MATH 731: INTRODUCTION TO HODGE THEORY

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These are course notes for MATH731 "Introduction to Hodge Theory" taught by Professor Mircea Mustaţă, taken by Zhan Jiang, who is responsible for any and all errors. Please email zoeng@umich.edu with any corrections. All texts in blue are comments by myself.

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Course Website: http://www-personal.umich.edu/~mmustata/731-2019.html

Goal of the course: Give an introduction to the basic results in Hodge theory.

Prerequisites: Familiarity with algebraic varieties and sheaf cohomology (no familiarity with scheme theory is required) and with smooth manifolds (the tangent bundle, differential forms, integration).

Rough plan:

(1) The classical topology on a complex algebraic variety. Relation between algebraic properties and properties in the classical topology.
(2) Holomorphic functions in several complex variables. The analytic space associated to an algebraic variety. GAGA (statements). Complex manifolds.
(3) Hodge theory on Riemannian manifolds.
(4) The Hodge decomposition on complex K¨ahler manifolds.
(5) Polarizations and the Lefschetz decomposition. The Hodge Index theorem.
(6) The category of (polarized) Hodge structures.
(7) Local systems and vector bundles with integrable connection. Variations of Hodge structures.
(8) Some homological algebra: spectral sequences, hypercohomology, rudiments of derived categories.
(9) De Rham cohomology and Grothendieck's theorem (the compact case).
(10) The De Rham complex with log poles and Grothendieck's theorem (the general case).
(11) The mixed Hodge structure on the cohomology of a smooth variety.
(12) The category of mixed Hodge structures.

Depending on the time available, we might discuss also other topics, such as:

(13) Application to the proof of the Kodaira-Akizuki-Nakano vanishing theorem.
(14) The construction of the mixed Hodge structure on the cohomology of singular varieties.

Textbook: There is no official textbook, but most materials could be found in [Voi03].

Compiled on December 13, 2019.

Contents

1. September 04, 2019

1.1. **The classical topology on a complex algebraic variety.**

Setup 1.1. A complex algebraic variety X is the set of closed points of a reduced scheme of finite type over the complex numbers $\mathbb{C}$.

Suppose that X is affine. There is a closed immersion $X \hookrightarrow \mathbb{A}^N_{\mathbb{C}} = \mathbb{C}^N$. The space $\mathbb{C}^N$ carries the usual Euclidean topology. The *classical topology* on X is the induced subspace topology.

This is well-defined: Given two immersions

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{C}^N \\ & \searrow & \\ & & \mathbb{C}^{N'} \end{array},$$

there are polynomial functions $\mathbb{C}^N \to \mathbb{C}^{N'}$ and $\mathbb{C}^{N'} \to \mathbb{C}^N$ that makes the diagram commutative. Since polynomial functions $\mathbb{C}^m \to \mathbb{C}^n$ are continuous with respect to the Euclidean topology, the 2 embeddings induce the same topology. ("embedding" and "immersion" are synonym).

Remark 1.2. How to find such maps? An immersion of $X \hookrightarrow \mathbb{C}^N$ corresponds to a surjection $\mathbb{C}[x_1, \dots, x_N] \twoheadrightarrow \mathcal{O}(X)$. So the diagram is

$$\begin{array}{ccc} \mathcal{O}(X) & \twoheadleftarrow & \mathbb{C}[x_1, \dots, x_N] \\ & \nwarrow & \\ & & \mathbb{C}[y_1, \dots, y_{N'}] \end{array}.$$

We can lift the map as follows: for each y_i, there is some polynomial $f_i \in \mathbb{C}[x_1, \dots, x_N]$ maps to the image of y_i because of surjectivity, take the map to be $y_i \mapsto f_i$, and vice versa for the converse map.

Proposition 1.3. *Let X be an algebraic variety over $\mathbb{C}$.*

(1) The classical topology on X is finer than the Zariski topology.
(2) If X is affine, $Z \hookrightarrow X$ is a closed subvariety, then the classical topology on Z is the subspace topology with respect to the classical topology on X.
(3) The same holds for an open subvariety $U \hookrightarrow X$ if U is affine.

Proof. (1): For this, enough to check it for $\mathbb{C}^N$. The definition of Zariski topology and the fact that polynomial functions are continuous in Euclidean topology finish the proof.

(2): By definition.

(3): Let's assume that U is basic affine, i.e. $U = \{x \in X | f(x) \neq 0\}$ for some $f \in \mathcal{O}(X)$. Let $g \in \mathbb{C}[x_1, \dots, x_N]$ such that $g|_X = f$. Given a closed immersion $U \hookrightarrow X \overset{\eta}{\hookrightarrow} \mathbb{C}^N$, we can construct an immersion $U \hookrightarrow \mathbb{C}^{N+1}$ by sending $u \in U$ to $\{(\eta(u), t) | g(u) \cdot t = 1\}$. So we have following diagram:

$$\begin{array}{ccc} X & \overset{\eta}{\hookrightarrow} & \mathbb{C}^N \\ \uparrow & & \beta \uparrow \downarrow \alpha \\ U & \hookrightarrow & \mathbb{C}^{N+1} \end{array}$$

where $\beta : \mathbb{C}^{N+1} \to \mathbb{C}^N$ is the projection onto 1st component and $\alpha : \mathbb{C}^N \setminus V(g) \to \mathbb{C}^{N+1}, u \mapsto (u, \frac{1}{g(u)})$. The two maps α, β are both continuous with respect to Euclidean topology. Hence the topology on U as subspace of $\mathbb{C}^N$ and $\mathbb{C}^{N+1}$ coincide.

If U is not basic open, we can cover it by basic open subsets $U = \cup U_i$. The classical topology on U_i is the subspace topology on U_i with respect to both classical topology on U and X from what we have proved. So the classical topology on U concide with the subspace topology from the classical topology of X. $\square$

Let's glue the construction: Let X be an algebraic variety over $\mathbb{C}$. Take an affine open cover $X = \cup_{i=1}^r U_i$. Then we know that

(1) $U_i \cap U_j$ is open in both U_i, U_j with respect to the classical topology.
(2) By covering $U_i \cap U_j$ by affine open subsets. Proposition 1.3 (3) implies that the classical topology on U_i and on U_j induce the same subspace topology on $U_i \cap U_j$. In this case, there is a unique topology on X s.t. the subspace topology on each U_i is the classical topology on U_i, i.e. $U \subseteq X$ is open in the classical topology if $U \cap U_i$ is open in the classical topology for each i.

Note that each U_i is open in X in this topology. Below are easy to check.

(1) The definition is independent of the choice of the cover. Write X^{an} for the topological space X with the classical topology.
(2) Proposition 1.3 extends from affine case to arbitrary complex algebraic varieties..

Remark 1.4. Let X, Y be two complex algebraic varieties.

(1) If $f : X \to Y$ is a morphism of algebraic varieties, then $f : X^{\mathsf{an}} \to Y^{\mathsf{an}}$ is continuous. In particular, regular functions on X are continuous with respect to the classical topology.
(2) Every point has a countable basis of open neighbourhoods in the classical topology.
(3) We have $\Rightarrow (X \times Y)^{\mathsf{an}} = X^{\mathsf{an}} \times Y^{\mathsf{an}}$, i.e. the classical topology on $X \times Y$ is the product of the classical topology on X, Y.

Proof of Remark 1.4. (1) reduces to that both X, Y are affine. Given closed immersions

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{C}^m \\ \downarrow f & & \vdots\, g \\ Y & \hookrightarrow & \mathbb{C}^n \end{array},$$

we have a lift g such that g is a polynomial map. Since polynomial maps are continuous, we are done.

(2) is trivial.

(3) reduces to that both X, Y are affine. Using the definition of the classical topology on complex varieties, we reduce to the case $X = \mathbb{C}^m$ and $Y = \mathbb{C}^n$. Then it follows by definition. □

Theorem 1.5. *If X is an irreducible complex algebraic variety and a nonempty subset $U \subseteq X$ is Zariski open, then U is dense in the classical topology.*

Corollary 1.6. *If X is any complex algebraic variety and $U \subseteq X$ is open and dense in Zariski topology, then U is dense in the classical topology.*

Proof. If $X = X_1 \cup \cdots \cup X_r$ is the irreducible decomposition, then U Zariski dense implies that $U \cap X_i \neq \varnothing, \forall i$. Theorem 1.5 implies that $U \cap X_i$ is dense in X_i in the classical topology. Hence U is dense in X^{an}. □

2. September 06, 2019

2.1. The classical topology on a complex algebraic variety(continued).

Proof of Theorem 1.5. **Step 1**: Given an open affine cover $X = U_1 \cup \cdots \cup U_r$, for each U_i, we have $U \cap U_i \neq \varnothing$. If we know that $U \cap U_i$ is dense in U_i^{an}, then U is dense in X^{an}. So we may assume X affine.

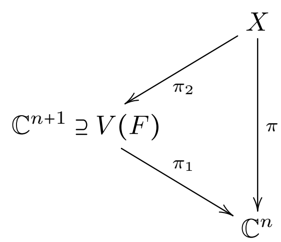
Step 2: Apply Noether normalization to get a finite surjective map $\pi : X \to \mathbb{C}^n$. Need to show that given any $p \in Z = X \smallsetminus U$, we can find sequence $y_m \in U$ such that y_m converges to p. Note that $\pi(X)$ is a closed proper subset of $\mathbb{C}^n$. So there is some nonzero element $g \in \mathbb{C}[x_1, \ldots, x_n]$ such that $\pi(Z) \subseteq V(g)$.

We consider following map

$$\varphi : \mathbb{R} \to \mathbb{C}$$
$$t \mapsto g\left(tu + (1-t)w\right)$$

where $u = \pi(p)$ and $w \in \mathbb{C}^n$ is a point such that $g(w) \neq 0$. So $\varphi(0) \neq 0$. Since φ is a polynomial, it only vainishes at finitely many points. So there is a sequence $\{t_i\}$ such that $t_i \to 1$ and $\varphi(t_i) \neq 0$. So there is a sequence u_m converges to u where $g(u_m) \neq 0$. The goal is that after passing to a subsequence, we want to find $y_m \in \pi^{-1}(u_m)$ such that y_m converges to p. Since $u_m \notin V(g) \Rightarrow y_m \in U$. Then $p \in \overline{U}$.

Step 3: Let $\pi^{-1}(u) = \{p = p_1, \ldots, p_r\}$. Choose $g \in \mathcal{O}(X)$ such that $g(p) = 0$ but $g(p_j) \neq 0$ for $j \geq 2$. Since π is finite, there is some $F \in \mathcal{O}(\mathbb{C}^n)[s]$ monic such that the image in $\mathcal{O}(X)$ satisfies $F(x, g) = 0$ where $x \in X$ and $g \in \mathcal{O}(X)$. Write $F(x,s) = s^d + a_1(x)s^{d-1} + \cdots + a_d(x)$. Since $\mathcal{O}(X)$ is a domain, we may assume that F is irreducible. Consider following map:



where $\pi_2(x) = (\pi(x), g(x))$. Since π is finite, π_2 must be finite. Since π is surjective, so is π_2 (otherwise $\pi_2(X)$ has dimension $< n$, so does $\pi(X)$). Since $g(p) = 0$, we have $a_d(u) = 0$. Since $|a_d(x)|$ is the absolute value of product of the roots of $F(x, -)$ and $u_m \to u$, we can choose s_m such that $F(u_m, s_m) = 0$ for any m and $s_m \to 0$. Now choose $y_m \in \pi_2^{-1}(u_m, s_m)$ arbitrarily. ($s_m = g(y_m)$).

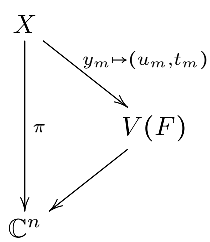
Claim. *After passing to a subsequence, may assume y_m converges to some y.*

Since $\pi(y_m) = u_m \to u \Rightarrow y \in \pi^{-1}(u)$. Since $\lim g(y_m) = g(y) = 0$, we have $y \neq p_j$ for $j \geq 2$. Hence $y = p$. □

Proof of Claim. Choose generators of $h_1, \ldots, h_s$ of $\mathcal{O}(X)$ to get a closed immersion $X \overset{(h_1,\ldots,h_s)}{\hookrightarrow} \mathbb{C}^s$. Use the fact that each h_i satisfies a monic equation:

$$t^{d_i} + a_{i,1}(x)t^{d_i - 1} + \cdots = 0$$

We want each $(h_i(y_m))_{m\geq 1}$ to be bounded.



