mathrm{S}(M)$ is smooth.

idea: Choose $U \subseteq X$ open such that $U \cap \mathrm{S}(M)$ is dense open in $\mathrm{S}(M)_{\mathrm{sm}}$.

$$\mathsf{R}\Gamma_{X-U}(M) \to M \to j_*(M|_U)$$

If $M|_U$ is holonomic, then $j_*(M|_U)$ is so (LEM1). So $\underline{\mathsf{R}\Gamma}_{X-U}(M)$ is coherent and fiberwise perfect, so $\underline{\mathsf{R}\Gamma}_{X-U}(M)$ is holonomic by induction. So M is holonomic.

Step 2: May assume that $\mathrm{S}(M)$ is smooth by Step 1, get $g : \mathrm{S}(M) \hookrightarrow X$. By Kashiwara $\Rightarrow M \cong g_+(g^+(M))$.

M coherent $\Rightarrow g^*(M)$ coherent

M fiberwise perfect $\Rightarrow g^+(M)$ fiberwise perfect

So may replace M with $g^+(M)$ to assume $\mathrm{S}(M) = X$

Step 3: Choose $U \subseteq X$ dense open such that $M|_U$ has projective coherent sheaves over $\mathcal{O}_U$ (Lem 5)

Choose $x \in U^{\mathrm{cl}} \subseteq X^{\mathrm{cl}}$.

$i_x^*(M)$ perfect $\Rightarrow$ each $\mathrm{H}^i(M|_U)$ is finite projective over $\mathcal{O}_U$.

So $M|_U$ is holonomic, now proceed by induction (as in step 1)

Proof of Lemma 5: look at the notes

Remark 18.1. The proof above shows that the following are equivalent for $M \in \mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_X)$:

(1) M is holonomic
(2) there exists a sheafificatin $\{k_i : Z_i \hookrightarrow X\}$ of X by locally closed smooth subvarieties such that each $k_i^+ M$ has finite locally free $\mathcal{O}_X$-coherent sheaves.

Lemma 18.2. *For any map $f : X \to Y$ of smooth varieties over k, f_+ preserves $\mathsf{D}^b_{\mathrm{Hol}}(-)$.*

Proof. By a Cech complex argument, may assume X is affine.

We may then factor f as $f : X \overset{i}{\hookrightarrow} P \overset{g}{\to} X$ where i is a locally closed immersion and g a proper smooth map.

Lem 1+Lem2 $\Rightarrow i_+$ preserves $\mathsf{D}^b_{\mathrm{Hol}}(-)$.

May replace X with P to assume f is proper smooth.

Fix $M \in \mathsf{D}^b_{\mathrm{Hol}}(X)$, so we know $f_+(M)$ is $\mathscr{D}$-coherent.

Using Lemma 4, it sufficies to show for any $y \in Y^{\mathrm{cl}}$, we have $i_y^* f_+(M) \in \mathsf{D}^b_{\mathrm{f.d.}}(k(y))$.

As f is smooth, we know that

$$f_+(M) = f_*(\mathrm{dR}_{X/Y}(M)) = f_*(\Omega^\bullet_{X/Y} \otimes_{\mathcal{O}_X} M)$$

So

$$\begin{aligned} i_y^* f_+(M) &= f_{y*}\left(i_{X_y}^*\left(\Omega^\bullet_{X/Y} \otimes_{\mathcal{O}_X} M\right)\right) \\ &= \mathsf{R}\Gamma(X_y, \mathrm{dR}_{X_y/y}(i_{X_y}^* M)) \\ &= f_{y+}(i_{X_y}^* M) \end{aligned}$$

where

$$\begin{array}{ccc} X_y & \xhookrightarrow{i_{X_y}} & X \\ \Big\downarrow{\scriptstyle f_y} & & \Big\downarrow{\scriptstyle f} \\ \{y\} & \xhookrightarrow{i_y} & Y \end{array}$$

$i_{X_y}^* M$ is holonomic (Lem 2) $\Rightarrow i_{X_y}^* M$ is $\mathscr{D}$-coherent over X_y.

X_y is proper over $\{y\} \Rightarrow f_{y+}(i_{X_y}^* M)$ is $\mathscr{D}$-coherent over $\{y\} \Leftrightarrow \mathcal{O}$-coherent over $\{y\}$.

So $i_y^* f_+(M) \in \mathsf{D}^b_{\mathbf{Coh}}(\{y\})$. □

Corollary 18.3 (Lyubeznik)**.** *Say R is a smooth k-algebra, $I \subseteq R$ ideal. Then each $\mathrm{H}^i_I(R)$ has finitely many associated primes.*

Proof. We know that $\mathsf{R}\Gamma_I(R) \in \mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_R)$, so each $\mathrm{H}^i_I(R)$ is holonomic.

So it sufficies to show that if M is holonomic $\mathscr{D}_R$-module, then $\mathrm{Ass}(M)$ is finite. For each $P \in \mathrm{Ass}(M)$, may choose a simple subquotient N_P of M supported at $\overline{\{P\}}$. But M has only finitely many isomorphic classes of simple subquotients, so there are only finitely many possibilities for p. □

Open question: R finite type k-algebra where k is field of characteristic 0. $I \subseteq R$ ideal. Does $\mathrm{H}^i_I(R)$ have finitely many minimal primes?

18.1. **The b-function Lemma.**

Theorem 18.4. *Let $X = \mathrm{Spec}(R)$ be smooth affine over k, $U = D(f) \overset{j}{\hookrightarrow} X$, $0 \neq f \in R$. $M \in \mathrm{Hol}(\mathscr{D}_U)$ generated by 1 element $u \in M$. Then there exists differential operator $d_0(n) \in \mathscr{D}_R[n]$ and $b_0(n) \in k[n]$ such that we have*

$$d_0(n)(f^{n+1}u) = b_0(n) \cdot f^n u \qquad \forall n \in \mathbb{Z}$$

In particular, there exists some $c \in \mathbb{Z}$ such that for any $m \leqslant c$, we have $\frac{u}{f^m} \in \mathscr{D}_R \cdot \frac{u}{f^c} \subseteq M \Rightarrow M$ is coherent over $\mathscr{D}_R$.

Example 18.5. Say $R = k[x_1, \ldots, x_m]$, $\partial_i = \frac{\partial}{\partial x_i}, f \in R, M = R_f, u = 1$.

(1) $f = x_1$: $d_0(n) = \frac{\partial}{\partial x_1}, b_0(n) = n + 1$.

$$\Rightarrow \frac{\partial}{\partial x_1}\left(x_1^{n+1}\right) = (n+1)x_1^n$$

(2) $f = \sum_{i=1}^m x_i^2$. One can show $\nabla = \sum_i \partial_i^2$ satisfies

$$\nabla(f^{n+1}) = 4(n+1)(n + \frac{m}{2}) \cdot f^n$$

(3) $f = x_1^2 + x_2^3$: $\nabla = \frac{1}{12}x_2\partial_1^2\partial_2 + \frac{1}{27}\partial_2^3 + \frac{n}{4}\partial_1 + \frac{3}{8}\partial_1^2$

$$\nabla(f^{n+1}) = (n+1)\left(n + \frac{5}{8}\right)\left(n + \frac{7}{6}\right)f^n$$

The minimal degree $b_0(n)$ that shows up in the Theorem is called the Bernstein-Sato polynomial of f, denoted $b_f(n)$.

Goal: Prove the theorem ans derive Lemma 1

Construction: (Complex pairs of f as a $\mathscr{D}$-module):

Let s be a formal variable, X, U as in Theorem.

$X_s = X \otimes_k k(s), U_s = U \otimes_k k(s), \mathscr{D}_{X_s} = \mathscr{D}_X \otimes_k k(s), \mathrm{T}^* X_s = (\mathrm{T}^* X)_s$.

We have a projection $\pi : X_s \to X$ So we have $\pi^* : \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_X} \to \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_{X_s}}$.

Have a naturally defined 1-form.

$$\mathrm{S}\mathrm{d}\log(f) = \frac{S\mathrm{d}f}{f} \in \Omega^1_{U_s}$$

Define a rank 1 locally free $\mathcal{O}$-Coh $\mathscr{D}$-module

$$\mathcal{O}_U \cdot f^s = (\mathcal{O}_{U_s} \cdot f^s, \mathrm{d} + s\mathrm{d}\log(f)) \in \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_{U_s}}$$

Explicitly,

- functions act as usual
- $a \in \mathrm{T}_U$ have

$$a(f^s) = \underbrace{''s \cdot f^{s-1} a(f)''}_{\text{Does not make sense}} = s \cdot \frac{a(f)}{f} \cdot f^s$$

May define a functor

$$B : \mathrm{L}\,\mathbf{Mod}_{sD_U} \to \mathrm{L}\,\mathbf{Mod}_{sD_{U_s}}$$
$$M \mapsto M \cdot f^s$$

as follows: (3 equivalent descriptions)

(1) $Mf^s = (\pi^* M) \otimes_{\mathcal{O}_{U_s}} \mathcal{O}_{U_s} \cdot f^s$ where $\pi : U_s \to U$.
(2) $Mf^s = (M_s \cdot f^s, \nabla + s\mathrm{d}\log(f))$.
(3) $Mf^s = M_s \cdot f^s$, functions act as usual, $a \in \mathrm{T}_U$ acts via: $a(M \cdot f^s) = \Big(a(M) + s\frac{a(f)}{f} M\Big) \cdot f^s$.

19. 11/09/2020

19.1. The b-function lemma (cont).

Theorem 19.1. *Let $X = \mathrm{Spec}(R)$ smooth affine over k. $U = \mathrm{D}(f) \subseteq X$ for $0 \neg f \in R$, $M \in \mathrm{Hol}(\mathscr{D}_U)$ generated by $u \in M$. Then $d_0(s) \in \mathrm{D}_R[s]$ and $b_0(s) \in k[s]$ such that $d_0(n) f^{n+1} u = b_0(n) f^n u$ for any $n \in \mathbb{Z}$.*

So there exist a $c \in \mathbb{Z}$ such that $\forall m \geqslant c$, we have $\frac{u}{f^m} \in \mathrm{D}_R \cdot \frac{u}{f^c} \subseteq M$. In particular, M is coherent over $\mathscr{D}_R$.

Last time: write $X_s = X \otimes_k k(s)$, $U_s = U \otimes_k k(s)$, etc.

So we have $\pi : X_s \to X$ or $\pi : U_s \to U$

We had a functor:

$$\begin{aligned} B : \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_U} &\to \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_{U_s}} \\ M &\mapsto M \cdot f^s = (M_s \cdot f^s, d * s\mathrm{d}\log(f)) \\ &= (\pi^* M) \otimes_{\mathcal{O}_U} \mathcal{O}_U \cdot f^s \end{aligned}$$

Strategy:

(1) Show $M \cdot f^s$ is holonomic over $\mathrm{D}_{R_s} = \mathrm{D}_{X_s}$
(2) Specialize $s \mapsto n \in \mathbb{Z}$ for most n to derive that $M = M \cdot f^n$ is holonomic over D_R.

Properties of $M \mapsto M \cdot f^s$

(1) If M is a coherent $\mathscr{D}_U$-module, then $\mathrm{SS}(M \cdot f^s) \subseteq T^*U_s = (T^*U)_s$ is simply $\pi^{-1}(\mathrm{SS}(M)) \subseteq \mathrm{T}^*U$. (Idea: if $\{\mathrm{F}_i\}$ is a good fit of M, then $\{(\mathrm{F}_i)_s \cdot f^s\}$ is a good filtration of $M \cdot f^s$). So

$$M \cdot f^s \in \mathrm{Hol}(\mathscr{D}_{U_s}) \Leftrightarrow M \in \mathrm{Hol}(\mathscr{D}_U).$$

(2) For $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_U}$, the following $\mathscr{D}_{U_s}$-modules are isomorphic

$$Mf^s = (M_s f^s, \mathrm{d} + s\mathrm{d}\log(f)) \cong M \cdot f^{s+k} = (M_s f^{s+k}, \mathrm{d} + (s+k)\mathrm{d}\log(f))$$
$$M \cdot f^s \mapsto \frac{M}{f^k} \cdot f^{s+k}$$

(3) This construction refines to a functor

$$\mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_U} \to \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_U[s]/k[s]}$$
$$M \mapsto \widetilde{M \cdot f^s} = (M \otimes_k k[s], \mathrm{d} + s\mathrm{d}\log(f)) \subseteq M_s \cdot f^s$$

So it makes sense to specialize $s \in k[s]$ to $\lambda \in k$: Given a map $e_\lambda : k[s] \to k$ sending s to λ, we can identify $e_\lambda^*(\widetilde{\mathcal{O}_U \cdot f^s}) = (\mathcal{O}_U, \mathrm{d} + \lambda \mathrm{d}\log(f))$ as a rank 1 $\mathscr{D}_U$-module with connection.

In particular, take $\lambda = n \in \mathbb{Z}$, we get

$$e_n^*(\widetilde{\mathcal{O}_U \cdot f^s}) = \mathcal{O}_U \cdot f^n = (\mathcal{O}_U, \mathrm{d} + n\mathrm{d}\log(f)) \cong \mathcal{O}_U = (\mathcal{O}_U, \mathrm{d})$$

So

$$e_n^*(\widetilde{Mf^s} = M$$

as a $\mathscr{D}_U$-module.

Proof of b-function lemma. Have $j_s : U_s \hookrightarrow X_s$. Let $N = j_{s*}(M \cdot f^s) \in \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_{X_s}}$. We know $N|_{U_s}$ is holonomic (property 1)

Extension lemma $\Rightarrow$ there exists a holonomic submodule $N' \subseteq N$ such that $N'|_{U_s} = N|_{U_s}$. So $N/N' \in \mathrm{L}\,\mathbf{Mod}_{\mathscr{D}_{X_s}}$ is set-theoretically supported on $V(f)_s \subseteq X_s$. So the element $uf^s \in N$ is killed in N/N' by some $f^k, k \geqslant 0$. So $f^k \cdot uf^s \in N'$ for some $k \geqslant 0$. So consider the descending sequence

$$N' \supseteq \mathrm{D}_{X_s} \cdot f^k uf^s \supseteq \mathrm{D}_{X_s} \cdot f^{k+1} uf^s \supseteq \cdots \supseteq \mathrm{D}_{X_s} f^{k+n} uf^s \supseteq \cdots$$

As N' has DCC, there exists a $m \gg 0$ such that

$$f^m \cdot uf^s \in \mathrm{D}_{X_s} \cdot f^{m+1} uf^s$$

So there exists $P \in \mathrm{D}_{X_s}$ such that $P \cdot (f^{m+1}uf^s) = f^m \cdot uf^s$.

Since $P = \frac{d_0(s)}{b_0(s)}$ for $d_0 \in \mathrm{D}_X[s]$ and $b_0 \in k[s]$, clear denominators to get

$$d_0(s) f^{m+1} uf^s = b_0(s) \cdot f^m \cdot u \cdot f^s \in \widetilde{M \cdot f^s} \subseteq M \cdot f^s.$$

Specialize $s \mapsto n \in \mathbb{Z}$ (as in Property 3) to get

$$\underbrace{d_0(n) \cdot f^{m+1} u \cdot f^n}_{= d_0(n) \cdot f^{m+n+1} u} = \underbrace{b_0(n) f^m u f^n}_{= b_0(n) \cdot f^{m+n} \cdot u} \in M$$

Change n to $m+n$ to get the lemma. ($\Rightarrow M$ is *coherent* over D_R). □

Remark 19.2 (A holonomic consequence).

Claim. *In the setup above, $M \cdot f^s$ is holonomic over D_{X_s}.*

In the previous proof, we saw

$$N' = \mathrm{D}_{X_s} \cdot f^k u f^s \subseteq M = \mathrm{D}_{U_s} \cdot u f^s$$

was a holonomic D_{X_s}-submodule.

Property (2) $\Rightarrow \forall c \in \mathbb{Z}$, we have an isomorphism

$$\mathrm{D}_{U_s} \cdot u \cdot f^s \cong \mathrm{D}_{U_s} \cdot u \cdot f^{s+k+c}$$
$$Puf^s \mapsto P \cdot \frac{u}{f^{k+c}} \cdot f^s$$

Transporting the holonomicity of $N' \subseteq M$ as above under this isomorphism, we get

$$\mathrm{D}_{X_s} \cdot \frac{u}{f^c} f^{s+k+c} \subseteq \mathrm{D}_{U_s} u f^{s+k+c}$$

is a holonomic $\mathscr{D}_{X_s}$-submodule for any c. By "transport of structure" along the automorphism $s \mapsto s+k+c$ of $k(s)$, we learn that $\mathrm{D}_{X_s} \cdot \frac{u}{f^c} \cdot f^s \subseteq \mathrm{D}_{U_s} \cdot u \cdot f^s$ is a holonomic D_{X_s}-submodule for any c.

Now

$$\cup_c \mathrm{D}_{X_s} \cdot \frac{u}{f^c} f^s = \mathrm{D}_{U_s} \cdot u \cdot f^s$$

and LHS is a finite union by the b-function lemma. So

$$(M \cdot f^s) = \mathrm{D}_U \cdot u \cdot f^s = \mathrm{D}_{X_s} \cdot \frac{u}{f^c} f^s$$

So Mf^s is holonomic over D_{X_s}.

Lemma 19.3. *For $j : U = \mathrm{D}(f) \hookrightarrow X$, M as in the b-function lemma, $j_* M$ is holonomic over D_X.*

Proof. We know $M \cdot f^s$ is cylic holonomic over D_{X_s}. Also, b-function lemma $\Leftarrow M$ is coherent over $\mathscr{D}_X$.

Goal: $\dim \mathrm{SS}(M) = n (= \dim(X))$.

Since Mf^s is cylic holonomic over D_{X_s}, we have $Mf^s = \mathrm{D}_{X_s} / \mathrm{D}_{X_s} \cdot \underbrace{(\{P\} r)}_{=I}$ where $P_i \in \mathrm{D}_{X_s}$ such that

(1) $P_i \in \mathrm{F}_{r_i}(\mathrm{D}_{X_s})$ and $\overline{P}_i \in \mathrm{gr}_{r_i}(\mathrm{D}_{X_s})$ gives a system of generators of $\mathrm{gr}(I) \subseteq \mathrm{gr}(\mathrm{D}_{X_s}) = \mathrm{Sym}^*(\mathrm{T}_{X_s})$.
(2) (holonomicity)$V(\overline{P}_1, \ldots, \overline{P}_r) \subseteq \mathrm{T}^* X_s$ has $\dim = n$.
 "Spreading out" to an open subset W of $\mathrm{Spec}(k[s])$ and specializing s to $n \in \mathbb{N} \subseteq W$ gives the desired conclusion using Property 3.

□

Remark 19.4 (Roskim). For any open injection $j : U \subseteq X$ and $M \in \mathrm{L}\,\mathbf{Mod}^{\mathbf{Coh}}_{\mathscr{D}_U}$, we have:

If $\dim(\mathrm{SS}(M)) \leqslant \dim(X) + \delta$, then for any coherent $\mathscr{D}$-submodules $N \subseteq j_*(M)$, we have $\dim(\mathrm{SS}(N)) \leqslant \dim(X) + \delta$.

19.2. **Verdier duality.**

Definition 19.5. For a smooth k-scheme X, set

$$\mathbb{D}_X(-) = \left(\underline{\mathsf{R}\,\mathrm{Hom}}_{\mathscr{D}_X}(-, \mathscr{D}_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}\right)[\dim X]$$

So $\mathbb{D}_X(-) : \mathrm{D}^b_{\mathbf{Coh}}(\mathscr{D}_X) \to \mathrm{D}^b_{\mathbf{Coh}}(\mathscr{D}_X)$

Note $\mathbb{D}_X(\mathrm{Hol}_X) \subseteq \mathrm{Hol}_X$ (no shifts)

Example 19.6. $\mathbb{D}_X(\mathcal{O}_X) = \mathcal{O}_X$:

Need to show $\underline{\mathrm{R\,Hom}}_{\mathscr{D}_X}(\mathcal{O}_X, \mathrm{D}_X)[n] \cong \omega_X$ as right $\mathscr{D}_X$-module where $n = \dim X$. Equivalently, need to show $\underline{\mathrm{R\,Hom}}_{\mathscr{D}_X^{\mathrm{op}}}(\omega_X, \mathrm{D}_X^{\mathrm{op}})[n] \cong \mathcal{O}_X$ as left D_X-module.

We use (recall) have dR resolution

$$dR(\mathscr{D}_X) = \left(\mathscr{D}_X \xrightarrow{\mathrm{d}} \Omega^1_X \otimes_{\mathcal{O}_X} \mathscr{D}_X \to \cdots \to \omega_X \otimes_{\mathcal{O}_X} \mathscr{D}_X\right) \cong \omega_X$$

Apply $\underline{\mathrm{R\,Hom}}_{\mathscr{D}_X^{\mathrm{op}}}(-, \mathrm{D}_X^{\mathrm{op}})$ to get:

$$\underbrace{\mathrm{Sp}}_{\text{Spencer complex}} = \left(\mathscr{D}_X \otimes_{\mathcal{O}_X} \bigwedge^n \mathrm{T}_X \to \mathscr{D}_X \otimes \bigwedge^{n-1} \mathrm{T}_X \to \cdots \to \mathscr{D}_X \otimes \mathrm{T}_X \to \mathscr{D}_X\right)$$

a complex of left $\mathscr{D}$-modules.

Obs: there exists a natural map

$$\begin{aligned} \mathrm{Sp}(\mathscr{D}_X) &\to \mathcal{O}_X \\ P \in \mathscr{D}_X &\mapsto P \cdot 1 \in \mathcal{O}_X \end{aligned}$$

Claim. *$\mathrm{Sp}(\mathscr{D}_X) \cong \mathcal{O}_X$ is a qis.*

Proof. Left as an Exercise(hint: reduces via filtrations to the std koszol resoluation of $\mathcal{O}_X$ over $\mathrm{Sym}^*_{\mathcal{O}_X} \mathrm{T}_X$). ☐

20. 11/11/2020

20.1. Verdier Duality (cont).

Exercise 20.1. Say $f : X \to Y$ is a smooth map of smooth k-varieties. Show that there exists resolution

$$\left(\mathscr{D}_X \otimes \bigwedge^n \mathrm{T}_{X/Y} \to \cdots \to \mathscr{D}_X \otimes_{\mathcal{O}_X} \bigwedge^2 \mathrm{T}_{X/Y} \to \mathscr{D}_X \otimes_{\mathcal{O}_X} \mathrm{T}_{X/Y} \to \mathscr{D}_X\right) \cong \mathscr{D}_{X\to Y}$$

The LHS is $\mathrm{Sp}_{X/Y}(\mathscr{D}_{X\to Y})$.

Example 20.2. X smooth variety, $x \in X$ closed point, δ_x = δ-functino D-module = $i_{x,+}(k(x))$. Then $\mathbb{D}(\delta_x) \cong \delta_x$.

Proof. δ_x =unique length 1 holonomic $\mathscr{D}_X$-module supported at x. So $\mathbb{D}(\delta_x)$ has same property. ($\Rightarrow$ $i_+(\mathbb{D}(\mathcal{O}_{\mathrm{pt}})) \cong \mathbb{D}(i_+(\mathcal{O}_{\mathrm{pt}}))$. ☐

Proposition 20.3. *Say $f : X \to Y$ projective map of smooth k-varieties. Then there exists a natural isomorphism $f_+\mathbb{D}(-) \cong \mathbb{D} \circ f_+(-)$ as $\mathrm{D}^b_{\mathrm{Hol}}(X)^{\mathrm{op}} \to \mathrm{D}^b_{\mathrm{Hol}}(Y)$.*

Proof. Assume f is a closed immersion.