Enough to show that $(a_{i,j}(\pi(y_m)))_{m\geq 1}$ is bounded. Since (u_m) is convergent, it is bounded. Hence all $(a_{i,j}(\pi(y_m)))_m$ are bounded. □

Corollary 2.1. *X is an algebraic variety over $\mathbb{C}$. $W \subseteq X$ is a constructible subset, then $\overline{W}^{\mathsf{Zar}} = \overline{W}^{\mathsf{an}}$.*

Proof. Since the classical topology is finer than the Zariski topology. We have $\overline{W}^{\mathsf{an}} \subseteq \overline{W}^{\mathsf{Zar}}$. Since W is constructible, there is some $U \subseteq W$ such that U is open and dense in $\overline{W}^{\mathsf{Zar}}$. By Corollary 1.6, U is dense in the classical topology in $\overline{W}^{\mathsf{Zar}}$. Thus $\overline{W}^{\mathsf{Zar}} \subseteq \overline{W}^{\mathsf{an}}$. □

Theorem 2.2. *Let X be an algebraic variety over $\mathbb{C}$. Then*

(1) X is separated if and only if X^{an} is Hausdorff.
(2) X is complete if and only if X^{an} is compact.
(3) If $f : X \to Y$ is morphism of separated varieties, then f is proper (in algebraic sense) if and only $f^{\mathsf{an}} : X^{\mathsf{an}} \to Y^{\mathsf{an}}$ is proper in the usual topological sense (i.e. the preimage of compact subsets is compact).

Proof of Theorem 2.2(1). Recall that in general, the diagonal map $\Delta : X \to X \times X$ is a locally closed immersion. X is separated if and only if Δ is a closed immersion if and only if $\Delta(X)$ is closed in $X \times X$. Corollary 2.1 tells us that $\Delta(X)$ is closed in Zariski topology if and only if it is closed in the classical topology. But $\Delta(X)$ is closed in $X^{\mathsf{an}} \times X^{\mathsf{an}}$ if and only if X^{an} is Hausdorff. □

Remark 2.3 (Results related to Theorem 1.5). If X is irreducible variety over $\mathbb{C}$, then X^{an} is connected.

3. September 09, 2019

3.1. The classical topology on a complex algebraic variety (continued).

Proof of Theorem 2.2(2). Note first that $(\mathbb{P}^n)^{\mathsf{an}}$ is compact: we have a continuous surjective map

$$\{z = (z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} |z_i|^2 = 1\} \longrightarrow (\mathbb{P}^n)^{\mathsf{an}}$$

and the source is compact.

Suppose now that X is complete, in particular, separated. Then X^{an} is Hausdorff. By Chow's lemma, there exists a surjective (birational) morphism $\pi : \widetilde{X} \to X$ with $\widetilde{X}$ projective. Then $\widetilde{X} \hookrightarrow \mathbb{P}^n$ is Zariski closed. Hence $(\widetilde{X})^{\mathsf{an}}$ is closed in $(\mathbb{P}^n)^{\mathsf{an}}$, hence compact. So $\pi(\widetilde{X}^{\mathsf{an}}) = X^{\mathsf{an}}$ is compact.

Conversely, suppose X^{an} is compact, in particular, Hausdorff. Then X is separated. Need to show that for every algebraic variety Y, the projection map $X \times Y \xrightarrow{f} Y$ is closed in the Zariski topology, i.e. $Z \subseteq X \times Y$ is Zariski closed $\Rightarrow f(Z)$ is Zariski closed.

Chevalley's theorem tells us that $f(Z)$ is constructible. By Corollary 2.1, $f(Z)$ is Zariski closed if and only if it is closed in the classical topology. Suppose $y_n \in f(Z)$ such that $\lim_{n \to \infty} y_n = b \in Y$. There exists $x_n \in X$ such that $(x_n, y_n) \in Z$ for any n. Since X^{an} is compcat, after passing to a subseqeunce, may assume that x_n converges to some $a \in X$. Since Z is closed in $X \times Y$, it is closed in the classical topology. Hence $(x_n, y_n) \in Z \Rightarrow (a, b) \in Z \Rightarrow b \in f(Z)$. □

Exercise 3.1. Show that if $f : X \to Y$ is a morphism of seprated algebraic variety, then f is proper if and only if f^{an} is proper.

From now on, all varieties over $\mathbb{C}$ will be assumed separated.

3.2. Holomorphic functions (Griffiths-Harris).

3.2.1. 1-*varaiable vase.* First we look at the case of 1 variable. All functions will be smooth, i.e. $\mathscr{C}^\infty$. Let $U \subseteq \mathbb{C}$ be open. Coordinate functions on U will be denoted $z = x + y\mathrm{i}$ and $\overline{z} = x - y\mathrm{i}$. Their differentials are

$$\mathrm{d}z = \mathrm{d}x + \mathrm{i}\mathrm{d}y,$$
$$\mathrm{d}\overline{z} = \mathrm{d}x - \mathrm{i}\mathrm{d}y,$$
$$\mathrm{d}z \wedge \mathrm{d}\overline{z} = (-2\mathrm{i})\mathrm{d}x \wedge \mathrm{d}y.$$

The dual basis of differentials given by $\mathrm{d}z, \mathrm{d}\overline{z}$ are

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}),$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}).$$

It's easy to verify that $\frac{\partial}{\partial z}(z^m) = mz^{m-1}, \frac{\partial}{\partial \overline{z}}(z^m) = 0$. By product rule, it's enough to check it for $m = 1$.

If $f : U \to \mathbb{C}$ is smooth, then $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y = \frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}$.

Exercise 3.2. Check that $\frac{\partial \overline{f}}{\partial \overline{z}} = \overline{\frac{\partial f}{\partial z}}$.

Proposition 3.3 (Cauchy's formula)**.** *Let Δ be a disc in $\mathbb{C}$. If f is a smooth function on an open neighbourhood of $\overline{\Delta}$, then*

$$f(z) = \frac{1}{2\pi \mathrm{i}}\int_{\partial\Delta} \frac{f(w)}{w - z}\mathrm{d}w + \frac{1}{2\pi \mathrm{i}}\int_{\Delta} \frac{\partial f}{\partial \overline{w}} \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w - z}$$

for all $z \in \Delta$.

Remark 3.4. From the statement of the Proposition 3.3, we know

(1) $\partial\Delta$ is oriented counterclockwise.
(2) Part of the statement is that the second integral is defined.

Proof of Proposition 3.3. Let Δ_ε be a disc of radius $\varepsilon << 1$ around z.

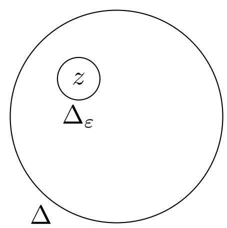


FIGURE 1. Small Neighbourhood

Let $\eta = \frac{f(w)}{w-z}\mathrm{d}w$. Then

$$\mathrm{d}\eta = -\frac{\partial}{\partial \overline{w}}\left(\frac{f(w)}{w - z}\right)\mathrm{d}w \wedge \mathrm{d}\overline{w}.$$

Since $\frac{\partial}{\partial \overline{w}}(\frac{1}{w-z}) = 0$. By quotient rule, we know that $\mathrm{d}\eta = -\frac{\partial f}{\partial \overline{w}}\frac{\mathrm{d}w\wedge \mathrm{d}\overline{w}}{w-z}$. Apply Stoke's formula for η on $\overline{\Delta} \setminus \Delta_\varepsilon$ and get

$$-\int_{\overline{\Delta}\setminus\Delta_\varepsilon} \frac{\partial f}{\partial \overline{w}} \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w - z} = \int_{\partial\Delta} \frac{f(w)}{w - z}\mathrm{d}w - \int_{\partial\Delta_\varepsilon} \frac{f(w)}{w - z}\mathrm{d}w$$

For the last term, change variable $w = z + \varepsilon e^{\mathrm{i}\theta}$ where $\theta \in [0, 2\pi]$, we get

$$\int_0^{2\pi} \frac{f(z + \varepsilon e^{\mathrm{i}\theta})}{\varepsilon e^{\mathrm{i}\theta}}\varepsilon \mathrm{i} e^{\mathrm{i}\theta}\mathrm{d}\theta = \mathrm{i}\int_0^{2\pi} f(z + \varepsilon e^{\mathrm{i}\theta})\mathrm{d}\theta$$

If ε goes to 0, then this integral goes to $2\pi \mathrm{i} f(z)$.

Then look at $\int_{\overline{\Delta}\setminus\Delta_\varepsilon} \frac{\partial f}{\partial \overline{w}}\frac{\mathrm{d}w\wedge \mathrm{d}\overline{w}}{w-z}$, via changing of variable $w = z + re^{\mathrm{i}\theta}$ where $r \geq 0, \theta \in [0, 2\pi]$, we have

$$\mathrm{d}w = e^{\mathrm{i}\theta}\mathrm{d}r + \mathrm{i}re^{\mathrm{i}\theta}\mathrm{d}\theta,$$
$$\mathrm{d}\overline{w} = e^{-\mathrm{i}\theta}\mathrm{d}r - \mathrm{i}re^{-\mathrm{i}\theta}\mathrm{d}\theta.$$

Hence $\mathrm{d}w \wedge \mathrm{d}\overline{w} = -2\mathrm{i}r\mathrm{d}r \wedge \mathrm{d}\theta$ and we have

$$\frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w - z} = -2\mathrm{i}e^{-\mathrm{i}\theta}\mathrm{d}r \wedge \mathrm{d}\theta.$$

This is clearly integrable on any compact subset of $\mathbb{C}$. Hence $\frac{\partial f}{\partial \overline{w}} \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w-z}$ is integrable over $\overline{\Delta}$. □

Definition 3.5. A smooth function $f : U \to \mathbb{C}$ is

- holomorphic if $\frac{\partial f}{\partial \overline{z}} = 0$ (If $f = u + \mathrm{i}v$, this is equivalent to the C-R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$).
- analytic if $\forall a \in U$, there exists a open disc $\Delta_r(a)$ centerted at a inside U such that for $z \in \Delta_r(a)$,

$$f(z) = \sum_{n\geq 0} c_n (z - a)^n$$

  for some $c_n \in \mathbb{C}$ where the convergence is absolute and uniform.

Theorem 3.6. *A function f is holomorphic if and only if f is analytic.*

Theorem 3.7 ($\overline{\partial}$-lemma in 1 variable)**.** *If $\overline{\Delta} \subseteq U$ is a disc and g is a smooth function on U, then there is a smooth function f on Δ such that*

$$\frac{\partial f}{\partial \overline{z}} = g \text{ on } \Delta.$$

4. September 11, 2019

4.1. Holomorphic functions (continued).

4.1.1. 1-*varaiable vase (continued).*

Proof of Theorem 3.6. Assume that f is holomorphic. Given $a \in U$, let Δ be a disc with radius R centered at a such that $\overline{\Delta} \subseteq U$ and let Δ' be a smaller disc with raidus $R' < R$ also centered at a.

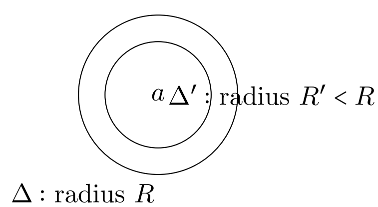


Figure 2. Two Neighbourhoods

Cauchy's formula implies that

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial\Delta} \frac{f(w)}{w - z} \mathrm{d}w$$

Since

$$\frac{f(w)}{w - z} = \frac{f(w)}{(w - a) - (z - a)} = \frac{f(w)}{(w - a)\left(1 - \frac{z - a}{w - a}\right)}$$
$$= \sum_{n\geq 0} \frac{f(w)}{(w - a)^{n+1}} (z - a)^n$$

where $|\frac{z-a}{w-a}| \leq \frac{R'}{R} < 1$ converges absolutely and uniformly for $z \in \Delta', w \in \partial\Delta$. We have $f(z) = \sum_{n\geq 0} c_n(z - a)^n$ where

$$c_n = \frac{1}{2\pi\mathrm{i}} \int_{\partial\Delta} \frac{f(w)}{(w - a)^{n+1}} \mathrm{d}w$$

converges absolutely and uniformly for $z \in \Delta'$. Hence f is analytic.

Now we assume that f is analytic. Choose around $a \in U$, a small disc Δ such that $\overline{\Delta} \subseteq U$. Then we have $f(z) = \sum_{n \geq 0} c_n (z-a)^n$ converges uniformly and absolutely for $z \in \Delta$.

Since $\frac{\partial P}{\partial \bar{z}} = 0$ for any polynomial P. Let P_n be the nth partial sum of the Taylor expansion. Then

$$P_n(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial \Delta} \frac{P_n(w)}{w-z} \mathrm{d}w \Rightarrow f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial \Delta} \frac{f(w)}{w-z} \mathrm{d}w.$$

So

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2\pi \mathrm{i}} \int_{\partial \Delta} \underbrace{\frac{\partial}{\partial \bar{z}} \left(\frac{f(w)}{w-z} \right)}_{0} \mathrm{d}w = 0.$$

Hence f is holomorphic. □

Theorem 4.1 (Theorem 3.7)**.** *Let $g : \mathbb{C} \supseteq U \to \mathbb{C}$ be a smooth function. If Δ is a disc such that $\overline{\Delta} \subseteq U$ and*

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Delta} g(w) \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{w-z}, z \in \Delta,$$

then f is a smooth function and $\frac{\partial f}{\partial \bar{z}} = g$ on Δ.