(1) Construct a comp map: $\eta_M : f_+\mathbb{D}(M) \to \mathbb{D}f_+(M)$ for any $M \in \mathrm{D}^b_{\mathrm{Hol}}(\mathscr{D}_X)$.

$$\begin{aligned} f_+(\mathbb{D}(M)) &= f_+(\underline{\mathrm{R\,Hom}}_{\mathscr{D}_X}(M \cdot \mathscr{D}_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}[d_X]) \\ &= f_+(\underline{\mathrm{R\,Hom}}_{\mathscr{D}_X}(M \cdot \mathscr{D}_X)[d_X]) \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \\ &= f_*(\underline{\mathrm{R\,Hom}}_{\mathscr{D}_X}(M \cdot \mathscr{D}_X)[d_X] \otimes_{\mathscr{D}_X} \mathscr{D}_{X\to Y}) \otimes \omega_Y^{-1} \\ &= f_*(\underline{\mathrm{R\,Hom}}_{\mathscr{D}_X}(M \cdot \mathscr{D}_{X\to Y})[d_X]) \otimes \omega_Y^{-1} \end{aligned}$$

The target of η_M is:

$$\mathbb{D}f_+M = \underline{\mathrm{R\,Hom}}_{\mathscr{D}_Y}(f_+M, \mathscr{D}_Y) \otimes \omega_Y^{-1}[d_Y]$$

So it suffices to define a canonical map

$$\begin{array}{ccccc} f_+(\mathscr{D}_{X\to Y})[d_X] & \longrightarrow & \mathscr{D}_Y[d_Y] & \Leftrightarrow & f_+(f^*\mathscr{D}_Y[d_X - d_Y]) \longrightarrow \mathscr{D}_Y \\ \big\downarrow{\scriptstyle =} & & & & \big\downarrow{\scriptstyle =} \quad \nearrow{\scriptstyle \mathrm{tr}_{X\to Y}} \\ f_+(f^*\mathscr{D}_Y)[d_X] & & & & f_+f^+(\mathscr{D}_Y) \end{array}$$

where $\mathrm{tr}_{X\to Y}$ comes from adjunction between f_+ and f^+.

(2) $\eta_M : f_+\mathbb{D}(M) \to \mathbb{D}f_+(M)$ is an isomorphism for any $M \in \mathrm{D}^b_{\mathrm{Hol}}(\mathscr{D}_X)$.

Above, we constructed a map

$$f_*\underline{\mathsf{R}\,\mathrm{Hom}}_{\mathscr{D}_X}(M, \mathscr{D}_{X\to Y})[d_X] \to \underline{\mathsf{R}\,\mathrm{Hom}}_{\mathscr{D}_Y}(f_*M, \mathscr{D}_Y)[d_Y]$$

By localizing on Y, it sufficies to show the equation above is an isomorphism on global sections, i.e.,

$$\begin{array}{ccc} \mathsf{R}\,\mathrm{Hom}_{\mathscr{D}_X}(M, \mathscr{D}_{X\to Y})[d_X] & \xrightarrow{\quad\eta\quad} & \mathsf{R}\,\mathrm{Hom}_{\mathscr{D}_Y}(f_+M, \mathscr{D}_Y)[d_Y] \\ {\scriptstyle f_+}\searrow & & \nearrow{\scriptstyle t=\text{trace map}} \\ & \mathsf{R}\,\mathrm{Hom}_{\mathscr{D}_Y}(f_+M, f_+\mathscr{D}_{X\to Y})[d_X] & \end{array}$$

So it suffices to show f_+ and t are isomorphisms

- f_+ isomorphism: Kashiwara (f closed immersion)
- For t, it suffices to show

$$\mathsf{R}\,\mathrm{Hom}_{\mathscr{D}_Y}(f_*M, \mathrm{Cone}(f_+\mathscr{D}_{X\to Y}[d_X] \to \mathscr{D}_Y[d_Y]) = 0$$

But $f_+\mathscr{D}_{X\to Y}[d_X] = f_+(f^+\mathscr{D}_Y[d_Y])$ (as in Step 1) and $f_+\mathscr{D}_{X\to Y}[d_X] \to \mathscr{D}_Y[d_Y]$ is the standard map $f_+(f^+\mathscr{D}_Y[d_Y]) \to \mathscr{D}_Y[d_Y]$, so the cone is $j_*(\mathscr{D}_Y[d_Y]|_{Y-X})$.

So it suffice to observe $\mathsf{R}\,\mathrm{Hom}(f_+(-), j_*(-)) = 0$.

□

Proof of $f_+\mathbb{D} \cong \mathbb{D}f_+$ for f projective. By previous step, may assume $X = Y \times \mathbb{P}^n \overset{f=\mathrm{pr}}{\to} Y$. Following previous strategy, suffices to:

(1) Construct a trace map for $f_+\mathscr{D}_{X\to Y}[d_X] \to \mathscr{D}_Y[d_Y]$, gives a map $\eta_M : f_+\mathbb{D}M \to \mathbb{D}f_+(M)$ as before.
(2) Check η_M is an isomorphism for any M.

Step 1:

$$\begin{aligned} f_+(\mathscr{D}_{X\to Y}) &= f_*(dR_{X/Y}(\mathscr{D}_{X\to Y}))[d_X - d_Y] \text{ Here we use } \mathrm{Sp}_{X/Y}(-), \text{ since } f \text{ is smooth} \\ dR_{X/Y}(\mathscr{D}_{X\to Y}) &= \left(f^*\mathscr{D}_Y \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} f^*\mathscr{D}_Y \to \cdots\right) \\ \Rightarrow f_*(dR_{X/Y}(\mathscr{D}_{X\to Y})) &= \left(f_*\mathcal{O} \otimes \mathscr{D}_Y \to f_*\Omega^1_{X/Y} \otimes \mathscr{D}_Y \to \cdots\right) \\ \text{Recall: } f_*\Omega^i_{X/Y} &= \left(\mathsf{R}\Gamma(\mathbb{P}^n, \Omega^i)\right) \otimes_k \mathcal{O}_Y \\ &= k[-i] \otimes_k \mathcal{O}_Y = \mathcal{O}_Y[-i] \\ \text{so } f_+\mathscr{D}_{X\to Y}[d_X - d_Y] &= f_*dR_{X/Y}(\mathscr{D}_{X\to Y})[2(d_X - d_Y)] \\ (f_+\mathscr{D}_{X\to Y})[d_X - d_Y] &\in \mathrm{D}^{\leqslant 0} \\ \text{and } \mathrm{H}^0(f_+\mathscr{D}_{X\to Y}[d_X - d_Y]) &= \mathrm{H}^{2(d_X - d_Y)}(f_*dR_{X/Y}(\mathscr{D}_{X\to Y})) \\ &= \mathcal{O}_Y \otimes \mathscr{D}_Y = \mathscr{D}_Y \end{aligned}$$

So to define $f_+(\mathscr{D}_{X\to Y})[d_X - d_Y] \to \mathscr{D}_Y$, we may simply use

$$f_+\mathscr{D}_{X\to Y}[d_X - d_Y] \to \mathrm{H}^0(-) \cong \mathscr{D}_Y$$

(For details, see 2.7.2 in HTT)

2) To check η_M is an isomorphism for any $M \in \mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_X)$, use Beilinson-Berinstein to reduce to $M = \mathscr{D}_X$ itself. and then check it "by hand".

Example 20.4. $X = \mathbb{P}^n, Y = \mathrm{pt}, f : X \to \mathrm{pt}$ structure map

Goal: $f_+\mathbb{D}(\mathscr{D}_X) \cong \mathbb{D} f_+ \mathscr{D}_X$

LHS:

$$\begin{aligned}\mathbb{D}(\mathscr{D}_X) &= \underline{\mathrm{R}\,\mathrm{Hom}}_{\mathscr{D}_X}(\mathscr{D}_X, \mathscr{D}_X) \otimes \omega_X^{-1}[n] = \mathscr{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1}[n] \\ f_+\mathbb{D}(\mathscr{D}_X) &= f_+(\mathscr{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1}[n]) = f_*(\mathscr{D}_{Y \leftarrow X} \otimes_{\mathscr{D}_X} \mathscr{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1}[n]) \\ &= f_*(\omega_X \otimes_{\mathcal{O}_X} \omega_X^{-1}[n]) = f_*\mathcal{O}[n] = k[n]\end{aligned}$$

RHS:

$$\begin{aligned}f_+\mathscr{D}_X &= f_*(\mathscr{D}_{Y \leftarrow X} \otimes_{\mathscr{D}_X} \mathscr{D}_X) \\ &= f_*(\omega_X) \\ &= k[-n]\end{aligned}$$

So $\mathbb{D}(f_+\mathscr{D}_X) \cong f_+\mathbb{D}(\mathscr{D}_X)$.

□

Corollary 20.5. *Let $f : X \to Y$ be a projection map of smooth k-varieties. Then f_+ is left-adjoint to f^+.*

Goal:

$$\mathsf{R}f_*\underline{\mathrm{R}\,\mathrm{Hom}}_{\mathscr{D}_X}(M, f^+N) \cong \underline{\mathrm{R}\,\mathrm{Hom}}_{\mathscr{D}_Y}(f_+M, N)$$

for any $M \in \mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_X), N \in \mathsf{D}^b_{\mathbf{Coh}}(\mathscr{D}_Y)$.

$$\begin{aligned}\text{LHS} &= \mathsf{R}f_*\left(\underline{\mathrm{R}\,\mathrm{Hom}}_{\mathscr{D}_X}(M, \mathscr{D}_X) \otimes_{\mathscr{D}_X} f^+N\right) \\ &= \mathsf{R}f_*\left((\omega_X \otimes_{\mathscr{D}_X} \mathbb{D}(M))[-d_X] \otimes_{\mathscr{D}_X} f^*N[d_X - d_Y]\right) \\ &= \mathsf{R}f_*\left((\omega_X \otimes_{\mathscr{D}_X} \mathbb{D}(M))[-d_X] \otimes_{\mathscr{D}_X} \mathscr{D}_{X \to Y} \otimes_{f^{-1}\mathscr{D}_Y} f^{-1}N[d_X - d_Y]\right) \\ \text{(by Proj formula)} &= \mathsf{R}f_*\left((\omega_X \otimes_{\mathscr{D}_X} \mathbb{D}(M)) \otimes_{\mathscr{D}_X} \mathscr{D}_{X \to Y}\right) \otimes_{\mathscr{D}_Y} N[-d_Y] \\ &= f_+(\omega_X \otimes_{\mathscr{D}_X} \mathbb{D}(M)) \otimes_{\mathscr{D}_Y} N[-d_Y] \\ &= (f_+\mathbb{D}(M) \otimes \omega_Y) \otimes_{\mathscr{D}_Y} N[-d_Y] \\ &= (\mathbb{D}f_+(M) \otimes \omega_Y) \otimes_{\mathscr{D}_Y} N[-d_Y] \\ &= \underline{\mathrm{R}\,\mathrm{Hom}}_{\mathscr{D}_Y}(f_+M, N)\end{aligned}$$

Question: Can one prove the proper case of $f_+\mathbb{D} \cong \mathbb{D}f_+$ as follows:

(1) f_+ and f^+ are adjoint (Neeman's) (Idea: define a right adjoint f^b, check $f^b = f^+$ on $\mathcal{O}$-modules (where $\underline{\mathrm{R}\,\mathrm{Hom}}_{\mathscr{D}_X}(\mathscr{D}_X \otimes_{\mathcal{O}_X} K, -)$ for $K \in \mathsf{D}^b_{\mathbf{Coh}}()$) and $f^b = f^+$ as $\mathscr{D}$-modules.)
(2) Construct a trace map $f_+\mathscr{D}_{X \to Y}[d_X - d_Y] \to \mathscr{D}_Y$ using adjunction
(3) Resulting transformation $\eta_M : f_+\mathbb{D}(M) \to \mathbb{D}f_+(M)$ is an isomorphism in a) projective case, b) use Chow's lemma to handle proper case.

21. 11/16/2020

21.1. **The 6 functor formalism for $\mathscr{D}$-modules.** *So far*: $f : X \to Y$ map of smooth k-varieties

(1) $f_+, f^+, \mathbb{D}$ preserve holonomicity
(2) f_+ left adjoint to f^+ for f proper
(3) $f_+\mathbb{D} = \mathbb{D}f_+$ for f proper

Construction: New functor on $\mathbb{D}^b_{\mathrm{Hol}}(-)$

(1) $f_{\mathscr{D},*} = f_+$, and $\mathsf{R}\Gamma_{\mathscr{D}}(X,-) = f_{\mathscr{D},*}$ when $Y = \mathrm{pt}$.

Example 21.1.

$$\mathsf{R}\Gamma_{\mathscr{D}}(X,M) = \mathsf{R}\Gamma(X,(M\otimes\omega_X)\otimes_{\mathscr{D}_X}\Omega_X) = \mathsf{R}\Gamma(X,(M\otimes\omega_X)\otimes_{\mathscr{D}_X}\mathrm{Sp}(\mathscr{D}_X)) = \mathsf{R}\Gamma_{dR}(X,M)[d_X]$$

(2) $f^!_{\mathscr{D}} = f^+ : \mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_Y) \to \mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_X)$
(3) $f_{\mathscr{D},!} = \mathbb{D}_Y \circ f_{\mathscr{D},*} \circ \mathbb{D}_X(-) : \mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_X) \to \mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_Y)$
(4) $f^*_{\mathscr{D}} = \mathbb{D}\cdot f^!_{\mathscr{D}}\cdot\mathbb{D} : \mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_Y) \to \mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_X)$.

Example 21.2. Let $j : U \hookrightarrow X$ where $U = \mathbb{G}_m$ and $X = \mathbb{A}^1$. The goal is to describe $j_{\mathscr{D},!}\mathcal{O}_U$.

First, recall $j_{\mathscr{D},*}\mathcal{O}_U$ sits a SES:

$$0 \to \underbrace{\mathcal{O}_X}_{=k[x],\textit{simple}} \to \underbrace{j_{\mathscr{D},*}\mathcal{O}_U}_{=k[x,x^{-1}]} \to \underbrace{\delta_0}_{=k[x,x^{-1}]/k[x],\textit{simple}}$$

Dualizing, we get

$$0 \to \underbrace{\delta_0}_{\textit{simple}} \to \underbrace{\mathbb{D}(j_{\mathscr{D},*}\mathcal{O}_U)}_{=j_{\mathscr{D},!}\mathcal{O}_U} \to \underbrace{\mathcal{O}_X}_{\textit{simple}} \to 0$$

Note: This SES does NOT split as $\mathscr{D}$-modules (but does as $\mathcal{O}$-modules)

Explicitly:

$$j_{\mathscr{D},*}\mathcal{O}_U = \frac{k[x,\partial]}{k[x,\partial]\cdot\partial x}$$
$$j_{\mathscr{D},!}\mathcal{O}_U = \frac{k[x,\partial]}{k[x,\partial]\cdot x\partial}$$

Properties of these functors

(1) $\mathbb{D}f_{\mathscr{D},!} = f_{\mathscr{D},*}\mathbb{D}$: clear
(2) $bDf^!_{\mathscr{D}} = f^*_{\mathscr{D}}\mathbb{D}$: clear
(3) $f_{\mathscr{D},!} = f_{\mathscr{D},*}$ for f proper (result from last time)
(4) $f_{\mathscr{D},!}$ is left adjoint to $f^!_{\mathscr{D}}$: suffices to show separately for open immersions and proper maps
  - proper maps: use $f_{\mathscr{D},!} = f_{\mathscr{D},*}$ and result from last time.
  - open immersions: by duality, suffices to show $f_{\mathscr{D},*}$ is right adjoint to $f^*_{\mathscr{D}}$. BUt f is an open immersion, so $f_{\mathscr{D},*} = f_*$ and $f^*_{\mathscr{D}} = f^*$ where both are sheaf theoretic. Now use general result that f_* is right adjoint to f^*.

(5) $f_{\mathscr{D},*}$ is right adjoint to $f^*_{\mathscr{D}}$: (a) + duality
(6) f is an affine map $\Rightarrow f_{\mathscr{D},*}$ is right t-exact ($bD^{\leq 0}$ is preserved) and $f_{\mathscr{D},!}$ is left t-exact.

  Proof. $f_{\mathscr{D},*}(M) = f_+(M) = f_*(\mathscr{D}_{Y\leftarrow X}\otimes^{\mathsf{L}}_{\mathscr{D}_X} M)$. f is affine, so f_* exact for qc sheaves and $\otimes^{\mathsf{L}}$ is right exact. So $f_{\mathscr{D},*}$ is right t-exact. (rest is by duality) □

(7) f is finite $\Rightarrow f_{\mathscr{D},*}$ is t-exact: f finite $\Rightarrow f$ proper $\Rightarrow f_{\mathscr{D},!} = f_{\mathscr{D},*}$, so we can use (6). Now say $j : U \hookrightarrow X$ open immersion, $i : Z \hookrightarrow X$ complement (Z smooth).
(8) $j_{\mathscr{D},*}$ is left t-exact and $j_{\mathscr{D},!}$ is right t-exact: $j_{\mathscr{D},*} = j_*$ on underlying abelian sheaves, so it is left-exact, rest by duality

  RMK (6)+(8)$\Rightarrow j_{\mathscr{D},*}$ is t-exact if j is affine.

(9) $j^*_{\mathscr{D}} = j^!_{\mathscr{D}}$: duality is local on X.
(10) $j^!_{\mathscr{D}}j_{\mathscr{D},!} = \mathrm{id}, j^!_{\mathscr{D}}j_{\mathscr{D},*} = \mathrm{id}$: by duality + (9), suffices to show $j^*_{\mathscr{D}}j_{\mathscr{D},*} = \mathrm{id}$. But this is genral sheaf theory for the ringed space $(X,\mathscr{D}_X)$.
(11) $i^*_{\mathscr{D}}i_{\mathscr{D},*} = \mathrm{id}$ (and same thing for all 4 versions): Have $i_{\mathscr{D},*} = i_{\mathscr{D},!}$ because i is proper. So by duality, it is enough to show $i^!_{\mathscr{D}}i_{\mathscr{D},*} = \mathrm{id}$, which follows from Kashiwara.
(12) $i^*_{\mathscr{D}}j_{\mathscr{D},!} = 0$ and $i^!_{\mathscr{D}}j_{\mathscr{D},*} = 0$: these are equivalent by duality, and $i^!_{\mathscr{D}}j_{\mathscr{D},*} = i^*j_*[?]$ on $\mathcal{O}$-modules. So it suffices to observe that $i^*j_* = 0$ as functors on $\mathsf{D}_{\mathrm{qc}}(\mathcal{O}_U)$. (exercise)

(13) $j^*_{\mathscr{D}} i_{\mathscr{D},*} = 0$ (and same for any such functor obtained by replacing $*$ with !): Formation of $i_{\mathscr{D},*}$ commutes with restriction to an open in X, so the claim is clear as $i^{-1}(U) = 0$ (and $i_{\mathscr{D},*} = i_{\mathscr{D},!}, j^*_{\mathscr{D}} = j^!_{\mathscr{D}}$)

$$\begin{array}{ccc} i^{-1}U & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \xrightarrow{i} & X \end{array}$$

Corollary 21.3. *For any map $f : X \to Y$ of smooth k-varieties, there exists a natural map $\eta_f : f_{\mathscr{D},!} \to f_{\mathscr{D},*}$ of functors $\mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_X) \to \mathsf{D}^b_{\mathrm{Hol}}(\mathscr{D}_Y)$ such that*

(1) If f is proper, then $\eta_f = \mathrm{id}$ (under $f_{\mathscr{D},!} = f_{\mathscr{D},*}$)
(2) If f is an open immersion, then
$$\eta_f : f_{\mathscr{D},!} \to f_{\mathscr{D},*}$$
is adjoint to $\mathrm{id} \cong f^!_{\mathscr{D}} f_{\mathscr{D},*}$ from (10)
(3) η_f is compatible with compositions.

Proof left as exercise

21.2. **Intermediate extensions.** Say $f : Z \to X$ is a locally closed immersion of smooth k-varieties. Then $f_{\mathscr{D},!}$ is right t-exact, $f_{\mathscr{D},*}$ is left t-exact. So for any $M \in \mathrm{Hol}_{\mathscr{D}_Z}$, we have a factorization

$$\begin{array}{ccc} f_{\mathscr{D},!} & \xrightarrow{\eta_f} & f_{\mathscr{D},*}(M) \\ \downarrow & & \uparrow \\ \mathrm{H}^0(f_{\mathscr{D},!}(M)) & \xrightarrow{\alpha_M} & \mathrm{H}^0(f_{\mathscr{D},*}(M)) \end{array}$$

Here we are using $K \in \mathsf{D}^{\leqslant 0}$ and $L \in \mathsf{D}^{\geqslant 0}$, then $\mathrm{Hom}(K, L) = \mathrm{Hom}(\mathrm{H}^0 K, \mathrm{H}^0 L)$.

Definition 21.4. intermediate extension: $\mathrm{IC}_Z(M) \coloneqq f_{\mathscr{D},!*}(M) \coloneqq \mathrm{Im}(\alpha_M) \in \mathrm{Hol}_{\mathscr{D}_X}$

Example 21.5. For $j : U \hookrightarrow X$ where $U = \mathbb{G}_m$ and $X = \mathbb{A}^1$. We saw earlier:

$$j_{\mathscr{D},!}\mathcal{O}_U : 0 \to \delta_0 \to j_{\mathscr{D},!}(\mathcal{O}_U) \to \mathcal{O}_X \to 0$$
$$j_{\mathscr{D},*}\mathcal{O}_U : 0 \to \mathcal{O}_X \to j_{\mathscr{D},*}(\mathcal{O}_U) \to \delta_0 \to 0$$

The map $\alpha_M : j_{\mathscr{D},!}(\mathcal{O}_U) \to j_{\mathscr{D},*}(\mathcal{O}_U)$:

$\alpha_M(\delta_0) = 0 : j_{\mathscr{D},*}(\mathcal{O}_U)(= k[x, x^{-1}])$ has no x-torsion elements.

Have an induced map

$$\begin{array}{ccccc} & & j_{\mathscr{D},!}(\mathcal{O}_U) & \twoheadrightarrow & \mathcal{O}_X \\ & & \downarrow{\scriptstyle \alpha_M} & \swarrow & \\ \mathcal{O}_X & \longrightarrow & j_{\mathscr{D},*}(\mathcal{O}_U) & \longrightarrow & \delta_0 \end{array}$$

But $\mathrm{Hom}_{\mathscr{D}}(\mathcal{O}_X, \delta_0) = 0$ as $\mathcal{O}_X$ and δ_0 are distinct simples.

So α_M factors as

$$\begin{array}{ccc} & j_{\mathscr{D},!}(\mathcal{O}_U) \longrightarrow & \mathcal{O}_X \\ & \swarrow & \\ \mathcal{O}_X \longrightarrow & j_{\mathscr{D},*}(\mathcal{O}_U) & \end{array}$$

Now α is the multiplication by an element of k^*: scalar because $\mathcal{O}_X$ is simple, nonzero because $\alpha_M|_U = \mathrm{id}$.

22. 11/18/2020

22.1. **IC extensions.** Last time: $f : Z \to X$ locally closed immersion, induces

$$f_{\mathscr{D},!*}(-) = \mathrm{IC}_Z(-) : \mathrm{Hol}_{\mathscr{D}_Z} \to \mathrm{Hol}_{\mathscr{D}_X}$$
$$M \mapsto \mathrm{Im}(\mathrm{H}^0(f_{\mathscr{D},!}(M)) \overset{\alpha_M}{\to} \mathrm{H}^0(f_{\mathscr{D},*}(M)))$$

Example 22.1. For $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$, we have $\mathrm{IC}_U(\mathcal{O}_U) = \mathcal{O}_X$.

Today: Classify all simple holonomic $\mathscr{D}_X$-modules as IC extensions and intersection cohomology.