Proof. Given $z_0 \in \Delta$, choose discs Δ', Δ'' centered at z_0 such that $\overline{\Delta'} \subseteq \Delta''$ and $\overline{\Delta''} \subseteq \Delta$.

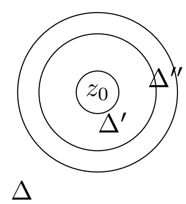


Figure 3. Discs centered at z_0

We can write $g = g_1 + g_2$ with g_1, g_2 smooth on U such that $g_1 = 0$ inside Δ' and $g_2 = 0$ outside Δ''.

Consider separately

$$f_i(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Delta} g_i(w) \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{w-z}.$$

For $z \in \Delta'$, f_1 is clearly smooth and

$$\frac{\partial f_1}{\partial \bar{z}} = \frac{1}{2\pi \mathrm{i}} \int_{\Delta} \underbrace{\frac{\partial}{\partial \bar{z}} \left(\frac{g_1(w)}{w-z} \right)}_{0} \mathrm{d}w \wedge \mathrm{d}\bar{w} = 0.$$

Consider

$$f_2(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{C}} g_2(w) \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{w-z}$$

because g_2 is zero outside Δ''. Use change of variable $w = z + r\mathrm{e}^{\mathrm{i}\theta}$, then

$$\frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{w-z} = -2\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta} \mathrm{d}r \wedge \mathrm{d}\theta$$

and

$$f_2(z) = -\frac{1}{\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\theta} \int_0^{\infty} g_2(z + r\mathrm{e}^{\mathrm{i}\theta}) \ \mathrm{d}r \ \mathrm{d}\theta.$$

This implies that f_2 is smooth on Δ and after going back via the change of variable

$$\frac{\partial f_2(z)}{\partial \overline{z}} = \frac{1}{2\pi \mathrm{i}} \int_{\Delta} \frac{\partial g_2(w)}{\partial \overline{w}} \cdot \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w - z}$$

Apply Cauchy's formula for g_2, we get

$$g_2(z) = \underbrace{\frac{1}{2\pi \mathrm{i}} \int_{\partial\Delta} \frac{g_2(w)}{w - z} \mathrm{d}w}_{0 \text{ since } g_2|_{\partial\Delta} = 0} + \frac{1}{2\pi \mathrm{i}} \int_{\Delta} \frac{\partial g_2}{\partial \overline{w}} \frac{\mathrm{d}w \wedge \mathrm{d}\overline{w}}{w - z}.$$

So we conclude that $\frac{\partial f_2}{\partial \overline{z}} = g_2$ on Δ. Since $\frac{\partial f_1}{\partial \overline{z}} = 0 = g_1$ on Δ', we have $\frac{\partial f}{\partial \overline{z}} = g$ on Δ'. So the same holds on Δ. ☐

Remark 4.2. The proof above also shows that if g is a smooth function of $z = z_1, \ldots, z_n$ on $U \times U_2 \times \cdots \times U_n$ where each $U_i \subseteq \mathbb{C}$ is open, then so is f. Moreover, if g is holomorphic (separately) in each of $z_2, \cdots, z_r$, so is f ($\frac{\partial f}{\partial \overline{z_i}} = 0$ for $2 \le i \le r$).

4.1.2. *Several Variable case.* Now we look at holomorphic functions of several variables. Let $U \subseteq \mathbb{C}^n$ be open with coordinates $z_1, \ldots, z_n$ and $z_j = x_j + \mathrm{i}y_j$ where $1 \le j \le n$.

Definition 4.3. A smooth function $f : U \to \mathbb{C}$ is *holomorphic* if it is holomorphic in each variable, i.e. $\frac{\partial f}{\partial \overline{z_i}} = 0$ on U. It is *analytic* if for every $a \in U$, there is a poly disc $B = B_r(a) = \{z \mid |z_i - a_i| < r, \ \forall i\}$ such that

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha (z - a)^\alpha$$

where $(z - a)^\alpha = \prod_{i=1}^n (z_i - a_i)^{\alpha_i}$.

Theorem 4.4. *Let $f : U \to \mathbb{C}$ be a smooth function. T.F.A.E.*

(1) f is holomorphic
(2) f is analytic
(3) For every poly disc $\Delta = \prod_{i=1}^n \{z_i \mid |z_i - a_i| \le r_i\} \subseteq U$,

$$f(z) = \left(\frac{1}{2\pi \mathrm{i}}\right)^n \int_{|z_i - a_i| = r_i} \frac{f(w)}{(w_1 - z_1) \cdots (w_n - z_n)} \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n$$

where the integral is over the product of circles with product orientation.

Proof. (2)⇒(1): It's clear that if f is analytic, it is analytic in each variable, hence holomorphic in each variable. So f is holomorphic.

(3)⇒(2): We have $f(z) = \sum_{\beta \in \mathbb{N}^n} c_\beta (z - a)^\beta$ where

$$c_\beta = \left(\frac{1}{2\pi \mathrm{i}}\right)^n \int_{|z_i - a_i| = r_i} \frac{f(w)}{(w_1 - z_1)^{\beta_1 + 1} \cdots (w_n - z_n)^{\beta_n + 1}} \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n.$$

Argue similarly as in 1 variable case.

(1)⇒(3): Use Cauchy's formula for holomorphic functions in each variable:

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{|z_n - a_n| = r_n} \frac{f(z_1, \ldots, z_{n-1}, w_n)}{z_n - w_n} \mathrm{d}w_n = \ldots$$

Use the fact that f is continuous, hence it satisfies Fubini's theorem. ☐

5.1. **Properties of holomorphic functions.** Let $\mathcal{O}(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic}\}$. Then

(1) $\mathcal{O}(U) \subseteq \mathscr{C}^\infty(U)$ is a $\mathbb{C}$-subalgebra.
(2) If $f \in \mathcal{O}(U)$ is such that $f(z) \neq 0\ \forall z \in U$, then $1/f \in \mathcal{O}(U)$.

Definition 5.1. Let $U \subseteq \mathbb{C}^n$ be an open subset. A map $f = (f_1, \dots, f_n) : U \to \mathbb{C}^m$ is holomorphic if all f_j are holomorphic.

Write $\mathbb{C}^n = \mathbb{R}^{2n}$ with coordinates $z_1, \dots, z_n$ and $\mathbb{C}^m = \mathbb{R}^{2m}$ with coordinates $z'_1, \dots, z'_m$. We have tangent maps

$$\mathrm{T}_p\mathbb{R}^{2n} \xrightarrow{df_p} \mathrm{T}_{f(p)}\mathbb{R}^{2m}.$$

Write $f_j = u_j + \mathrm{i}v_j$. Then the tangent maps is explicitly,

$$\frac{\partial}{\partial x_j}(p) \mapsto \sum_{k=1}^m \frac{\partial u_k}{\partial x_j}(p)\frac{\partial}{\partial x'_k}(f(p)) + \sum_{k=1}^m \frac{\partial v_k}{\partial x_j}(p)\frac{\partial}{\partial y'_k}(f(p))$$

and similar for $\frac{\partial}{\partial y_j}(p)$.

Exercise 5.2. Show that after we tensor with $\mathbb{C}$, we have

$$\frac{\partial}{\partial z_j}(p) \mapsto \sum_{k=1}^m \frac{\partial f_k}{\partial z_j}(p)\frac{\partial}{\partial z'_k}(f(p)) + \sum_{k=1}^m \frac{\partial \overline{f}_k}{\partial z_j}(p)\frac{\partial}{\partial \overline{z'}_k}(f(p))$$
$$\frac{\partial}{\partial \overline{z}_j}(p) \mapsto \sum_{k=1}^m \frac{\partial f_k}{\partial \overline{z}_j}(p)\frac{\partial}{\partial z'_k}(f(p)) + \sum_{k=1}^m \frac{\partial \overline{f}_k}{\partial \overline{z}_j}(p)\frac{\partial}{\partial \overline{z'}_k}(f(p))$$

Upshot: If f is holomorphic, then $\frac{\partial \overline{f}_k}{\partial z_j} = \overline{\frac{\partial f_k}{\partial \overline{z}_j}} = 0$. Hence the subspace spanned by $\frac{\partial}{\partial z_j}$ maps to the subspace spanned by $\frac{\partial}{\partial z'_j}$ and so do the $\overline{z}_j$ and $\overline{z}'_j$, i.e.

$$\mathrm{span}(\frac{\partial}{\partial z_j} \mid j) \to \mathrm{span}(\frac{\partial}{\partial z'_k} \mid k)$$
$$\mathrm{span}(\frac{\partial}{\partial \overline{z}_j} \mid j) \to \mathrm{span}(\frac{\partial}{\partial \overline{z}'_k} \mid k)$$

For functions $U \xrightarrow{f} V \xrightarrow{g} \mathbb{C}^p$. If g is holomorphic, then $g \circ f$ is holomorphic. In fact, for any g,

$$\frac{\partial(g \circ f)}{\partial \overline{z}_j}(p) = \sum_{k=1}^m \overline{\frac{\partial f_k}{\partial z'_j}} \left(\frac{\partial g}{\partial \overline{z}'_k} \circ f \right)$$

- f, g holomorphic, then $g \circ f$ is also holomorphic.
- If f is holomorphic and $g \circ f$ is holomorphic, and $\mathbb{C}^n \supseteq U \xrightarrow{f} V \subseteq \mathbb{C}^n$ is a diffeomorphism and the Jacobian matrix $\left(\frac{\partial f_i}{\partial z_j}\right)$ is invertible at every point, then g is holomorphic. Moreover, if both f, g are holomorphic where $g : V \to \mathbb{C}$, then

$$\frac{\partial(g \circ f)}{\partial z_j}(p) = \sum_{k=1}^m \frac{\partial f_k}{\partial z'_j} \left(\frac{\partial g}{\partial z'_k} \circ f \right)$$

Given $f : \mathbb{C}^n \supseteq U \to \mathbb{C}^n$ where both $\mathbb{C}^n$ have coordinates $z_1, \dots, z_n$, write $f = (f_1, \dots, f_n)$. We want to compare the real Jacobian of f with the complex Jacobian $\det\left(\frac{\partial f_i}{\partial z_j}\right)$. We know that

$$f^*(\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n) = (\text{real Jacobian of } f)\,\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

Note that

$$\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j = (\mathrm{d}x_j + \mathrm{i}\mathrm{d}y_j)(\mathrm{d}x_j - \mathrm{i}\mathrm{d}y_j) = (-2\mathrm{i})\mathrm{d}x_j \wedge \mathrm{d}y_j.$$

So we have

$$f^*(\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n) = (\text{real Jacobian of } f)\,\mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n$$

and the left-hand-side is $\mathrm{d}f_1 \wedge \mathrm{d}\overline{f}_1 \wedge \cdots \wedge \mathrm{d}f_n \wedge \mathrm{d}\overline{f}_n$. (Recall that $\mathrm{d}f = \sum_j \left(\frac{\partial f}{\partial x_j}\mathrm{d}x_j + \frac{\partial f}{\partial y_j}\mathrm{d}y_j\right) = \sum_j \left(\frac{\partial f}{\partial z_j}\mathrm{d}z_j + \frac{\partial f}{\partial \overline{z}_j}\mathrm{d}\overline{z}_j\right)$ for any smooth f.)

If f is holomorphic, then f_k is holomorphic and $\mathrm{d}f_k = \sum_{j=1}^n \frac{\partial f_k}{\partial z_j}\mathrm{d}z_j, \mathrm{d}\overline{f}_k = \sum_{j=1}^n \overline{\frac{\partial f_k}{\partial z_j}}\mathrm{d}\overline{z}_j$.

Hence

$$\begin{aligned}\mathrm{d}f_1 \wedge \mathrm{d}\overline{f}_1 \wedge \cdots \wedge \mathrm{d}f_n \wedge \mathrm{d}\overline{f}_n &= \left(\det \frac{\partial f_j}{\partial z_k}\right) \cdot \left(\det \overline{\frac{\partial f_j}{\partial z_k}}\right) \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n \\ &= |\det \frac{\partial f_j}{\partial z_k}|^2 \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n\end{aligned}$$

The conclusion is that the determinant of the real Jacobian matrix of f is the square of the determinant of the complex Jacobian of f. So

- it is always positive.
- one is 0 iff the other one is 0.

Theorem 5.3 (Holomorphic Inverse Function Theorem)**.** *Let $U \subseteq \mathbb{C}^n$ be open and let $f : U \to \mathbb{C}^n$ be holomorphic. For any point $p \in U$ such that $\det\left(\frac{\partial f_i}{\partial z_j}(p)\right) \neq 0$, there exists an open neighbourhood $U' \subseteq U$ of p and $V' \subseteq \mathbb{C}^n$ of $f(p)$ such that f gives a bijective map $U' \to V'$, and its inverse is holomorphic.*

Proof. The hypothesis tells us that the determinant of the real Jacobian of f is nonzero at p. By the Inverse Function Theorem for smooth maps, there exist open neighbourhoods U', V' of $p, f(p)$ respectively, such that $U' \xrightarrow{f} V'$ is an bijection whose inverse g is smooth. We may assume that $\det\left(\frac{\partial f_j}{\partial z_k}\right) \neq 0$ on U'. Since $f, g \circ f = \mathrm{id}$ are both holomorphic, we know that g is holomorphic. □

Remark 5.4. We have

(1) If $f : U \to \mathbb{C}$ is holomorphic, then $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}$ is holomorphic for all α. (Since $\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_k}$ commute).
(2) If $a \in U$ is such that $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a) = 0$ for all α, then $f = 0$ in a neighbourhood of a.

Proof of Remark 5.4(2). We know that if $B = \{z \mid |z_i - a_i| < \varepsilon,\ \forall i\}$ and $\overline{B} \subseteq U$, then

$$f = \left(\frac{1}{2\pi\mathrm{i}}\right)^n \int\limits_{|z_i - a_i| = \varepsilon} \frac{f(w_i)}{(w_1 - z_1)\cdots(w_n - z_n)} \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n$$

Using this, we get $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha (z - a)^\alpha$ where

$$c_\alpha = \left(\frac{1}{2\pi\mathrm{i}}\right)^n \int\limits_{|z_i - a_i| = \varepsilon} \frac{f(w_i)}{(w_1 - a_1)^{\alpha_1 + 1}\cdots(w_n - a_n)^{\alpha_n + 1}} \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_n = \frac{1}{\alpha!}\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a).$$

So $c_\alpha = 0$ for all α, and hence $f = 0$ in B. □

Proposition 5.5. *Let $f : U \to \mathbb{C}$ be holomorphic and U is connected. If $f = 0$ on some $V \subseteq U$ open, then $f = 0$ on U.*

Proof. Let $U' = \{z \in U \mid f = 0 \text{ on some open neighbourhood of } z\}$. Then U' is open and nonempty by assumption. It is enough to prove that U' is closed.

Let $\{z_n\} \subseteq U'$ be a sequence which converges to some a. For every α, $\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z_n) = 0 \Rightarrow \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(a) = 0$. This holds for every α. Then $f = 0$ in a heighbourhood of a. So $a \in U'$ and U' is closed. So $U' = U$ by connectedness. □

6. September 16, 2019

6.1. Properties of holomorphic functions (continued).

Theorem 6.1 (Maximum Modulus Theorem). *If $U \subseteq \mathbb{C}^n$ is open and connected, and $f : U \to \mathbb{C}$ is holomorphic such that $|f|$ has a local maximum at $a \in U$, then f is constant.*

Proof. By Proposition 5.5, it is enough to show that there is an open neighbourhood U_0 of a such that f is constant on U_0.

We prove this by reduction to $n = 1$. Take U_0 to be an open polydisk containing a, i.e. $U_0 = \{z \mid |z_i - a_i| < \varepsilon,\ \forall i\}$. For any $z \in U_0$, consider the 1-variable function $\mathbb{C} \ni w \mapsto f(wa + (1-w)z)$ for $|wa_i - (1-w)z_i - a_i| < \varepsilon$ defined on open subset of $\mathbb{C}$ that contains $0, 1$. It's a holomorphic function. Its absolute value has a local maximum at $w = 1$. The 1-variable case implies that this is constant. Hence $f(z) = f(a)$.

Now we are left with the 1-variable case. Let Δ be a disc concentrated at a, $\overline{\Delta} \subseteq U$ is such that $\Delta = B_R(a)$. Then Cauchy's formula tells us that

$$\begin{aligned} f(a) &= \frac{1}{2\pi \mathrm{i}} \int_{\partial\Delta} \frac{f(w)}{w-a} \mathrm{d}w \\ &\overset{w = a + Re^{2\pi\mathrm{i}\theta}}{=} \frac{1}{\cancel{2\pi}\mathrm{i}} \int_0^1 \frac{f(a + Re^{2\pi\mathrm{i}\theta})}{\cancel{Re^{2\pi\mathrm{i}\theta}}} \cancel{Re^{2\pi\mathrm{i}\theta}} \cdot \cancel{2\pi}\mathrm{i}\mathrm{d}\theta \\ &= \int_0^1 f(a + Re^{2\pi\mathrm{i}\theta}) \mathrm{d}\theta \end{aligned}$$

So we have

$$|f(a)| \le \int_0^1 \underbrace{|f(a + Re^{2\pi\mathrm{i}\theta})|}_{\le |f(a)|} \mathrm{d}\theta \le |f(a)| \int_0^1 \mathrm{d}\theta = |f(a)|$$

assuming that $|f(z)| \le |f(a)|$ in an open neighbourhood of $\overline{\Delta}$. This is fine if we take $R << 1$. So above inequalities are all equalities. But the last equality and the fact that f is continuous imply that $|f(z)| = |f(a)|,\ \forall z \in \partial\Delta$. The same holds for any R' such that $0 < R' \le R$. Hence $|f(z)|$ is constant in an open neighbourhood of a. By Exercise 6.2, f is itself a constant. □

Exercise 6.2. Show that if $f = u + iv$ is holomorphic on some open subset and $u^2 + v^2$ is a constant, then f is a constant. (Hint: Apply the differentials $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ and use C-R equations.)

Write $U_x = \frac{\partial u}{\partial x}$ to simplify notations. Apply $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ to $u^2 + v^2 = 0$ we get

$$\begin{aligned} U_x + V_x &= 0, \\ U_y + V_y &= 0. \end{aligned}$$

Using C-R equation we have

$$\begin{aligned} U_x - V_y &= 0, \\ U_y + V_x &= 0. \end{aligned}$$

Then we have

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} U_x \\ U_y \\ V_x \\ V_y \end{pmatrix} = 0.$$

Since the matrix is invertible, we conclude that all derivatives are zero. Therefore both u and v are constants and hence f is a constant.

6.2. **Complex manifolds.** If $U \subseteq \mathbb{C}^n$ is open and connected, consider the sheaf $\mathcal{O}_U$ on U such that $\mathcal{O}_U(V) = \{f : V \to \mathbb{C} \mid f \text{ holomorphic}\}$. Then this is a *sheaf*:

(1) We have restriction maps: if $V_1 \subseteq V_2$ are open subsets of U, then f holomorphic on V_2 implies that $f|_{V_1}$ is holomorphic.
(2) If $V = \cup_{i\in I} U_i$, and $\varphi_i : V_i \to \mathbb{C}$ are holomorphic functions such that $\varphi_i|_{V_i\cap V_j} = \varphi_j|_{V_i\cap V_j}$, then there is a unique $\varphi : V \to \mathbb{C}$ holomorphic such that $\varphi|_{V_i} = \varphi_i$, $\forall i$. (Key point: If $\varphi : V \to \mathbb{C}$ is such that $\varphi|_{V_i}$ is holomorphic for any i, then φ is holomorphic. It's clear from the definiton.)

Definition 6.3. Let $\mathscr{C}_{X,\mathbb{C}}$ be a sheaf of continuous functions from a topological space X to $\mathbb{C}$. A complex manifold of dimension n is a pair $(X, \mathcal{O}_X)$ such that

(1) X is a Hausdorff topological space having countable basis of open subsets.
(2) $\mathcal{O}_X \subseteq \mathscr{C}_{X,\mathbb{C}}$ is a subsheaf such that X can be written as $X = \cup_i U_i$ where $U_i \subseteq X$ is open and each $(U_i, \mathcal{O}_{U_i}) \cong (V_i, \mathcal{O}_{V_i})$ for some $V_i \subseteq \mathbb{C}^n$ open where $\mathcal{O}_{V_i}$ is the sheaf of holomorphic functions on V_i.

Remark 6.4. Suppose $V_1, V_2 \subseteq \mathbb{C}^n$ are open subsets. A morphism $f : (V_1, \mathcal{O}_{V_1}) \to (V_2, \mathcal{O}_{V_2})$ is an isomorphism if $f : V_1 \to V_2$ is a homeomorphism which induces an isomorphism between the corresponding sheaves, i.e. $\forall U \subseteq V_2$, the map

$$\mathcal{O}_{V_2}(U) \to \mathcal{O}_{V_1}(f^{-1}(U))$$
$$\varphi \mapsto \varphi \circ f$$

is an isomorphism. (This forces f, f^{-1} to be holomorphic, the converse is clearly true as well).

Definition 6.5. Let $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ be complex manifolds.

(1) The sections of $\mathcal{O}_X(X)$ are the *holomorphic functions* on X.
(2) A *holomorphic* map $(X, \mathcal{O}_X) \xrightarrow{f} (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ such that for any $V \subseteq Y$ open, and any $\varphi \in \mathcal{O}_Y(V)$, we have $\varphi \circ f \in \mathcal{O}_X(f^{-1}(V))$.

Note: If $X \subseteq \mathbb{C}^n, Y \subseteq \mathbb{C}^m$ are open subsets, then this concides with our previous definition. This is saying that the map between sheaves are automatically given by the topological map.

Remark 6.6. If $U \subseteq \mathbb{C}^n$ is open, $p \in U$, then $\mathcal{O}_{U,p} = \varinjlim_{V\ni p} \mathcal{O}_U(V)$ is a local ring. We can see this by using the "valuation at p" map. The kernel $\{(V,\varphi) \mid \varphi(p) = 0\} = \underline{\mathfrak{m}}$ is the unique maximal ideal: For any $(V,\varphi) \notin \underline{\mathfrak{m}}$, we may assume that $\varphi(z) \neq 0$ for any $z \in V$, hence $\frac{1}{\varphi} \in \mathcal{O}(V)$. So $(\mathcal{O}_{U,p}, \underline{\mathfrak{m}})$ is local. In particular, all such rings for manifolds of fixed dimension are isomorphic.

Remark 6.7. One can define complex manifolds using atlasses. Take X a topological space with suitable properties, an open cover $X = \cup_i U_i$ and homeomorphisms $\varphi_i : U_i \cong V_i \subseteq \mathbb{C}^n$ where V_i is open such that $\forall i, j$ the map $\varphi_i(U_i \cap U_j) \xrightarrow{\varphi_i\circ\varphi_j^{-1}} \varphi_j(U_i \cap U_j)$ is biholomorphic. Then one identifies two such objects $(X, \mathcal{A})$ and $(X, \mathcal{A}')$ if $\mathcal{A}$ and $\mathcal{A}'$ are compatible.

Remark 6.8. It's clear from the definition via atlasses, using the fact that holomorphic maps $\mathbb{C}^m \supseteq U \to \mathbb{C}^n$ are smooth, that every complex manifolds of complex dimension n has an underlying real smooth manifold structure (denote $X_{\mathbb{R}}$ if necessary) of real dimension $2n$.

We have an induced map $\mathcal{O}_X \subseteq \mathscr{C}^\infty_{X,\mathbb{C}}$.

Next time,

- vector bundles in the smooth/holomorphic category.
- submanifolds
- complex manifold structure associated to a smooth complex algebraic variety.

7. September 18, 2019

7.1. Vector bundles.

Definition 7.1. If M is a real smooth manifold, a real (resp. complex) vector bundle on M of rank r is a smooth manifold E, with a smooth map $\pi : E \to M$ such that for any $x \in M$, $\pi^{-1}(x)$ has a structure of vector space over $\mathbb{R}$ (resp. $\mathbb{C}$) of dimension r such that there exists open cover $M = \cup_i U_i$ such that

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow[\cong]{} & U_i \times \mathbb{R}^r(\text{ resp. } \mathbb{C}^r) \\ \downarrow & \swarrow & \\ U_i & & \end{array}$$

Given such E, get a sheaf $\mathscr{E}$ on M such that

$$\mathscr{E}(U) = \{s : U \to E \text{ smooth} \mid \pi \circ s = 1_U\}.$$

This gives an equivalence of categories

$$\left\{\begin{matrix}\text{real (complex) vector bundles on} \\ M \text{ of rank } r\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{locally free sheaves of rank } r \text{ of} \\ \mathscr{C}^\infty_{M,\mathbb{R}}\text{-modules } (\mathscr{C}^\infty_{M,\mathbb{C}}\text{-modules})\end{matrix}\right\}$$

We will consider the corresponding notion in the category of complex manifolds, e.g. let E be a holomorphic complex manifold and consider complex vector bundles.

The corresponding locally free sheaves are $\mathcal{O}_M$-modules. Note that associated to such E, we will have sheaves of smooth sections and holomorphic sections.

7.2. Submanifold.

Definition 7.2. Let X be a complex manifold of dimension n. A closed submanifold of codimension r of X is a closed subset $Y \subseteq X$ such that for any $p \in Y$, there exists a chart $p \in U \overset{\varphi}{\to} V \subseteq \mathbb{C}^n$ in X such that

$$\varphi(U \cap Y) = \{z \in V \mid z_1 = z_2 = \cdots = z_r = 0\}.$$

It is easy to check that by restricting such charts to Y, we get a holomorphic atlas on Y, making it a complex manifold with $\dim(Y) = n - r$. It also satisfies following universal property.

Given a holomorphic map $g : Z \to X$ such that $g(Z) \subseteq Y$. There exists a unique holomorphic map $g' : Z \to Y$ such that $i \circ g' = g$ where $i : Y \hookrightarrow X$ is the inclusion map.