Lemma 22.2. *Say $f : Z \to X$ locally closed immersion of smooth k-varieties*

(1) $f_{\mathscr{D},!*}\mathbb{D}_Z = \mathbb{D}_X f_{\mathscr{D},!*}$ ($\Rightarrow$ If $M \in \mathrm{Hol}_Z$ self-dual, then $\mathrm{IC}_Z(M)$ is self-dual)
(2) $f^!_{\mathscr{D}} f_{\mathscr{D},!*} = \mathrm{id} = f^*_{\mathscr{D}} f_{\mathscr{D},!*}$
(3) If $M \in \mathrm{Hol}_Z$ is simple, then $f_{\mathscr{D},!*}(M)$ is the unique simple $\mathscr{D}_X$-submodule of $\mathrm{H}^0 f_{\mathscr{D},*}(M)$ (dually, unique simple quotient of $\mathrm{H}^0 f_{\mathscr{D},!}(M)$).
(4) If f is an open immersion, then $f_{\mathscr{D},!*}(M)$ is the unique extensino of M with neither subobjects nor quotients supported on $X - Z$.
(5) $f_{\mathscr{D},!*}(-)$ preserves injectives/surjectives, and gives fully faithful map $\mathrm{Hol}_Z \hookrightarrow \mathrm{Hol}_X$.

Proof. (1): Clear (unwind definitions). $\alpha_{\mathbb{D}M} = \mathbb{D}(\alpha_M)$

(2): by duality, suffices to show $f^!_{\mathscr{D}} f_{\mathscr{D},!*} = \mathrm{id}$

By factoring f, we may assume f is either a closed immersion or open immersion.

- closed immersion: OK because $f_{\mathscr{D},!*} = f_{\mathscr{D},!} = f_{\mathscr{D},*}$ so use last time's lemma
- open immersion: we saw last time that $f^!_{\mathscr{D}} = f^*_{\mathscr{D}}$ and that they provide left-inverses to $f_{\mathscr{D},!}, f_{\mathscr{D},*}$.

(3) as above, may assume f is an open immersion. Assume f is affine (most likely not necessary) $\Rightarrow f_{\mathscr{D},!} = \mathrm{H}^0(f_{\mathscr{D},!}), f_{\mathscr{D},*} = \mathrm{H}^0(f_{\mathscr{D},*})$. Say $M \in \mathrm{Hol}_Z$ is simple ord $N \subseteq f_{\mathscr{D},*}(M)$ is simple. Goal: Show $f_{\mathscr{D},!*}(M) \subseteq N$ ($\Rightarrow$ the rest by duality)

$$0 \neq \mathrm{incl} \in \mathrm{Hom}(N, f_{\mathscr{D},*}(M)) = \mathrm{Hom}(f^*_{\mathscr{D}} N, M)$$

So $\mathrm{incl} : N \subseteq f_{\mathscr{D},*}(M)$ is adjoint to a nonzero map $f^*_{\mathscr{D}} N \to M$.

As M is simple, this map $f^*_{\mathscr{D}} N \to M$ is surjective (in fact, isomorphism).

Consider the diagram

$$\begin{array}{ccc} f_{\mathscr{D},!}(f^*_{\mathscr{D}} N) & \longrightarrow & f_{\mathscr{D},!}(M) \\ \big\downarrow & & \big\downarrow \alpha_M \\ N & \xrightarrow{\ \mathrm{incl}\ } & f_{\mathscr{D},*}(M) \end{array}$$

The diagram is commutative and the upper arrow is surjective (because $f_{\mathscr{D},!}$ is right t-exact), which imply that $N \supseteq \mathrm{Im}(\alpha_M) = f_{\mathscr{D},!*}(M)$.

(4): f open immersion, $M \in \mathrm{Hol}_Z \Rightarrow f_{\mathscr{D},!*}(M)$ is the unique extension with no subobjects/quotients supported on $X - Z$.

This is because $f_{\mathscr{D},!*}(M) \subseteq f_{\mathscr{D},*}(M)$, so it has no subobjects supported on $X - Z$. By duality, $f_{\mathscr{D},!*}(M)$ also has no quotients supported on $X - Z$. Conversely, say $N \in \mathrm{Hol}_X$ is an extension of $M \in \mathrm{Hol}_Z$ and N has no subobjects/quotients supported on $X - Z$.

Observation:

$$N \hookrightarrow f_{\mathscr{D},*}(f_{\mathscr{D},*} N) = f_{\mathscr{D},*}(M)$$

(1) because N has no subobjects supported on $X - Z$
(2) $f_{\mathscr{D},!}(M) = f_{\mathscr{D},!}(f_{\mathscr{D}}^{*}N) \to N$ is surjective, simplicity,

So we have

$$\begin{array}{ccc} f_{\mathscr{D},!} & \xrightarrow{\alpha_M} & f_{\mathscr{D},*}(M) \\ & \searrow \quad \nearrow & \\ & N & \end{array}$$

So $\mathrm{Im}(\alpha_M) = N$, which is $f_{\mathscr{D},!*}(M) = N$.

(5): $f_{\mathscr{D},!*}$ preserves injections/surjections and gives fully faithful embedding $\mathrm{Hol}_Z \hookrightarrow \mathrm{Hol}_X$.

$$f_{\mathscr{D},!*}(-) \subseteq \mathrm{H}^0(f_{\mathscr{D},*}(-))$$

which implies that $f_{\mathscr{D},!*}(-)$ preserves injections, similarily for surjections.

Full faithfullness: for $M, N \in \mathrm{Hol}_Z$, we have

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{D}_Z}(M,N) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}_X}(f_{\mathscr{D},!*}(M), f_{\mathscr{D},!*}(N)) \\ & \overset{\mathrm{id}}{\searrow} & \downarrow f^{!}_{\mathscr{D}} \\ & & \mathrm{Hom}_{\mathscr{D}_Z}(M,N) \end{array}$$

So faithfullness is OK

For fullness (reduce to open immersions): Say $\alpha : f_{\mathscr{D},!*}(M) \to f_{\mathscr{D},!*}(N)$ such that $f^{!}_{\mathscr{D}}(\alpha) = 0$. Then $\mathrm{Im}(\alpha)$ is supported on $X - Z$, so $\mathrm{Im}(\alpha) = 0$ by (4). So $\alpha = 0$. ☐

Remark 22.3. If at (3) shows the following: Say $f : Z \to X$ is an open immersion, and $M \in \mathrm{Hol}_Z$. Then $f_{\mathscr{D},!*}(M) \subseteq \mathrm{H}^0 f_{\mathscr{D},*}(M)$ is the smallest $\mathscr{D}_X$-submodule extending M on Z. (exercise) (and dually, $f_{\mathscr{D},!}(M) \to f_{\mathscr{D},!*}(M)$ is the smallest quotient extending M on Z)

Example 22.4. Say X smooth variety, $j : U \hookrightarrow X$ dense open, (E, ∇) vector bundle with flat cone on X $\Rightarrow E = j_{\mathscr{D},!*}(E|_U)$

Proof. By remark above, $j_{\mathscr{D},!*}(E|_U) \subseteq E$. Moreover, $E/j_{\mathscr{D},!*}(E|_U)$ is a vector bundle with falt cone on X and vanishes over U $\Rightarrow$must be 0. ☐

Example 22.5. Say $U = \mathrm{D}(f) \overset{j}{\hookrightarrow} X = \mathrm{Spec}(R)$, X smooth affine k-variety. For $M \in \mathrm{Hol}_{\mathscr{D}_U}, M_0 \subseteq M$ is finitely generated R-submodule which generates M as a $\mathscr{D}_U$-module.

Can show: $j_{\mathscr{D},!*}(M) = \mathrm{D}_R \cdot (f^k M_0) \subseteq j_{\mathscr{D},*}(M)$ for $k \gg 0$

22.2. **Classification.** :

Theorem 22.6. *X smooth k-variety. Every simple object in $\mathrm{Hol}_{\mathscr{D}_X}$ has the form $f_{!*}(E)$ for $f : Z \hookrightarrow X$ locally closed smooth subvariety and (E, ∇) a simple (vector bundle with flat connection) on Z.*

Proof. We already know that such $f_{!*}(E)$ are simple,

Conversely, take $M \in \mathrm{Hol}_{\mathscr{D}_X}$ simple.

(1): If $U \overset{j}{\subseteq} X$ affine open such that $M|_U \neq 0$, then $M = j_{\mathscr{D},!*}(M|_U)$.

Proof. $M|_U \neq 0$, so $j_{\mathscr{D},!}(M|_U) \to M$ is nonzero, thus surjective as M is simple. Dually $M \to j_{\mathscr{D},*}(M|_U)$ is injective. So $M = j_{\mathscr{D},!*}(M|_U)$. ☐

Upshot: Replace X with affine open U intersecting $\mathrm{Supp}(M)$ in $\mathrm{Supp}_{\mathrm{sm}}$ to assume that $\mathrm{Supp}(M) = Z$ is smooth.

(2): Use Kashiwara to replace X with $\mathrm{Supp}(Z)$ to assume $\mathrm{Supp}(M) = X$.

(3) Repeat argument in (1) to see $M = j_{\mathscr{D},!*}(M|_U)$ for any dense affine open $U \subseteq X$. But for U sufficiently small, we know that $M|_U$ is a vector bundle. □

Construction (Intersection cohomology)

Z possibly singular variety over k. $i : Z \hookrightarrow P$ closed immersion with P smooth.

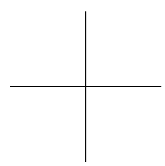
- $\mathrm{IC}_Z = \mathrm{IC}_{Z_{\mathrm{sm}}}(\mathcal{O}_{Z_{\mathrm{sm}}}) = f_{\mathscr{D},!*}(\mathcal{O}_{Z_{\mathrm{sm}}}) \in \mathrm{Hol}_P$ where $f : Z_{\mathrm{sm}} \subseteq Z \subseteq P$ (Can showL this is independent of P under Kashiwara invarionie of $\mathrm{Hol}_{Z,\mathscr{D}_P}$.)
- $\mathrm{IH}(Z) = \mathsf{R}\Gamma_Z(P, \mathrm{IC}_Z)$ (Intersection coh of Z)

(1) Z smooth: $\mathrm{IC}_Z = i_+(\mathcal{O}_Z) = i_{\mathscr{D},*}(\mathcal{O}_Z)$

$$\Rightarrow \mathrm{IH}(Z) = \mathsf{R}\Gamma_{\mathscr{D}}(P, i_+\mathcal{O}_Z) = \mathsf{R}\Gamma_{\mathscr{D}}(Z, \mathcal{O}_Z) = \mathsf{R}\Gamma_{\mathrm{dR}}(Z)[\dim Z]$$

(2) IC_Z is self-dual $\Rightarrow$ if Z is proper, then $\mathrm{IH}(Z)$ is self-dual. So $\mathrm{IH}^i(Z) \cong \mathrm{IH}^{-i}(Z)^\vee$. So we have "Poincaré duality" in IH.

Example 22.7. Let $Z = \mathbb{P}^1 \vee \mathbb{P}^1 \subseteq \mathbb{P}^2$, i.e.,



$\subseteq \mathbb{P}^2$

We have

$$\mathrm{H}^i_{\mathrm{sing}}(Z) = \begin{cases} \mathbb{C} & \deg = 0 \\ 0 & i = 1 \\ \mathbb{C}^2 & i = 2 \end{cases}$$

and

$$\mathrm{IH}_Z = \begin{cases} \mathbb{C}^2 \\ 0 \\ \mathbb{C}^2 \end{cases}$$

$\mathrm{IC}_Z = \pi_*(\mathcal{O}_{\widetilde{Z}})$ where $\pi : \widetilde{Z} \to Z$ is the normalization.