Proposition 7.3. *If $U \subseteq \mathbb{C}^n$ is open and $f_1, \ldots, f_r \in \mathcal{O}(U)$ are such that* $\operatorname{rank}\left(\frac{\partial f_i}{\partial z_j}(p)\right) = r \le n$ *for any $p \in U$, then $\{z \in U \mid f_1(z) = \cdots = f_r(z) = 0\}$ is a closed submanifold of U of codimension r.*

Proof. Given $p \in Y$, we may assume that $\det\left(\frac{\partial f_i}{\partial z_j}(p)\right)_{1 \le i,j \le r} \neq 0$. Define a function $\varphi : U \to \mathbb{C}^n$ to be $\varphi(z) = (f_1(z), \cdots, f_r(z), z_{r+1}, \cdots, z_n)$. Then $\det\left(\frac{\partial f_i}{\partial z_j}(p)\right) \neq 0$ and we can apply the inverse function theorem. We see that φ is biholomorphic in some neighbourhood of p. In such neighbourhood, φ is the desired chart. □

Basic properties of holomorphic functions we discussed extend to this setting. Let X be a complex manifold.

(1) If $f \in \mathcal{O}(X)$ such that $f|_U = 0$ for some $U \subseteq X$ open and X is connected, then $f = 0$.
(2) (Maximum Modulus Principal) If $f \in \mathcal{O}(X)$ is such that $|f|$ has a local maximum and X is connected, then f is a constant.

Corollary 7.4. *If X is a connected compact complex manifold, then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$.*

Proof. For any $f \in \Gamma(X, \mathcal{O}_X)$, if X is compact, then $|f|$ must have a maximum. Then by Maximum Modulus Principal, f is a constant. □

7.3. **The complex manifold associated to a smooth complex algebraic variety.** Let X be a smooth complex algebraic variety of pure dimension n. Choose $U \subseteq X$ affine open subset and let $U \subseteq \mathbb{C}^N$ be a closed immersion. Let $r = N - n$. Since $U, \mathbb{C}^N$ are both smooth, we can cover $\mathbb{C}^N$ by Zariski open subsets V_i. If $V_i \cap U \neq \varnothing$, then $V_i \cap U \hookrightarrow V_i$ is cut out by r equations $f_1, \dots, f_r \in \mathcal{O}(V_i)$ with $\operatorname{rank}\left(\frac{\partial f_i}{\partial z_j}(p)\right) = r$ for all $p \in V_i \cap U$. Because stalkwise we can do it. Apply Proposition 7.3, each $V_i \cap U \hookrightarrow V_i$ is a closed complex submanifold of codimension r.

Exercise 7.5. Check that the resulting transition maps are holomorphic, using the fact that rational maps are holomorphic.

Exercise 7.6. Show that if $f : X \to Y$ between smooth complex algebraic varieties, then $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is holomorphic.

Theorem 7.7. *If X is a connected complex algebraic variety, then X^{an} is connected.*

Proof. First prove this when X is a smooth connected projective curve over $\mathbb{C}$. We know that X^{an} is a 1-dimensional complex manifold. It is compact since X is complete by Theorem 2.2. Suppose that $X^{\mathrm{an}} = U \coprod V$ with U, V open and nonempty in X^{an}.

Take $P \in U$. If $n >> 0$ ($n \geq 2\mathrm{genus}(X)$), then $\mathcal{O}_X(nP)$ is globally generated. There exists $s \in \Gamma(X, \mathcal{O}_X(nP))$ which does not vanish at P. Equivalently we have $nP \sim Q_1 + \cdots + Q_n$ where $Q_i \neq P, \forall i$. So there exists $\varphi \in \mathbb{C}(X)^*$ (Rational functions) such that $\operatorname{div}(\varphi) = (Q_1 + \cdots + Q_n) - nP$. So φ gives a regular function on $X \smallsetminus \{P\} \to \mathbb{C}$. (This is holomorphic) By restricting to V we get holomorphic map $g = \varphi|_V : V \to \mathbb{C}$ where V is a compact complex manifold.

Hence g is a constant on the connected component of V. In particular, φ takes the same value infinitely many times. Hence φ is constant (Because V is 1-dimensional, φ is a regular function. So the preimage of 0 under φ is a finite set) and $\operatorname{div}(\varphi) = 0$, contradiction!

We now need to reduce to this case:

- We may assume that X is irreducible. (Since by hypothesis we can go from any irreducible component to any other one via points of intersection.)
- (Proposition 8.1) If X is an irreducible algebraic variety over k, where $k = \overline{k}$. For any $x, y \in X$, there exists an irreducible curve $C \subseteq X$ such that $x, y \in C$.

□

8. September 20, 2019

8.1. **The complex manifold associated to a smooth complex algebraic variety (continued).** We continue the proof of Theorem 7.7, first we need following proposition.

Proposition 8.1. *Let X be an irreducible algebraic variety over $k = \overline{k}$, $x_1, x_2 \in X$. Then there exists an irreducible curve $C \subseteq X$ such that $x_1, x_2 \in C$.*

Proof. We may assume that $\dim X = n \geq 2$. By Chow's lemma, there exists a surjective morphism $\pi : \widetilde{X} \to X$ where $\widetilde{X}$ is irreducible, quasi-projective. If $\widetilde{x}_1, \widetilde{x}_2$ lie above x_1, x_2, then it is enough to find curve $\widetilde{C}$ on $\widetilde{X}$ through $\widetilde{x}_1, \widetilde{x}_2$ and take $C = \pi(\widetilde{C})$. So we may assume that X is quasiprojective.

Next we choose a locally closed immersoin $X \hookrightarrow \mathbb{P}^N$. It is enough to prove the statement for $\overline{X}$. So we may assume that X is projective.

Let $Y = \mathrm{Bl}_{\{x_1, x_2\}} X \xrightarrow{p} X$. Write $E_i = p^{-1}(x_i)$ for the exceptional divisors. Then $\dim E_i = n - 1$. The blowup Y is projective since X is. Choose an embedding $Y \hookrightarrow \mathbb{P}^N$. Cut Y with $n-1$ general hyperplanes $H_1, \dots, H_{n-1}$ so that we can make the intersection dimension go down by $n-1$. By Bertini's theorem (See Remark 8.2)), the intersection is irreducible and the intersection $E_i \cap H_1 \cap \cdots \cap H_{n-1} \neq \varnothing$ for any i. Let $Z = Y \cap H_1 \cap \cdots \cap H_{n-1}$. Then $C = p(Z)$ satisfies the requirements. □

Remark 8.2. (Bertini's theorem) A general hyperplane intersects an irreducible projective variety of dimension at least 2 at an irreducible subvariety.

Proof of Theorem 7.7. We may assume that X is irreducible. If $X^{\mathrm{an}} = U \coprod V$ with both U, V open and nonempty. Let $X_1, \ldots, X_r$ be all irreducible components. Then the assertion for irreducible variety tells us that if $X_i \cap U \neq \varnothing$, then $X_i \cap V = \varnothing$ and $X_i \subseteq U$. Let $I = \{i \in [r] \mid X_i \cap U \neq \varnothing\}$. Then $X = (\cup_{i \in I} X_i) \coprod (\cup_{i \notin I}) X_i$, which contradicts with connectedness.

Now X is irreducible and by Proposition 8.1, it is enough to treat the case when X is an irreducible curve.

If $\widetilde{X} \to X$ is the normalization, then it is enough to show that $\widetilde{X}^{\mathrm{an}}$ connected. We may assume that X is smooth. By [Har77, Chapter I, Corollary 6.10], X is isomorphic to an open subset of a smooth projective connected variety $\overline{X}$. By the proof from last time, we know that $\overline{X}^{\mathrm{an}}$ is connected. Since $\overline{X} \smallsetminus X$ consists of only finitely many points. The claim below finishes the proof.

Claim. *If M is smooth real manifold of dimension ≥ 2, and p is a point in M, then M is connected $\Rightarrow M \smallsetminus \{p\}$ is connected.*

Note that $\overline{X}$ is of real dimension 2, we see that X must be connected. □

Proof of Claim. If $M \smallsetminus \{p\} = U \coprod V$ where U, V are both nonempty and open in $M \smallsetminus \{p\}$, then $p \in \overline{U} \cap \overline{V}$. Choose W an open ball neighbourhood of p. Then $W \smallsetminus \{p\}$ is not connected. But this is clearly path-connected. □

8.2. More examples of complex manifolds.

Condition 8.3. Suppose X is a complex manifold of dimension n and G is a group acting on X via holomorphic maps. Suppose following two conditions:

(1) For any $x \in X$, there exists $U \ni x$ open neighbourhood such that for any $g \neq e$, we have $U \cap gU = \varnothing$.
(2) For any $x, y \in X$ such that x, y are not in the same orbit, there exists $U \ni x$ and $V \ni y$ open such that $gU \cap V = \varnothing$ for all $g \in G$.

Condition 8.3(1) implies that the quotient map $\pi : X \to X/G$ is a covering map. Moreover, the transition maps are holomorphic. So there exists a unique complex manifold structure on X/G such that π is holomorphic. Condition 8.3(2) implies that X/G is Hausdorff.

Example 8.4. (Complex tori) Let V be a n-dimensional complex vector space, and $\Lambda \subseteq V$ a lattice, i.e. a free abelian group of rank $2n$ such that $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$. Then $V \to Z := V/\Lambda$ gives a complex manifold structure on Z.

The action of Λ by translation satisfies Condition 8.3(1) and (2). For (1), since Λ is a discrete subgroup of V of finite rank. For any point v in Λ, the number $d = \inf\{\mathrm{dist}(u, v) \mid u \in \Lambda\}$ is positive. For any point $x \in V$, the open neighbourhood $B_{d/2}(x)$ satisfies the condition.

For (2), we can always find a fundamental domin which contains these two points x, y. If at least one of them is at the boundary, then shift our fundamental domain slightly so that both points are in the interior of our domain. Then choose nonintersect open neighbourhoods of each point, and they won't intersect under the translation of Λ.

We'll see that for $n \geq 2$, most of these do not come from algebraic varieties. Although topologically they are $(S^1)^{2n}$.

Example 8.5. (Hopf surface) Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 \smallsetminus \{(0,0)\}$ where the generator γ of $\mathbb{Z}$ acts by $(z_1, z_2) \mapsto (2z_1, 2z_2)$. It's easy to check that Condition 8.3(1) and (2) are satisfied. We can get a complex manifold structure on the quotient.

We have diffeomorphism:

$$\mathbb{C}^2 \smallsetminus \{(0,0)\} \simeq S^3 \times \mathbb{R}$$

$$(z_1, z_2) \mapsto \left(\frac{1}{\sqrt{|z_1|^2 + |z_2|^2}} (z_1, z_2), \log \sqrt{|z_1|^2 + |z_2|^2} \right).$$

The action of γ under this diffeomorphism is $(u,t) \mapsto (u, t+\log 2)$. Hence the Hopf surface $\mathbb{C}^2 \smallsetminus \{(0,0)\}/\mathbb{Z} \overset{\text{diffeo}}{\simeq} S^3 \times S^1$.

8.3. **Orientation.** Let V be a 1-dimensional vector space. Then the orientation of V is a choice of an element in $V/\mathbb{R}^*_{>0}$. Given such a choice, there is a canonical choice of orientation on V^*.

If V is an n-dimensional vector space, an orientation on V is an orientation on $\bigwedge^n V$. If X is a smooth real manifold and E is a real vector bundle on X, an orientation on E is a compatible system of orientations on $E_{(x)}$ for each $x \in X$, i.e. locally have trivialization $\pi^{-1}(U) \simeq U \times \mathbb{R}^r$ preserving orientation on the fibers where $\pi : E \to X$.

We know that orientations on E correspond to orientations on E^*.

Definition 8.6. An orientation on a smooth real manifold X is an orientation on the tangent bundle $\mathrm{T}X$ (or equivalently, on T^*X).

Giving an orientation is equivalent to giving a system of charts such that if for each transition maps $f = (f_1, \dots, f_n) : U \to \mathbb{R}^n$, $\det\left(\frac{\partial f_i}{\partial x_j}\right) > 0$. Recall that if X is a complex manifold and we consider the smooth manifold structure, then we see that if we take the system of holomorphic charts, then for the transition maps $f = (f_1, \dots, f_n) : \mathbb{C}^n \supseteq U \to \mathbb{C}^n$, the determinant of the real Jacobian is $|\det \frac{\partial f_i}{\partial x_j}|^2 > 0$. We have a canonical orientation on X. Then convention is that given $f : U \to \mathbb{C}^n$, the orientation on U corresponds to the orientation on $\mathbb{C}^n$ given by $\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n$.

9. September 23, 2019

9.1. The analytic space associated to an algebraic variety.

Setup 9.1. Let $U \subseteq \mathbb{C}^n$ be an open subset in the classical topology, and $f_1, \dots, f_r \in \mathcal{O}(U)$. A *local model* is a subset $Z = \{x \in U \mid f_1(x) = \cdots = f_r(x) = 0\}$. Consider on Z the sheaf given by

$$\mathcal{O}_Z(V) = \{f : V \to \mathbb{C} \mid \text{ locally } f \text{ extends to a holomorphic function on an open subset in } \mathbb{C}^n\}$$

Consider the inclusion map $Z \overset{j}{\hookrightarrow} U$, we have $\mathcal{O}_U \twoheadrightarrow j_*\mathcal{O}_Z$ and the kernel is $\mathscr{I}_{Z/U}$ where $\mathscr{I}_{Z/U}(V) = \{f : V \to \mathbb{C} \mid f|_{V\cap Z} = 0\}$. So $(Z, \mathcal{O}_Z)$ is a locally ringed space.

Definition 9.2. A reduced analytic space is a locally ringed space $(X, \mathcal{O}_X)$ such that

(1) X is a Hausdorff topological space with countable basis for the topology.
(2) There exists an open cover $X = \cup_i W_i$ such that each $(W_i, \mathcal{O}_{W_i})$ is isomorphic as a locally ringed space to a local model in Setup 9.1

Sections of $\mathcal{O}_X$ are holomorphic functions on X.

Definition 9.3. A holomorphic map between analytic spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ is a continuous map $f : X \to Y$ such that for any $V \subseteq Y$ open, and any $\varphi \in \mathcal{O}_Y(V)$, we have $\varphi \circ f \in \mathcal{O}_X(\varphi^{-1}(V))$.

Because the fact that you can recover your function from its value at each point, the induced sheaf map is given by pre-compose with φ.

Example 9.4. Every manifold is canonically an analytic space.

Example 9.5. If X is a separated algebraic variety over $\mathbb{C}$, then we have a sheaf $\mathcal{O}_{X^{\mathsf{an}}}$ on X^{an} that makes it an analytic space. Choose affine open subsets covering X. Each such open subset U has a closed immersion $U \hookrightarrow \mathbb{C}^N$. Hence U is cut out by finitely many polynomials. So we have a sheaf $\mathcal{O}_{U^{\mathsf{an}}}$ on U^{an} making it an analytic space. It's easy to check that these sheaves are compatible on intersections. So we get $\mathcal{O}_{X^{\mathsf{an}}}$.

In this way, we get a functor

$$\{\text{complex algebraic variety}\} \longrightarrow \{\text{analytic spaces}\}$$

9.2. **Comparison results.** Fix X a complex algebraic variety, we get X^{an} an analytic space. We have a morphism of locally ringed spaces

$$\left(\varphi, \varphi^\sharp\right) : (X^{\mathsf{an}}, \mathcal{O}_{X^{\mathsf{an}}}) \to (X, \mathcal{O}_X)$$
$$\varphi : x \mapsto x$$
$$\varphi^\sharp : \mathcal{O}_X \to \varphi_* \left(\mathcal{O}_{X^{\mathsf{an}}}\right)$$

where for each $U \subseteq X$, the map $\varphi^\sharp(U) : \mathcal{O}_X(U) \to \mathcal{O}_{X^{\mathsf{an}}}(U)$ is just viewing each regular function on U as a holomorphic function on U. Hence the corresponding ring homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathsf{an}},x}$ is a local homomorphism.

Given an $\mathcal{O}_X$-module $\mathscr{F}$, let $\mathscr{F}^{\mathsf{an}} := \varphi^*(\mathscr{F}) = \varphi^{-1}(\mathscr{F}) \otimes_{\varphi^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{\mathsf{an}}}$ be the pullback $\mathcal{O}_{X^{\mathsf{an}}}$-module. In particular, for every $x \in X$, we have canonical isomorphism $\left(\mathscr{F}^{\mathsf{an}}\right)_x \cong \mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X^{\mathsf{an}},x}$. In fact, we will see that $\mathcal{O}_{X^{\mathsf{an}},x}$ is a Noetherian ring and $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathsf{an}},x}$ is flat. In particular, this will imply that the functor $\mathscr{F} \to \mathscr{F}^{\mathsf{an}}$ is *exact*.

Note that in general, we have canonical maps

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\mathcal{O}_{X^{\mathsf{an}}}}(\mathscr{F}^{\mathsf{an}}, \mathscr{G}^{\mathsf{an}}).$$

If we take cohomology, we get

$$\mathrm{H}^i\left(X, \mathscr{F}\right) \to \mathrm{H}^i\left(X^{\mathsf{an}}, \mathscr{F}^{\mathsf{an}}\right).$$

More generally, we have canonical maps

$$\mathrm{Ext}^i_{\mathcal{O}_X}\left(\mathscr{F}, \mathscr{G}\right) \to \mathrm{Ext}^i_{\mathcal{O}_{X^{\mathsf{an}}}}\left(\mathscr{F}^{\mathsf{an}}, \mathscr{G}^{\mathsf{an}}\right).$$

Theorem 9.6 (GAGA, 1st part)**.** *If X is a complete variety, then the functor $\mathscr{F} \to \mathscr{F}^{\mathsf{an}}$ is fully faithful on coherent sheaves. Moreover, for all coherent sheaves $\mathscr{F}, \mathscr{G}$, the map $\mathrm{Ext}^i_{\mathcal{O}_X}\left(\mathscr{F}, \mathscr{G}\right) \to \mathrm{Ext}^i_{\mathcal{O}_{X^{\mathsf{an}}}}\left(\mathscr{F}^{\mathsf{an}}, \mathscr{G}^{\mathsf{an}}\right)$ is an isomorphism.*

The theorem is proved due to Serre for X projective and Grothendieck for X complete. There is also a real version for proper morphisms. We will prove this theorem later when X is projective.

Definition 9.7. If $(X, \mathcal{O}_X)$ is a locally ringed space, an $\mathcal{O}_X$-module $\mathscr{F}$ is *locally finitely generated* if $\forall x \in X$, there exists open neighbourhood $U \ni x$ and $s_1, \ldots, s_n \in \mathscr{F}(U)$ such that $s_{1,y}, \cdots, s_{n,y} \in \mathscr{F}_y$ generate $\mathscr{F}_y$ over $\mathcal{O}_{X,y}$ for any $y \in U$.

Definition 9.8. An $\mathcal{O}_X$-module $\mathscr{F}$ is coherent if

- it is locally generated, and
- for every open subset $U \subseteq X$, $s_1, \ldots, s_r \in \mathscr{F}(U)$, the kernel $\mathrm{Ker}\left(\mathcal{O}_U^{\oplus r} \to \mathscr{F}\right)$ where $e_i \mapsto s_i$ is locally finitely generated on U.

Exercise 9.9. Check that on an algebraic variety, this coincides with the definition in [Har77].

Theorem 9.10 (OKa)**.** *If X is an analytic space, then $\mathcal{O}_X$ is* coherent. *In particular, any locally free $\mathcal{O}_X$-modules of finite rank are coherent.*

If X is an algebraic variety over $\mathbb{C}$, then any coherent sheaf $\mathscr{F}$ on X locally has a finite presentation. Then $\mathscr{F}^{\mathsf{an}}$ is coherent by Theorem 9.10.

Theorem 9.11 (GAGA, 2^{nd} part). *If X is complete, then the functor*

$$\{\textit{coherent } \mathcal{O}_X\textit{-modules}\} \longrightarrow \{\textit{coherent } \mathcal{O}_{X^{\mathsf{an}}}\textit{-modules}\}$$
$$\mathscr{F} \mapsto \mathscr{F}^{\mathsf{an}}$$

is an equivalence of categories.

Remark 9.12. In particular, in this case we have equivalence of categories

$$\{\text{locally free } \mathcal{O}_X\text{-modules}\} \to \{\text{locally free } \mathcal{O}_{X^{\mathsf{an}}}\text{-modules}\}.$$

This requires to show that if $\mathscr{F}^{\mathsf{an}}$ is locally free, then $\mathscr{F}$ is locally free. Need the fact that $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathsf{an}},x}$ is faithfully flat.

Remark 9.13. So we have

(1) $(\mathcal{O}_X)^{\mathsf{an}} = \mathcal{O}_{X^{\mathsf{an}}}$.
(2) If X is a complex algebraic variety, and $E \xrightarrow{\pi} X$ is an algebraic vector bundle with sheaf of sections $\mathscr{E}$, then $E^{\mathsf{an}} \xrightarrow{\pi^{\mathsf{an}}} X^{\mathsf{an}}$ is a holomorphic vector bundle with sheaf of holomorphic sections $\mathscr{E}^{\mathsf{an}}$.
(3) Applying the Theorem 9.11 for coherent ideal sheaves, we get that every closed analytic subspace of X^{an} is equal to Y^{an} for some closed subset Y of X. Need to check that if you start with the defining ideal of an algebraic subset, then the analytification of that is the defining ideal of the analytification of the algebraic subset.
(4) The functor

$$\{\text{complete algebraic varieties}\} \longrightarrow \{\text{compact analytic space}\}$$
$$X \mapsto X^{\mathsf{an}}$$

is fully faithful.

10. September 25, 2019

10.1. The ring $\mathcal{O}_{\mathbb{C}^n,0}$.

Definition 10.1. Let $\mathbb{C}\{z_1,\dots,z_n\} \subseteq \mathbb{C}[\![z_1,\dots,z_n]\!]$ be the set of convergent power series, i.e. the set of elements f where there exists R such that $f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ converges uniformly and absolutely for all $|z_i| < R, \forall i$.

It is easy to check that $f = \sum a_\alpha z^\alpha$ converges if and only if there exists $R > 0$ such that $\{|a_\alpha| R^{|\alpha|}\}_\alpha$ is bounded if and only if $\limsup_{|\alpha|\to\infty} |a_\alpha|^{1/|\alpha|} < \infty$.

The ring $\mathcal{O}_{\mathbb{C}^n,0}$ is called the *ring of germs of holomorphic functions* at 0. We have a map $\mathcal{O}_{\mathbb{C}^n,0} \to \mathbb{C}[\![z_1,\dots,z_n]\!]$ sending f to its power series expansion $\sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ where $a_\alpha = \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial z^\alpha}(0)$, and we know that the expansion converges absolutely and uniformly in a heighbourhood of 0. The image is by definition in $\mathbb{C}\{z_1,\dots,z_n\}$ and the map is clearly injective. It is also easy to check that this is a ring homomorphism. So it is a ring isomorphism. We conclude that if M is a complex manifold and $p \in M$, we have isomorphisms $\mathcal{O}_{M,p} \cong \mathbb{C}\{z_1,\dots,z_n\}$ where $n = \dim M$.

Goal of this lecture is to show that the ring $\mathbb{C}\{z_1,\dots,z_n\}$ is Noetherian. The key ingredient is Weierstrass Preparation Theorem (Theorem 10.3).

Definition 10.2. A Weierstrass polynomial with repsect to z_n is an element of $\mathbb{C}\{z_1,\dots,z_n\}$ of the form

$$z_n^d + a_1(z_1,\dots,z_{n-1})z_n^{d-1} + \cdots + a_d(z_1,\dots,z_{n-1})$$

such that $a_i(0) = 0$ for $1 \le i \le d$.

Theorem 10.3 (Weierstrass Preparation Theorem). *Given $f \in \mathbb{C}\{z_1,\dots,z_n\}$ such that $f(0,\cdots,0,z_n) \neq 0$, there exist unique $g, h \in \mathbb{C}\{z_1,\dots,z_n\}$ with $h(0) \neq 0$ such that g is a Weierstrass polynomial and $f = gh$.*

Remark 10.4. In the case $n = 1$, let $0 \neq f \in \mathbb{C}\{z\}$. Then $f = z^d h$ where $h(0) \neq 0$. This also implies that if $f \in \mathcal{O}(U)$, then the zeros of f do not accumulate in U.

Remark 10.5. The condition $f(0, ..., 0, z_n) \neq 0$ can always be achieved if $f \neq 0$ by a linear change of variable.

Recall (a special case of) Residue Theorem. Suppose that $\varphi \in \mathcal{O}(U \setminus \{a_1, \dots, a_r\})$, take a polydisc Δ such that $\overline{\Delta} \subseteq U$ and $a_i \in \Delta, \forall i$. Then

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial \Delta} \varphi(z) \mathrm{d}z = \sum_{i=1}^{r} \mathrm{Res}_{a_i}(\varphi)$$

In fact, we will only need this when φ is meromorphic at a_i with pole of order ≤ 1. Using Stokes' theorem and the fact that $\mathrm{d}(\varphi(z)\mathrm{d}z) = 0$, we reduce to the computation of the integral of the case $r = 1$. In this case, we have $\varphi = \frac{\psi}{z-a}$. We also know that $\mathrm{Res}_a(\varphi) = \psi(a)$, because of Cauchy's formula

$$\frac{1}{2\pi \mathrm{i}} \int_{|w-a|<r} \frac{\psi(w)}{w-a} \mathrm{d}w = \psi(a).$$

In our case, we will have $f \in \mathcal{O}(U)$ with $\overline{\Delta} \subseteq U$. Consider

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial \Delta} z^j \frac{f'(z)}{f(z)} \mathrm{d}z$$

where f has no zeros on $\partial \Delta$.