For $f : X \to Y$ proper where both X and Y are smooth.

$$\begin{aligned} \mathsf{R}f_{\mathscr{D},*}(\mathrm{IC}_X) &= \oplus_i \mathrm{H}^i(\mathsf{R}f_{\mathscr{D},*}\mathrm{IC}_X)[-i] \\ &= \oplus_i \left(\oplus_j \mathrm{IC}_{Z_j}(\underbrace{L_j}_{\text{simple local system (=vector bundle with flat conn)}}) \right)[-i] \end{aligned}$$

23. 11/30/2020

23.1. **D-modules in char p.** In char 0, we have seen two interesting examples of $\mathscr{D}$-modules:

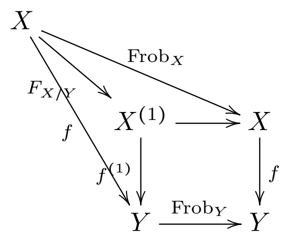
(1) If $I \subseteq \mathcal{O}_X$, then $\mathrm{H}^i_J(\mathcal{O}_X)$ has a $\mathscr{D}$-module structure, which has applications to commutative algerba.
(2) Any "flat vector bundle" (E, ∇) gives a $\mathscr{D}$-module. E.g., $f : Y \to X$ smooth proper $\Rightarrow \mathsf{R}^i f_{\mathscr{D},*}(\mathcal{O}_Y) = \left(\mathrm{H}^i(\mathsf{R}f_*\Omega^\bullet_{Y/X}), \nabla_{\mathrm{GM}}\right)$ which is a flat vector bundle

In char p, these lead to different notions:

(1) Grothedieck $\mathscr{D}$-modules
(2) Crystalline $\mathscr{D}$-modules

Review Y scheme of characteristic p, $\mathrm{Frob}_Y : Y \to Y$

- $E \in \mathbf{QCoh}(Y) \Rightarrow E^{(1)} = \mathrm{Frob}_Y^*(E)$
- Given $f : X \to Y$, have the full diagram:



where $F_{X/Y}$ =relative Frobenius exercise f smooth $\Rightarrow F_{X/Y}$ is flat
- k perfect field of characteristic p, R smooth k-algebra.

Example 23.1. $R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Let $\sigma(f)$ be the map applying Frobenius to the coefficients of f. Then $R^{(1)} = k[x_1, \ldots, x_n]/(\sigma(f)_1, \ldots, \sigma(f)_r)$.

The relative Frobenius

$$R^{(1)} = \frac{k[x_1, \ldots, x_n]}{(\sigma(f)_1, \ldots, \sigma(f)_r)} \to \frac{k[x_1, \ldots, x_n]}{(f_1, \ldots, f_r)} = R$$

is the unique k-algebra map sending x_i to x_i^p.

23.2. **Grothendieck differential operators.**

Definition 23.2. R/k as before. $\mathrm{D}^G_{R/k} \subseteq \mathrm{End}_k(R)$ is a filtered ring defined as follows:

- $\mathrm{F}_0 D^G - R/k = R \subseteq \mathrm{End}_k(R)$
- $\mathrm{F}_i D^G_{R/k} = \{f \in \mathrm{End}_k(R) | [f, \mathrm{F}_0 D^G_{R/k}] \in \mathrm{F}_{i-1} D^G_{R/k}\}$

and

$$D^G_{R/k} = \cup_n \mathrm{F}_n D^G_{R/k} \subseteq \mathrm{End}_k(R)$$

Proposition 23.3. *(1) $(D^G_{R/k}, \mathrm{F}_\bullet)$ is an almost commutative filtered ring with* $\mathrm{gr}_* D^G_{R/k} = \Gamma^*_R(\mathrm{T}_{R/k})$
(2) $D^G_{R/k} = \cup_{e \geqslant 0} \mathrm{Hom}_{R^{(e)}}(R, R) \subseteq \mathrm{End}_k(R)$

Proof. (1): Use $D^G_{R/k} = \cup_n \mathrm{Hom}_R(\mathrm{P}^n(R/k), R)$ where $\mathrm{P}^n(R/k) = (R \otimes_k R)/I^n_\Delta$ and $\mathrm{gr}_{I^\bullet}(R \otimes_k R) = \mathrm{Sym}^*_R(I/I^2)$ (By smoothness of R/k)

(2): The projective systems $\{\frac{R \otimes_k R}{I^n_\Delta}\}_{n \geqslant 1}$ and $\{\frac{R \otimes_k R}{I^{[p^e]}_\Delta}\}_{e \geqslant 1}$ are pro-isomorphic, so

$$D^G_{R/k} = \cup_e \mathrm{Hom}_R(\frac{R \otimes_k R}{I^{[p^e]}_\Delta}, R)$$

Now use $R \otimes_k R/I^{[p^e]}_\Delta = R \otimes_{R^{(e)}} R$ to get

$$D^G_{R/k} = \cup_e \mathrm{Hom}_R(R \otimes_{R^{(e)}} R, R) = \cup_e \mathrm{Hom}_{R^{(e)}}(R, R)$$

□

Example 23.4. Let $R = k[x]$. For all $n \geqslant 1$, the operator "$\frac{1}{n!}\left(\frac{\partial}{\partial x}\right)^n(-)$" makes sense as an operator on R. In fact, one can show that

$$D^G_{R/k} = k[x, \{\frac{1}{n!}\left(\frac{\partial}{\partial x}\right)^n\}_{n\geqslant 1}]$$

Definition 23.5 (F-divided modules)**.** We havea a projective system of rings:

$$R^{(\bullet)} = \{\cdots R^{(e+1)} \overset{F_{R^{(e)}/k}}{\to} R^{(e)} \to \cdots \to R^{(1)} \overset{F_{R/k}}{\to} R\}$$

The category of F-divisible modules are

$$\mathbf{Mod}^b_R = \varprojlim \mathbf{Mod}_{R^{(\bullet)}} = \{(M_e, \alpha_e) | M_e \in \mathbf{Mod}_{R^{(e)}}, \alpha_e : F^*_{R^{(e)}/k}(M_{(e+1)}) \cong M_e\}$$

exercise: If $(M_e, \alpha_e) \in \mathbf{Mod}^b_R$ with M_0 finitely generated over R, then each M_e is a vector bundle.

Theorem 23.6. *The following category are equivalent*

(1) $\mathbf{Mod}_{R,\mathrm{strat}}$ *is the collections of* $M \in \mathbf{Mod}_R$ *with* $\alpha_n : \mathrm{pr}_1^* M \cong \mathrm{pr}_2^* M$ *over* $R \otimes_k R/I^n_\Delta$ *satisfying cocycle condition on* $R \otimes R \otimes R/I^n_\Delta$*. (=*$\mathbf{QCoh}((\mathrm{Spec}(R)/k)_{\mathrm{strat}})$*)*
(2) $\mathbf{Mod}^b_R$
(3) $\mathbf{Mod}_{D^G_{R/k}}$.

Proof. (2) ⇔ (3) is due to Katz-Greseber in locally free cases

(1) ⇔ (2): Let $R^\bullet$ be the 2-truncated Cech rsolution of $k \to R$, i.e.,

$$R \rightrightarrows R \otimes R \mathrel{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}} R \otimes R \otimes R$$

Take $I^\bullet \subseteq R^\bullet$ to be the ideal of diagonals, i.e.

$$0 \rightrightarrows I_\Delta \mathrel{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}} I_\Delta$$

Since

$$\mathbf{Mod}_{R,\mathrm{strat}} = \varprojlim_\bullet \varprojlim_n \mathbf{Mod}_{R^\bullet/(I^\bullet)^n} = \varprojlim_\bullet \varprojlim_n (\ \mathbf{Mod}_R \rightrightarrows \mathbf{Mod}_{R\otimes R/I^n_\Delta} \mathrel{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}} \mathbf{Mod}_{R\otimes R\otimes R/I^n_\Delta}\ )$$

Now $\{R^\bullet/(I^\bullet_\Delta)^n\}_{n\geqslant 1}$ and $\{R^\bullet/(I^\bullet_\Delta)^{[p^e]}\}_{e\geqslant 1}$ are pro-isomorphic, so

$$\mathbf{Mod}_{R,\mathrm{strat}} = \varprojlim_\bullet \varprojlim_e \mathbf{Mod}_{R^\bullet/(I^\bullet)^{[p^e]}}$$

But

$$R^\bullet/(I^\bullet_\Delta)^{[p^e]} = \left(\ R \rightrightarrows R \otimes_{R^{(e)}} R \mathrel{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}} R \otimes_{R^{(e)}} R \otimes_{R^{(e)}} R\ \right)$$

This is the Cech nerve if $R^{(e)} \to R$ and $R^{(e)} \to R$ is faithfully flat.

(if $A \to B$ if faithfully flat, then

$$\mathbf{Mod}_A \cong \varprojlim (\ \mathbf{Mod}_B \rightrightarrows \mathbf{Mod}_{B\otimes_A B} \mathrel{\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}} \mathbf{Mod}_{B\otimes_A B\otimes_A B}\ )$$

) So we have faithfully flat descent, and $\mathbf{Mod}_{R,\mathrm{strat}} = \varprojlim_e \mathbf{Mod}_{R^{(e)}} = \mathbf{Mod}^{b-}_R$.

(2) ⇔ (3):

$$D^G_{R/k} = \varinjlim_e \mathrm{Hom}_{R^{(e)}}(R, R)$$

So

$$\mathbf{Mod}_{D^G_{R/k}} \cong \varprojlim_e \mathbf{Mod}_{\mathrm{Hom}_{R^{(e)}}(R,R)}$$

As R/k is smooth, the map $R^{(e)} \to R$ is finite locally free. R commutative rings, E vector bundle over R (supported everywhere), then $\mathbf{Mod}_R \cong \mathbf{Mod}_{\mathrm{End}(E)}$ via $M \mapsto M \otimes_R E$

Monrita theory:

$$\mathbf{Mod}_{R^{(e)}} \cong \mathbf{Mod}_{\mathrm{Hom}_{R^{(e)}}(R,R)}$$
$$M \mapsto M \otimes_{R^{(e)}} R$$

So we have

$$\mathbf{Mod}_{D^G_{R/k}} \cong \varprojlim_{e} \mathbf{Mod}_{R^{(e)}} = \mathbf{Mod}^b_R$$

□

Example 23.7. Let $k = \mathbb{F}_p$.

(1) (Unit Frobenius modules): Say $M \in \mathbf{Mod}_R$ equipped with $\varphi_M : \mathrm{Frob}^* M \cong M$. So (M, φ_M) gives an object $(M_e, \alpha_e) \in \mathbf{Mod}^b_R$: $M_e = M, \alpha_e = \alpha_M$.
(2) (Local systems): Katz showed that (M, φ_M) as above with M finite locally free correspond to étale $\mathbb{F}_p$-local systems L on $\mathrm{Spec}(R)_{\text{ét}}$ via $L \mapsto \big(L \otimes_{\mathbb{F}_p} \mathcal{O}_X, \text{obvious}\big)$
(3) Say $I \subseteq R$ is an ideal. Then $\mathrm{Frob}^* \mathsf{R}\Gamma_I(R) \cong \mathsf{R}\Gamma_I(R)$ naturally. As Frob is flat, this gives $\mathrm{Frob}^* \mathrm{H}^i_I(R) \cong \mathrm{H}^i_I(R)$. So each $\mathrm{H}^i_I(R)$ gives a unit Frobenius module and thus an object of $\mathbf{Mod}^b_R$ or $\mathbf{Mod}_{D^G_{R/k}}$.

Remark 23.8 (The Gauss Morita connection is not natually a $\mathscr{D}$-module). Say $f : X \to S$ is a proper smooth map of k-schemes. $M = M_{X/S} = \mathsf{R}f_*(\Omega^\bullet_{X/S}) \in \mathsf{D}^b_{\mathbf{Coh}}(S)$. The Coh groups $\mathrm{H}^i(M)$ carry the Gauss-Morita connection

Claim. *There is no functorial (in X) $\mathscr{D}$-module structure on* $\mathrm{H}^i(M)$

We have an S-linear relative Frob map

$$\begin{array}{ccc} X & \xrightarrow{F_{X/S}} & X^{(1)} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^{(1)}} & \\ & S & \end{array}$$

$$H : M_{X^{(1)}/S} \overset{F^*_{X/S}}{\to} M_{X/S} = M$$

Now, assumeing S is smooth, thus gives

$$H : \mathrm{Frob}^*_S \mathrm{H}^i(M_{X/S}) = \mathrm{H}^i(M_{X^{(1)}/S}) \overset{F^*}{\to} \mathrm{H}^i(M_{X/S})$$

Example 23.9. Let $X \overset{f}{\to} S$ be a universal family of dimension 1 of elliptic curves. Then $H : \mathrm{Frob}^*_S \mathrm{H}^1_{\mathrm{dR}}(X/S) \overset{F^*}{\to} \mathrm{H}^1_{\mathrm{dR}}(X/S)$ (rank 2 vector bundle)

Fact 23.10.

$$\{s \in S | H \otimes k(s) = H(s) = 0\} = \{s \in S | X_S \textit{ is a ss ell curve}/k(s)\}$$

So H is a map of (rk 2) vector bundle with finitely many zeros. But any map of vector bundles supported on a $\mathscr{D}$-module structure cannot be 0 at only finitely many points (exercise)

So $\mathrm{H}^1(M_{X/S})$ does not support on a $\mathscr{D}$-module structure.