Suppose a is a zero of f, we can write $f(z) = (z-a)^m h(z)$ where $h(a) \neq 0$, then $z^j \frac{f'(z)}{f(z)} = z^j \left(\frac{m}{z-a} + \frac{h'(z)}{h(z)} \right)$. So we have $\mathrm{Res}_a z^j \frac{f'(z)}{f(z)} = m a^j$. The conclusion is that

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial \Delta} z^j \frac{f'(z)}{f(z)} \mathrm{d}z = \lambda_1^j + \cdots + \lambda_m^j \tag{10.1}$$

where $\lambda_1, \dots, \lambda_m$ are the roots of f in Δ, counted with multiplicity. Now we are ready to prove Theorem 10.3.

Proof of Theorem 10.3. Write $z' = (z_1, \dots, z_{n-1})$, $f_{z'}(z_n) = f(z', z_n)$. If f is a holomorphic function in $U \ni 0$, let ε_n be such that $f(0, ..., 0, z_n)$ has no zero with $0 < |z_n| \leq \varepsilon_n$. Choose $\varepsilon' > 0$ such that if z' satisfies $|z_i| < \varepsilon'$ for $1 \leq i \leq n-1$ and $|z_n| = \varepsilon_n$, then $f(z', z_n) \neq 0$ and $\{z \mid |z_i| < \varepsilon', \ \forall i \leq n-1, \ |z_n| < \varepsilon_n\} \subseteq U$.

Given z' such that $|z_i| < \varepsilon'$ for $i \leq n-1$, let $\lambda_1(z'), \cdots, \lambda_m(z')$ be the zeros of $f_{z'}(z_n)$ in $\{z_n \mid |z_n| < \varepsilon_n\}$ counted with multiplicities. Then (10.1) implies that

$$\sum_{i=1}^{m} \lambda_i(z')^j = \frac{1}{2\pi \mathrm{i}} \int_{|w|=\varepsilon_n} w^j \frac{\frac{\partial f}{\partial z_n}(z', w)}{f(z', w)} \mathrm{d}w.$$

Note that right-hand side is a holomorphic function. If $j = 0$, then left-hand side is an integer. Hence it's an constant and we have $m = \mathrm{ord}_{z_n} f(0, \cdots, 0, z_n)$ if we take $z' = 0$.

If $\sigma_1(z'), \cdots, \sigma_m(z')$ are the symmetric functions of $\lambda_1(z'), \cdots, \lambda_m(z')$, then each σ_i is holomorphic for $|z_j| < \varepsilon'$ for $j \leq n-1$ and $\sigma_i(0) = 0$ for $1 \leq i \leq m-1$.

Let $g = z_n^m - \sigma_1(z') z_n^{m-1} + \cdots + (-1)^m \sigma_m(z')$. It is a Weierstrass polynomial and it is clear that the function $\frac{f}{g}$ is well-defined and holomorphic in $\{z \mid |z_j| < \varepsilon', \ j \leq n-1, \ |z_n| < \varepsilon_n\} \setminus \{g = 0\}$.

For every z', $\frac{f(z', -)}{g(z', -)}$ extends to a holomorphic function of z_n for $|z_n| < \varepsilon_n$. It's an exercise to check that therefore $h = \frac{f}{g}$ is holomorphic in a neighbourhood of 0 and $h(0) \neq 0$. So the existence is proved.

For uniqueness, if $f = g'h'$ as in the theorem where $g' = z_n^{d'} + \cdots$. Then $f(0, \cdots, 0, z_n) = z_n^{d'} \cdot h(0, \cdots, 0, z_n) \Rightarrow d' = m$. For every z', $f(z', -)$ has d roots in $|z_n| < \varepsilon_n$. So $g'(z', -)$ vanishes on these roots with the right multiplicities. By degree constrain, we must have $g' = g$. ☐

11.1. **The ring $\mathcal{O}_{\mathbb{C}^n,0}$ (continued).**

Corollary 11.1. *For any n, the ring $\mathbb{C}\{z_1,\dots,z_n\}$ is Noetherian.*

Proof. We prove by induction on n. The base case $n=0$ is trivial since it's a filed.

Suppose $0 \neq I \subseteq \mathbb{C}\{z_1,\dots,z_n\}$, Let $(f_\lambda)_{\lambda\in\Lambda}$ be a set of generators for I. Fix $\alpha_0 \in \Lambda$ such that $f_{\lambda_0} \neq 0$. Do a linear change of variable to get $f_{\lambda_0}(0,\cdots,0,z_n) \neq 0$. For $\lambda \neq \lambda_0$, if $f_\lambda(0,\cdots,0,z_n) = 0$, then replace f_λ by $f_\lambda + f_{\lambda_0}$.

Hence we may assume that $f_\lambda(0,\cdots,0,z_n) \neq 0$ for any λ. By Weierstrass Preparation Theorem (Theorem 10.3), $f_\lambda = (\text{invertible element}) \times (\text{element of } \mathbb{C}\{z_1,\dots,z_{n-1}\}[z_n])$. Then this implies that I is generated by $I \cap \mathbb{C}\{z_1,\dots,z_{n-1}\}[z_n]$. But by induction hypothesis, $\mathbb{C}\{z_1,\dots,z_{n-1}\}$ is Noetherian, hence so is $\mathbb{C}\{z_1,\dots,z_{n-1}\}[z_n]$. So I is finitely generated as well. ☐

Proposition 11.2. *If X is a smooth algebraic variety, then the ring homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathsf{an}},x}$ is (faithfully) flat for any $x \in X$.*

Proof. **Step 1**: Suppose $X = \mathbb{A}^n$. Then $R = \mathbb{C}\{z_1,\dots,z_n\} \supseteq \underline{m} = \{f \mid f(0) = 0\}$. It is easy to check that $\underline{m} = \{z_1,\dots,z_n\}$. Moreover, $R/\underline{m}^N \cong \mathbb{C}[z_1,\dots,z_n]/(z_1,\dots,z_n)^N$. Hence

$$\widehat{R} = \varprojlim R/\underline{m}^N \cong \varprojlim \mathbb{C}[z_1,\dots,z_n]/\left(z_1,\dots,z_n\right)^N \cong \mathbb{C}[\![z_1,\dots,z_n]\!].$$

Now for an exact sequence of $\mathbb{C}[z_1,\dots,z_n]_{(z_1,\dots,z_n)}$-modules, it is exact after we base change to $\mathbb{C}[\![z_1,\dots,z_n]\!] \cong \widehat{R}$. But since R is Noetherian by Corollary 11.1, the map $R \to \widehat{R}$ is faithfully flat. We conclude that base change to R is also exact. So $\mathbb{C}[z_1,\dots,z_n]_{(z_1,\dots,z_n)} \to R$ is flat. ☐

Remark 11.3. In the proof, since $\widehat{R}$ is regular of dimension n, we also know that $R = \mathbb{C}\{z_1,\dots,z_n\}$ is a regular ring of dimension n.

Example 11.4. If $X \subseteq Y$ are smooth algebraic varieties with X defined by $\mathscr{J}$, and we consider $X^{\mathsf{an}} \subseteq Y^{\mathsf{an}}$, then the ideal of $\mathcal{O}_{Y^{\mathsf{an}}}$ vanishing on X^{an} is $\mathscr{J}^{\mathsf{an}}$.[Key point: reduce to $X = (x_1 = \cdots = x_r = 0) \subseteq \mathbb{C}^n = Y$]

This has two consequences:

(1) In general, if $X \subseteq \mathbb{C}^N$ is a smooth algebraic variety defined by the ideal I, then

$$\begin{array}{ccc} \mathcal{O}_{X,x} & \xrightarrow{\varphi} & \mathcal{O}_{X^{\mathsf{an}},x} \\ \uparrow & & \uparrow \\ \mathcal{O}_{\mathbb{C}^N,x} & \xrightarrow{\psi} & \mathcal{O}_{(\mathbb{C}^N)^{\mathsf{an}},x} \end{array}$$

where $\mathcal{O}_{X,x} = \mathcal{O}_{\mathbb{C}^N,x}/I\mathcal{O}_{\mathbb{C}^N,x}$ and $\mathcal{O}_{X^{\mathsf{an}},x} = \mathcal{O}_{(\mathbb{C}^N)^{\mathsf{an}},x}/I\mathcal{O}_{(\mathbb{C}^N)^{\mathsf{an}},x}$. Then ψ flat implies that φ is flat.

(2) If $i : X \hookrightarrow Y$ is a closed immersion of smooth algebraic varieties, for any sheaf $\mathscr{F}$ on X, we have $(i_*(\mathscr{F}))^{\mathsf{an}} \xrightarrow{\cong} (i^{\mathsf{an}})_*(\mathscr{F}^{\mathsf{an}})$. It is an exercise to show that we have such a morphism which gives isomorphism on stalks.

Theorem 11.5. *Let X be a smooth projective complex algebraic variety. Then*

(1) The functor $\mathscr{F} \to \mathscr{F}^{\mathsf{an}}$ is exact
(2) If $\mathscr{F},\mathscr{G}$ are coherent sheaves on X, then we have isomorphisms

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}) \xrightarrow{\cong} \operatorname{Ext}^i_{\mathcal{O}_X^{\mathsf{an}}}(\mathscr{F}^{\mathsf{an}},\mathscr{G}^{\mathsf{an}})$$

We need following fact

Fact 11.6. *The cohomology of* $(\mathbb{P}^n)^{\mathsf{an}}$

$$\mathrm{H}^i\left((\mathbb{P}^n)^{\mathsf{an}}, \mathcal{O}_{(\mathbb{P}^n)^{\mathsf{an}}}\right) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i > 0 \end{cases}$$

We will see this via Hodge Theory when we compute $\mathrm{H}^*\left((\mathbb{P}^n)^{\mathsf{an}}, \mathbb{C}\right)$.

Proof of Theorem 11.5. Have $X \hookrightarrow \mathbb{P}^n$. First we treat $\mathscr{F} = \mathcal{O}_X$. Need to show that $\mathrm{H}^i(X, \mathscr{G}) \to \mathrm{H}^i(X^{\mathsf{an}}, \mathscr{G}^{\mathsf{an}})$ is an isomorphism. By pushing-forward to X, we may assume $X = \mathbb{P}^n$. Next suppose $\mathscr{G} = \mathcal{O}_{\mathbb{P}^n}(m)$ and argue by induction on n, The case $n = 0$ is trivial. The key exact sequence is

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0$$

tensor with $\mathcal{O}(m)$. We know the assertion for $\mathcal{O}_{\mathbb{P}^{n-1}}(m)$, then 5-lemma implies that the assertion holds for $\mathcal{O}_{\mathbb{P}^n}(m)$ if and only if it holds for $\mathcal{O}_{\mathbb{P}^n}(m-1)$. Another induction on m finishes the proof. But we know it for $m = 0$, so we are done.

For general $\mathscr{G}$, argue by decreasing induction on i to show $\mathrm{H}^i(X, \mathscr{G}) \to \mathrm{H}^i(X^{\mathsf{an}}, \mathscr{G}^{\mathsf{an}})$ is an isomorphism. For $i >> 0$, both are zero hence it's OK. For induction step, given $\mathscr{G}$, there exists a short exact sequence

$$0 \to \mathscr{G}' \to \mathscr{E} \to \mathscr{G} \to 0$$

where $\mathscr{E}$ is a direct sum of $\mathcal{O}(m)$. So we know the assertion for $\mathscr{E}$. Look at the induced diagram:

$$\begin{array}{ccccccccc}
\mathrm{H}^i(X, \mathscr{G}') & \longrightarrow & \mathrm{H}^i(X, \mathscr{E}) & \longrightarrow & \mathrm{H}^i(X, \mathscr{G}) & \longrightarrow & \mathrm{H}^{i+1}(X, \mathscr{G}') & \longrightarrow & \mathrm{H}^{i+1}(X, \mathscr{E}) \\
\downarrow \beta & & \downarrow \cong & & \downarrow \alpha & & \downarrow \cong \text{by ind.} & & \downarrow \cong \\
\mathrm{H}^i(X^{\mathsf{an}}, (\mathscr{G}')^{\mathsf{an}}) & \longrightarrow & \mathrm{H}^i(X^{\mathsf{an}}, (\mathscr{E})^{\mathsf{an}}) & \longrightarrow & \mathrm{H}^i(X^{\mathsf{an}}, (\mathscr{G})^{\mathsf{an}}) & \longrightarrow & \mathrm{H}^{i+1}(X^{\mathsf{an}}, (\mathscr{G}')^{\mathsf{an}}) & \longrightarrow & \mathrm{H}^{i+1}(X^{\mathsf{an}}, (\mathscr{E})^{\mathsf{an}})
\end{array}.$$

Now by 5-lemma, we know that α is surjective for every $\mathscr{G}$. Hence β is surjective. Again by 5-lemma, α is injective.