Groth "Crystals and dR coh of schemes"

24. 12/02/2020

24.1. **Crystalline differential operators.** ref:

- Berthelot-Ogus "Notes on crystalline cohomology"
- N. Katz: "Nilpotent connections"
- Bezrukonikov-Mirakovic-rymynin: "Localization of modules"
- Ogus-Vologodsky: "Non-abelian Hodge theory in char p"

k perfect field of characteristic p, X/k smooth variety, R/k smooth k-algebra

Definition 24.1. Recall $\mathrm{T}_{X/k} = \underline{\mathrm{Der}}_k(\mathcal{O}_X, \mathcal{O}_X)$ is still a Lie subalgebra of $\underline{\mathrm{End}}_k(\mathcal{O}_X)$ (in fact, Lie algebraoid over $\mathcal{O}_X$)

Let $D_{X/k}$ be the universal enveloping algebra of the Liesubalgebra. Then

$$D_{X/k} = \left(\oplus_{n\geqslant 0} \mathrm{T}_{X/k}^{\otimes_{\mathcal{O}_X} n}\right)/\sim$$

where $\sim$ is spanned by two kinds of relations

- $a \in \mathrm{T}_{X/k}$, $f \in \mathcal{O}_X$, $[a, f] = a(f)$.
- $a, b \in \mathrm{T}_{X/k}$, Then $[a, b]_{\mathrm{T}_{X/k}} = [a, b]_{D_{X/k}}$.

Lemma 24.2. *(1) $D_{X/k}$ is naturally an almost commutative filtered ring with $\mathrm{gr}_* D_{X/k} = \mathrm{Sym}^*_{\mathcal{O}_X}(\mathrm{T}_{X/k})$.*
(2) $\mathbf{QCoh}(D_{X/k}) \cong \{(M, \nabla)\}$ where $M \in \mathbf{QCoh}(X)$ and $\nabla : M \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} M$ is a flat connection.
(3) $\mathcal{O}_{X^p} \subseteq \mathcal{O}_X \subseteq \mathscr{D}_{X/k}$ is contained in the center.

Proof. (1)+(2): same as before

(3): STS $[f^p, g] = 0$ for any $g \in \mathcal{O}_X$ and $[f^p, a] = 0$ for any $a \in \mathrm{T}_{X/k}$.

The first one is clear

The second one

$$\begin{aligned}[f^p, a] &= -[a, f^p] \\ &= -a(f^p) \\ &= 0\end{aligned}$$

□

Example 24.3 (The Gauss-Marnim connectoin; See Katz's paper)**.** Say $f : Y \to X$ is a smooth map, assume X is curve

Claim. *Each $\mathsf{R}^i f_*(\Omega^\bullet_{Y/X})$ admits a natural flat connection over X.*

Proof. There exists a natural SES (of complexes)

$$f^*\Omega^1_X \otimes \Omega^\bullet_{Y/X}[-1] \to \Omega^\bullet_{Y/k} \to \Omega^\bullet_{Y/X}$$

of $f^{-1}\mathcal{O}_X$-modules on Y (in deg 1, get usual SES for Ω^1)

Apply $\mathsf{R}f_*$ and look at bdy map:

$$\mathsf{R}f_*(\Omega^\bullet_{Y/X}) \xrightarrow{\delta} \mathsf{R}f_*(f^*\Omega^1_X \otimes_{\mathcal{O}_Y} \Omega^\bullet_{Y/X}) = \Omega^1_X \otimes_{\mathcal{O}_X} \mathsf{R}f_*(\Omega^\bullet_{Y/X})$$

Here δ gives GM connection on Coh groups. □

Remark 24.4. (Berthelot-Osus) Let (D, J) be the PD-envelope of $(X \times X, I_\Delta)$. Then $\mathcal{O}_D$ carries a natural filtration by $J^{[n]} = n$th-divided power of J. One can show:

$$D_{X/k} = \mathrm{Colim}_n \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_{D/J^{[n]}}, \mathcal{O}_X)$$

(in char 0, $\mathcal{O}_{D/J^{[n]}} = \mathrm{P}^n(\mathcal{O}_X/k)) \Rightarrow \mathbf{QCoh}(D_{X/k}) \cong$ the collection of $M \in \mathbf{QCoh}(X)$ with $\varepsilon_n : \mathrm{pr}_1^*(M) \cong \mathrm{pr}_2^*(M)$ on $\mathcal{O}_{D/J^{[n]}}$ and compatible conditions.

Example 24.5.

$$\mathscr{D}_{k[x_1,\dots,x_n]/k} = k[x_1,\dots,x_n] \otimes_k k[\partial_1,\dots,\partial_n]$$

where $\partial_i = \frac{\partial}{\partial x_i}$ (same as char 0)

Warning: $D_{X/k}$ does not act faithfully on $\mathcal{O}_X$. For example: $X = \mathbb{A}^1$, $D = x\frac{\mathrm{d}}{\mathrm{d}x} \in \mathrm{T}_X \in \mathscr{D}_{X/k}$. Then $D^p(f(x)) = D(f(x))$, but $D^p \neq D$.

(1) $D^n(x^m) = D^{n-1}(D(x^m)) = D^{n-1}(mx^m)$. So $D^p(x^m) = m^p x^m \overset{\mathrm{FLT}}{=} mx^m = D(x^m)$
(2) $D^p \neq D$: $D^p \in \mathrm{F}_p D_{X/k} - \mathrm{F}_{p-1} D_{X/k}$ so $D \in \mathrm{F}_1 D_{X/k} \subseteq \mathrm{F}_{p-1} D_{X/k}$. So $D^p \neq D$.

24.2. **Relation between $D_{R/k}$ and $D^G_{R/k}$.** Observation: The action of $D_{R/k}$ on R gives a map $D_{R/k} \to \mathrm{End}_k(R)$ whose image is inside $D^G_{R/k}$ (check on generators).

So we get a map $D_{R/k} \xrightarrow{\mu} D^G_{R/k}$ of rings.

(1) μ is not injective (previous warning)
(2) μ is not surjective. In fact, μ factors as

$$D_{R/k} \to \mathrm{Hom}_{R^{(1)}}(R,R) \subseteq \bigcup_e \mathrm{Hom}_{R^{(e)}}(R,R) = D^G_{R/k}$$

Proof. Suffices to show $\mu(f)\mu(a) \in \mathrm{Hom}_{R^{(1)}}(R,R)$ for all $f \in R, a \in \mathrm{T}_{R/k}$.
- $\mu(f) \in \mathrm{Hom}_R(R,R) \subseteq \mathrm{Hom}_{R^{(1)}}(R,R)$.
- $\mu(a) \in \mathrm{Hom}_{R^{(1)}}(R,R)$: Need to show that $a(f^p g) = f^p a(g)$ for all $f, g \in R$, which is clear by Leibnize rule.

□

(3) μ carries the order filtration on $D_{R/k}$ into that of $D^G_{R/k}$

Proof. Clear in deg $\leqslant 1$, rest follows by multiplicatively as $\mathrm{F}_i D_{R/k}$ is generated by F_1^i. □

(4) The induced map $\mathrm{gr}(\mu) : \mathrm{gr}_*(D_{R/k}) \to \mathrm{gr}_*(D^G_{R/k})$ is the natural map $\mathrm{Sym}^*_R(\mathrm{T}_{R/k}) \to \Gamma^*_R(\mathrm{T}_{R/k})$. (Pf: check on gr^1) So μ has a huge kernel.

Exercise 24.6. We saw a map

$$D_{R/k} \to \mathrm{Hom}_{R^{(1)}}(R,R) \subseteq D^G_{R/k}$$

so we get a restriction of scalars.

$$\begin{array}{ccc}
\mathbf{Mod}_{\mathrm{Hom}_{R^{(1)}}(R,R)} & \longrightarrow & \mathbf{Mod}_{D_{R/k}} \\
{\scriptstyle N \mapsto N \otimes_{R^{(1)}} R}\uparrow\ \downarrow{\scriptstyle \simeq(\text{Morita theory})} & & \downarrow{\scriptstyle \simeq} \\
\mathbf{Mod}_{R^{(1)}} & \xrightarrow[N \mapsto (N \otimes_{R^{(1)}} R, \mathrm{id} \otimes \mathrm{d})]{} & \{(M,\nabla) \mid M \in \mathbf{Mod}_R, \nabla \text{ flat conn}\}
\end{array}$$

where $\mathrm{id} \otimes \mathrm{d}$ is the "Frobenius descent connection".

p-curvature: R/k smooth algebra:

Obs: Given $a \in \mathrm{T}_{R/k} \subseteq \mathrm{End}_k(R)$, the pth power $a^p \in \mathrm{End}_k(R)$ is also a derivation (so lies in $\mathrm{T}_{R/k}$)

Proof.

$$\begin{aligned} a^p(fg) &= \sum_{i=0}^{p} \binom{p}{i} a^i(f) a^{p-i}(g) \\ &= \binom{p}{0} a^0(f) a^p(g) + \binom{p}{p} a^p(f) a^0(g) \\ &= f a^p(g) + a^p(f) g \end{aligned}$$

So get p-power map $\mathrm{T}_{R/k} \overset{a \mapsto a^{[p]}}{\to} \mathrm{T}_{R/k}$. □

Example 24.7. $R = k[x]$

(1) $a = \frac{\mathrm{d}}{\mathrm{d}x}$: $a^{[p]} = 0$ (check on $x \in k[x]$)
(2) $a = x\frac{\mathrm{d}}{\mathrm{d}x}$: $a^{[p]} = a$ (Pf: $a^{[p]}(x) = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^p(x) = x = a(x)$)

Definition 24.8. For any $a \in \mathrm{T}_{R/k}$, write

(1) $\psi(a) = a^p - a^{[p]} \in D_{R/k}$. Since $a^p \in \mathrm{F}_p(D_{R/k})$ and $a^{[p]} \in \mathrm{F}_1 D_{R/k} \in \mathrm{F}_p D_{R/k}$.
(2) A $D_{R/k}$-module M has *p-curvature 0* if $\psi(a)$ kills M for any $a \in \mathrm{T}_{R/k}$.
(3) A $D_{R/k}$-module M has *nilpotent p-curvature* if there exists a finite filtration of M whose graded pieces has 0 p-curvature.

Example 24.9. (1): Given $N \in \mathbf{Mod}_{R^{(1)}}$, we had $(M, \nabla) = (N \otimes_{R^{(1)}} R, \mathrm{id} \otimes \mathrm{d}) \in \mathbf{Mod}_{D_{R/k}}$.

Claim. *This has p-curvature 0*

Proof. $N = R^{(1)} \rightsquigarrow (M, \nabla) = (R, \mathrm{d})$. (R, d) has p-curvature 0 because $D_{R/k} \to \mathrm{End}_k(R), \psi(a) \mapsto a^p - a^p = 0$ is a ring map. In general, write N as a quotient of a free $R^{(1)}$-module. □

(2): Say $f : X \to \mathrm{Spec}(R)$ is a smooth morphism. Then $(\mathsf{R}^i f_*(\Omega^\bullet_{X/R}), \nabla_{\mathrm{GM}})$ has nilpotent p-curvature.

(Pf idea: use the conjugate spectral sequence. $E_2^{ij} : \mathrm{Frob}_S^* \mathsf{R}^i f_* \Omega^j_{X/R} \Rightarrow \mathsf{R}^{i+j} f_* \Omega^\bullet_{X/R}$. Can show : this is a SS in $\mathbf{Mod}_{D_{R/k}}$ where E_2 term has Frob descent connection with p-curvature 0). See Katz's paper

(3) $R = k[x]$, $M = R \cdot e$ $\nabla = \mathrm{d} + \mathrm{d}x$. So $\frac{\mathrm{d}}{\mathrm{d}x} e = e$.

$$\begin{aligned} \psi\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)(e) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p (e) - \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{[p]} (e) \\ &= e - 0 = e \end{aligned}$$

So $\psi(\frac{\mathrm{d}}{\mathrm{d}x})$ is not nilpotent, so M does not have nilpotent p-curvature.

$$\begin{array}{ccc} \mathbb{P}^1 & \hookleftarrow & \mathbb{A}^1 \\ \uparrow & & \\ \mathbb{A}^1 & & \end{array}$$

25. 12/07/2020

25.1. **The p-curvature.** Last time: R smooth k-algebra (k perfect field of char p)

$$\begin{aligned} \psi : \mathrm{T}_{R/k} &\to D_{R/k} \\ a &\mapsto a^p - a^{[p]} \in \mathrm{F}_p D_{R/k} \subseteq D_{R/k} \end{aligned}$$

Lemma 25.1. *(1) ψ is p-linear, i.e., ψ induces an $R^{(1)}$-linear map*

$$\mathrm{Frob}_k^* \mathrm{T}_{R/k} = \mathrm{T}_{R^{(1)}/k} \to \mathrm{Frob}_{R/k,*} D_{R/k}$$

(2) $\mathrm{Im}(\psi) \subseteq Z(\mathrm{Frob}_{R/k,*} D_{R/k}) = \mathrm{Frob}_{R/k,*} Z(D_{R/k})$

Proof. We shall use: Given $D_1, D_2 \in \mathrm{F}_p D_{R/k}$, we have $D_1 = D_2$ if and only if

(1) $D_1 = D_2$ in $\mathrm{gr}_p D_{R/k}$
(2) D_1 and D_2 act in the same way on R.

(Pf idea: $\mathrm{F}_{p-1} D_{R/k} \hookrightarrow D^G_{R/k} \subseteq \mathrm{End}_k(R)$)

(1) ψ is additive: Fix $a, b \in \mathrm{T}_{R/k}$. WTS: $\psi(a+b) = (a+b)^p - (a+b)^{[p]} = \psi(a) + \psi(b)$.

Using the criterion above, we must how

(1) $a^p + b^p = (a+b)^p$ in $\mathrm{gr}_p D_{R/k}$ (because $c^{[p]} = 0$ in gr_p) clear because $\mathrm{gr}_* D_{R/k}$ is commutative of char p.
(2) Clear because a^p and $a^{[p]}$ act in the same way on R by definitino of $a^{[p]}$.

ψ is p-linear, i.e., $\psi(fa) = f^p \psi(a)$ for $f \in R, a \in \mathrm{T}_{R/k}$. This is proven similarily.

(2): $\psi(a) \in Z(D_{R/k})$ for all $a \in \mathrm{T}_{R/k}$.

The claim is local on $\mathrm{Spec}(R)$. So we may choose étale coordinates $x_1, \ldots, x_n$. So $\mathrm{T}_{R^{(1)}/k} = \bigoplus_{i=1}^n R^{(1)} \partial_i$ where $\partial_i = \frac{\mathrm{d}}{\mathrm{d}x_i}$.

By (1), suffices to show $\psi(\partial_i) \in Z(D_{R/k})$.

Recall: $\psi(\partial_i) = \partial_i^p$ (because $\partial_i^{[p]} = 0$)

Must show $\partial_i^p \in Z(D_{R/k}) \Leftrightarrow [\partial_i^p, f] = 0$ for any $f \in R$.

We must show $\mathrm{ad}(\partial_i^p)(f) = 0$ for any $f \in R$.

check Obs: $\mathrm{ad}(a^p) = \mathrm{ad}(a)^p$ in char associated $\mathbb{F}_p$-algebra

Must show $\mathrm{ad}(\partial_i)^p(f) = 0$.

But $\mathrm{ad}(\partial_i)^p(f) = (\partial_i)^p\,(f) = 0$ because in char p. So $\psi(\mathrm{T}_{R^{(1)}/k}) \in Z(D_{R/k})$. □

Lemma 25.2. *get an $R^{(1)}$-algebra map*

$$\mathrm{Sym}_{R^{(1)}}(\mathrm{T}_{R^{(1)}/k}) \to Z(\mathrm{Frob}_{R/k,*} D_{R/k}) \subseteq F_{R/k,*} D_{R/k}$$

Upshot: For any $M \in \mathbf{Mod}_{D_{R/k}}$, get an action of $\mathrm{Sym}^*_{R^{(1)}}(\mathrm{T}_{R^{(1)}/k})$ or $\mathrm{Frob}_{R/k,*} M$

$$\rightsquigarrow \Theta : M \to \mathrm{Frob}^*_{R/k} \Omega^1_{R^{(1)}/k} \otimes_R M$$

"F-Higgs field": R-linear, $\Theta \wedge \Theta = 0$.

Theorem 25.3 (Bezrukavnikov-Mirakovic-Rumyrim)**.** $E = \mathrm{Frob}_{R/k,*} D_{R/k} \supseteq Z(E)$

(1) The p-curvature map $\psi : \mathrm{Sym}_{R^{(1)}}(\mathrm{T}_{R^{(1)}/k}) \to Z(E)$ is an isomorphism
(2) E is an Azumaya algebra over $Z(E)$, i.e., there exists a faithfully flat map $Z(E) \to Z'$ such that $E \otimes_{Z(E)} Z' \cong M_{p \dim(X)}(Z')$

Proof Sketch. (1) Claim is étale local, so we may assume $R = k[x_1, \ldots, x_n]$, $E = k[x_1, \ldots, x_n] \otimes_k k[\partial_1, \ldots, \partial_n]$.

Since $\psi(\partial_i) = \partial_i^p$, we must show

$$Z(E) = k[x_1^p, \dots, x_n^p] \otimes k[\partial_1^p, \dots, \partial_n^p] \\ \subseteq E = k[x_1, \dots, x_n] \otimes k[\partial_1, \dots, \partial_n]$$

Say $D = \sum_{I \subseteq S} f_I \partial^I \in Z(E)$, where $S \subseteq \mathbb{N}^n$ finite set of multiindices and $f_I \in k[x_1, \dots, x_n]$.

Must show: $S \subseteq (p\mathbb{N})^n \subseteq \mathbb{N}^n$ (and same for x_i and ∂_i-swapped) Check by hand (hint: $[D, x_j] = 0$ implies $I_j \in p\mathbb{N}$)

(2): $Z(E) = \mathrm{Sym}^*_{R^{(1)}}(\mathrm{T}_{R^{(1)}/k}) \to E$ is an Azumaya algebra

(1) Z_R =centralizer of $R \subseteq E$
Obs: $Z_R \supseteq Z(E)$, $Z_R \supseteq R$.
In fact: $Z(E) \to Z_R$ identifies with $Z(E) \otimes_{R^{(1)}} (R^{(1)} \to R)$. So $Z(E) \to Z_R$ is faithfully flat of $\deg p^{\dim(X)}$.
(2) $Z_R \subseteq E$ is a subring. Z_R acts on E via left multiplication. In fact, E is a vector bundle of rank $p^{\dim(X)}$ over Z_R.
(3) The natural map
$$R \otimes_{R^{(1)}} E \cong Z_R \otimes_{Z(E)} E \to \mathrm{End}_{Z_R}(E)$$
is an isomorphism of algebras.

In particular, $Z(E) \to E$ becomes End(v.b.) after a faithfully flat extension of $Z(E)$. □

Remark 25.4. (1) $D_{R/k}$ is *not* a split Azumaya algebra.
Proof. $D_{R/k}$ is a domain (because gr is so) while matrix algebras have zero divisors. □
(2) $D_{R/k}$ splits after base change along $\mathrm{Sym}^*_{R^{(1)}} * (\mathrm{T}_{R^{(1)}/k}) \to R^{(1)}$ (zero section) Geometrically: $D_{X/k}$ splits along $X^{(1)} \hookrightarrow \mathrm{T}^* X^{(1)}$.

This implies

Theorem 25.5 (Cartler)**.** *The following cats are equivalent:*

(a) $D_{R/k}$-modules with p-curvature $\mathcal{O}$ ($\Leftrightarrow$ modules over $D_{R/k} \otimes_{\mathrm{Sym}_{R^{(1)}}(\mathrm{T}_{R^{(1)}/k})} R^{(1)}$)
(b) $R^{(1)}$-modules

The functors are

$$(M, \nabla) \in (a) \mapsto M^{\nabla = 0} \in (b) \\ N \in (b) \mapsto \left(\mathrm{Frob}^*_{R/k} N, \nabla_{\mathrm{can}}\right) \in (a)$$

Ogus-Vologodsky: Splitting property above actually extends to the $(p-1)$-st inf nbhd of $X^{(1)} \subseteq \mathrm{T}^* X^{(1)}$. (provided X lifts to $W_2(k)$).

get "non-abelian Hodge theory in char p"

25.2. The p-curvature conjecture.

Conjecture 25.6 (Groth-Katz)**.** *R finitely generated $\mathbb{Z}$-algebra domain flat over $\mathbb{Z}$, $K = \mathrm{Frac}(R)$, S/R smooth algebra. (M, ∇)= flat vector bundle on S/R. Assume for all maximal ideals $\mathfrak{p} \subseteq R$, the connection $(M/\mathfrak{p}M, \nabla)$ on $S/\mathfrak{p}$ has a full set of solutions (i.e., p-curvature). Then there exists a finite étale cover $S_K \to T$ such that $(M, \nabla) \otimes_{S_K} T$ has a full set of solutions.*

Example 25.7. $R = \mathbb{Z}[\frac{1}{N}], S = R[x, x^{-1}]$. $(M, \nabla) = (S, \mathrm{d} - \frac{1}{Nx}\mathrm{d}x)$.

So solutions of $\nabla \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{1}{Nx} f(x)$.

Obs: (M, ∇) satisfies the hyp of conj: if $a \in \mathbb{Z}$ is such that $aN \equiv 1 \mod p$, then $x^a \in S$ provides a solution.

So conj ⇒ (M,∇) has a solution after a finite étale covers T pf $S_{\mathbb{Q}}$. In fact, $T = S_{\mathbb{Q}}[x^{1/N}]$.

Evidence:

(1) "virtually solvable monodramy" for $(M,\nabla)\otimes_S S_K$ (Chednoskyum, andré, bost)
(2) $\operatorname{Spec}(S)$ alg group (Bost)
(3) "$\operatorname{Spec}(S)$" is some class of locally symmetric case (Farb-Kisin)
(4) Gauss-Manim connections (katz)

Theorem 25.8 (Katz)**.** *(M,∇) as in the conj. Assume $(M,\nabla) = (\mathsf{R}^i f_*\Omega^{\bullet}_{Y/S}, \nabla_{\mathrm{GM}})$ for some smooth proper map $f : Y \to S$. Then conj holds true*

Proof. Hodge-to-dR SS gives a filtration on M with gr^* being

$$\mathrm{gr}^*_{\mathrm{Hodge}}(\nabla_{\mathrm{GM}}) : \mathrm{gr}^*(M) \to \Omega^1_{S/R} \otimes \mathrm{gr}^{*-1}(M)$$

Kodaira-Spencer class of f.

Step 1: Show the conclusion of the conj is implied by triviality of $\mathrm{gr}^*_{\mathrm{Hodge}}(\nabla_{\mathrm{GM}})$. The idea is to use the map $S \to$moduli space of Hodge structure $s \mapsto \mathrm{H}^i(Y_s)$. and identify $\mathrm{gr}^*_{\mathrm{Hodge}}(\nabla_{\mathrm{GM}})$ with tangent space of the above

Step 2: Reduce mod $\mathfrak{p}$ and use the conjugate filtration p-curvature map for $M/\mathfrak{p}$ preserves conj filtration and lowers deg by 1. because it's 0 on $\mathrm{gr}_* \rightsquigarrow \mathrm{gr}^{\mathrm{conj}}_*(M/\mathfrak{p}) \to \Omega^1_{S/R} \otimes \mathrm{gr}^{\mathrm{conj}}_{*-1}(M/\mathfrak{p})$. □

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