For every $\mathscr{F}$ locally free, we have $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \cong \mathrm{H}^i(X, \mathscr{G} \otimes \mathscr{F}^\vee)$. Hence we have isomorphisms $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \cong \mathrm{Ext}^i_{\mathcal{O}_{X^{\mathsf{an}}}}((\mathscr{F})^{\mathsf{an}}, (\mathscr{G})^{\mathsf{an}})$.

For general $\mathscr{F}$, use increasing induction and write

$$0 \to \mathscr{F}' \to \mathscr{E} \to \mathscr{F} \to 0$$

where $\mathscr{E}$ is locally free. Take long exact sequence of Ext and use again 5-lemma. ☐

12. October 07, 2019

12.1. **The tangent bundle of a complex manifold.** Let $V_{\mathbb{R}}$ be a finite dimensional vector space over $\mathbb{R}$. To give a complex vector space structure on V is equivalent to give a linear map $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$ such that $J^2 = -\mathrm{id}$. Then i will acts as J on $V_{\mathbb{R}}$ and we can view it as a $\mathbb{C}$-vector space.

Given such J, write V for corresponding $\mathbb{C}$-vector space structure and $V_{\mathbb{C}}$ for the base change $\mathbb{C}$-vector space. Then J induces $\mathbb{C}$-linear map $J_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$ where $v \otimes \lambda \mapsto J(v) \otimes \lambda$. Then $J^2 = -\mathrm{id}$ implies that $V_{\mathbb{C}} = V' \oplus V''$ where $V' = \{v \in V_{\mathbb{C}} \mid J_{\mathbb{C}}(v) = \mathrm{i}v\}$ and $V'' = \{v \in V_{\mathbb{C}} \mid J_{\mathbb{C}}(v) = -\mathrm{i}v\}$ are $\mathbb{C}$-subspaces of $V_{\mathbb{C}}$. Since J satisfies $x^2 + 1 = 0$, it has only two eigenvalues $\pm\mathrm{i}$. Then V' (resp. V'') is just its i (resp. $-$i) eigenspace.

Explicitly, let us consider $V = \mathbb{C}$ viewed as a two-dimensional $\mathbb{R}$-vector space, write $\begin{pmatrix} a \\ b \end{pmatrix}$ for a complex number $a + b\mathrm{i}$. Then J acts by $\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} -b \\ a \end{pmatrix}$. In $V_{\mathbb{C}} = \{\begin{pmatrix} a_1 + a_2\mathrm{i} \\ b_1 + b_2\mathrm{i} \end{pmatrix}\}$, $J_{\mathbb{C}}(v) = \mathrm{i}v$ gives us

$$\begin{pmatrix} -b_1 - b_2\mathrm{i} \\ a_1 + a_2\mathrm{i} \end{pmatrix} = \begin{pmatrix} -a_2 + a_1\mathrm{i} \\ -b_2 + b_1\mathrm{i} \end{pmatrix} \Leftrightarrow a_2 = b_1, a_1 = -b_2.$$

So the subspace $V' = \{\begin{pmatrix} a_1 + a_2\mathrm{i} \\ a_2 - a_1\mathrm{i} \end{pmatrix}\}$ and $V'' = \{\begin{pmatrix} a_1 + a_2\mathrm{i} \\ -a_2 + a_1\mathrm{i} \end{pmatrix}\}$

Denote by $u \mapsto \overline{u}$ the conjugate linear map: $V_{\mathbb{C}} \to V_{\mathbb{C}}, v \otimes \lambda \mapsto v \otimes \overline{\lambda}$. Then $V_{\mathbb{R}} \hookrightarrow V_{\mathbb{C}}$ is the invariant subspace of the conjugation map.

Claim 12.1. *The composition $V \overset{j}{\hookrightarrow} V_{\mathbb{C}} \overset{\mathrm{pr}_1}{\to} V'$ is a complex isomorphism. and $V \overset{j}{\hookrightarrow} V_{\mathbb{C}} \overset{\mathrm{pr}_2}{\to} V''$ is a conjugate linear isomorphism. Hence $V'' = \overline{V'}$.*

Proof. If $v \in V$, write $v \otimes 1 = v' + v''$ where $v' \in V'$ and $v'' \in V''$. Then $J(v \otimes 1) = \mathrm{i}v' - \mathrm{i}v''$. Hence

$$v' = \frac{1}{2}\left(v - \mathrm{i}J_{\mathbb{C}}(v)\right)$$
$$v'' = \frac{1}{2}\left(v + \mathrm{i}J_{\mathbb{C}}(v)\right).$$

Note that $J_{\mathbb{C}}$ is the extended linear map, it still maps $V_{\mathbb{R}}$ to $V_{\mathbb{R}}$. So $J_{\mathbb{C}}(v) \in V_{\mathbb{R}}$. So $v'' = \overline{v'}$. It's an exercise to check that $v \mapsto v'$ is $\mathbb{C}$-linear and $v \mapsto v''$ is conjugate linear. We need to check that under this map, $J(v) \mapsto \mathrm{i}v'$ and $J(v) \mapsto \overline{\mathrm{i}v'}$. But this is more or less by definition: e.g.

$$J(v) \mapsto \frac{1}{2}\left(J(v) - \mathrm{i}J(J(v))\right) = \frac{1}{2}\left(J(v) + \mathrm{i}v\right) = \mathrm{i}\left(\frac{1}{2}\left(v - \mathrm{i}J(v)\right)\right)$$

Then above form shows that theses two maps are injective. Hence they are bijection by dimension arguement. ∎

Let $U = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. If V has a complex structure, then U also have one induced by V, that is given $\varphi \in U$, $(\lambda\varphi)(v) = \varphi(\lambda v)$ for $\lambda \in \mathbb{C}$. By Claim 12.1, we have a decomposition $U_{\mathbb{C}} = U' \oplus U''$. On the other hand, we have $(V_{\mathbb{C}})^* = \mathrm{Hom}_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong U \otimes \mathbb{C}$ as $\mathbb{C}$ vector spaces. So $(V_{\mathbb{C}})^* \cong U_{\mathbb{C}}$.

Exercise 12.2. Check that via these isomorphisms, $(J_{V,\mathbb{C}})^*$ corresponds to $J_{U,\mathbb{C}}$.

I feel like that this is tautologically true.

Then this implies that $U' = (V')^*$ and $U'' = (V'')^*$.

The decomposition $U_{\mathbb{C}} = U' \oplus U''$ induces a decomposition $(\overset{p}{\wedge} U)_{\mathbb{C}} = \overset{p}{\wedge}(U_{\mathbb{C}}) = \overset{p}{\wedge}(U' \oplus U'') = \oplus_{i+j=p}(\overset{i}{\wedge} U' \otimes \overset{j}{\wedge} U'')$. The conjugation on $U_{\mathbb{C}}$ that maps U' to U'' (and U'' to U') induces conjugation on $(\overset{p}{\wedge} U)_{\mathbb{C}}$ which maps $\overset{i}{\wedge} U' \otimes \overset{j}{\wedge} U''$ to $\overset{j}{\wedge} U' \otimes \overset{i}{\wedge} U''$.

Because we want to talk about $\mathrm{d}z$ and $\mathrm{d}\overline{z}$, we need twice the usual complex dimension. Hence we have to view the complex vector space as a real vector space and then base change to $\mathbb{C}$. But then we have two complex structures, that is where the magic happens.

This globalizes as follows. Suppose M is a smooth real manifold and E is a smooth real vector bundle on M. To give a complex structure on E is equivalent to give a morphism of vector bundles $J : E \to E$ such that $J^2 = -\mathrm{id}$.

In this case, the previous discussion globalizes to give a decomposition:

$$E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C} = E' \oplus E''$$

and we have an isomorphism of complex vector bundles $E \hookrightarrow E_{\mathbb{C}} \overset{\mathrm{pr}_1}{\to} E'$ and a complex conjugate isomorphism $E \hookrightarrow E_{\mathbb{C}} \overset{\mathrm{pr}_2}{\to} E''$, and conjugation operator on $E_{\mathbb{C}}$ mapping E' to E''. The dual E^* also has a complex structure, get corresponding decomposition of $(\overset{p}{\wedge} E^*)_{\mathbb{C}}$ etc.

Definition 12.3. Let M be a smooth real manifold. An *almost complex structure* on M is a complex structure on the tangent bundle $\mathrm{T}M$, i.e. a morphism $J : \mathrm{T}M \to \mathrm{T}M$ of vector bundles such that $J^2 = -\mathrm{id}$.

Proposition 12.4. *If M is a complex manifold, then M carries a canonical almost complex structure. Moreover, if the corresponding decomposition is $\mathrm{T}M_{\mathbb{C}} = \mathrm{T}^{1,0}M \oplus \mathrm{T}^{0,1}M$ then $\mathrm{T}^{1,0}M$ is a holomorphic vector bundle.*

Proof. Enough to treat the case of open subsets of $\mathbb{C}^n$ and then show that biholomorphic maps preserve this sturcture

Let us discuss the decription of the decomposition $V_{\mathbb{C}} = V' \oplus V''$ in terms of basis. Suppose that $x_1, \dots, x_n$ give a basis of V over $\mathbb{C}$. If $y_j = J(x_j)$, then $x_1, \dots, x_n, y_1, \dots, y_n$ form a basis of $V_{\mathbb{R}}$.

Consider these in $V_{\mathbb{C}}$ via the inclusion $V \hookrightarrow V_{\mathbb{C}}$. Let e_j be the V'-component of x_j. Then

$$\begin{aligned} e_j &= \frac{1}{2}(x_j - \mathrm{i}y_j) \\ \overline{e_j} &= \frac{1}{2}(x_j + \mathrm{i}y_j) \end{aligned}$$

So $e_1, \dots, e_n$ form a basis of V' and $\overline{e}_1, \dots, \overline{e}_n$ form a basis for V''. Let $x_1^*, \dots, x_n^*, y_1^*, \dots, y_n^*$ be the dual basis of $U = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. Then $y_j^* = -J_{U,\mathbb{C}}(x_j^*)$ (CHECK) and hence $x_j^* + \mathrm{i}y_j^*$ (resp. $x_j^* - \mathrm{i}y_j^*$,) where $1 \leq j \leq n$ form a basis of U' (resp. U'').

Equip $\mathbb{C}^n$ with the coordinates $z_1, \dots, z_n$ where $z_j = x_j + \mathrm{i}y_j$. If $U \subseteq \mathbb{C}^n$ is open, then $\mathrm{T}U$ is trivialized by $\frac{\partial}{\partial x_1}, \cdots \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}$. Define a complex structure by

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

Then we have a decomposition

$$\mathrm{T}U_{\mathbb{C}} = \underbrace{\mathrm{T}^{1,0}U}_{\text{trivialized by } \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}} \bigoplus \underbrace{\mathrm{T}^{0,1}U}_{\text{trivialized by } \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n}}.$$

The key point is that we have shown that if $f : U \to V$ is a holomorphic map, then the canonical map $\mathrm{T}U_{\mathbb{C}} \to f^*\mathrm{T}V_{\mathbb{C}}$ maps $\mathrm{T}^{1,0}U$ to $f^*\mathrm{T}^{1,0}V$ and $\mathrm{T}^{0,1}U$ to $f^*\mathrm{T}^{0,1}V$. In particular, if f is biholomorphic, the isomorphism $\mathrm{T}U_{\mathbb{C}}$ and $f^*\mathrm{T}V_{\mathbb{C}}$ preserves the decomposition, i.e. it respects the two complex structures. Here we use the fact that if $\varphi : V \to W$ is an $\mathbb{R}$-linear isomorphism between complex vector spaces such that $\varphi \otimes 1 : V_{\mathbb{C}} \to W_{\mathbb{C}}$ maps V' to W' and V'' to W'', then φ is $\mathbb{C}$-linear. This is because of following diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow{\cong} & & \downarrow{\cong} \\ V' & \xrightarrow{\varphi \otimes 1} & W'. \end{array}$$

Note that $\varphi \otimes 1$ is clearly $\mathbb{C}$-linear. Hence φ is $\mathbb{C}$-linear.

This finishes the proof of the first statement. For the second part, if $f : \mathbb{C}^n \supseteq V \to W \subseteq \mathbb{C}^m$ is holomorphic, then $f^* : \mathrm{T}^{1,0}V \to \mathrm{T}^{1,0}W$ is given by the Jacobian matrix $(\frac{\partial f_i}{\partial z_j})$. In particular, the transition maps of $\mathrm{T}^{1,0}M$ are holomorphic functions. □

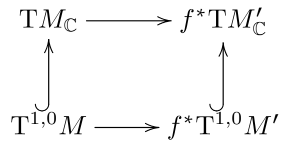
13. October 09, 2019

13.1. **The Dolbeault complex.** *Last time*: If M is a complex manifold, then $\mathrm{T}M$ has a canonical complex structure and a decomposition

$$\mathrm{T}M_{\mathbb{C}} = \mathrm{T}^{1,0}M \oplus \mathrm{T}^{0,1}M$$

where $\mathrm{T}^{1,0}M$ is a holomorphic vector bundle. It is called the *holomorphic tangent bundle* of M.

If $f : M \to M'$ is a holomorphic map, then



Dually, we have a decomposition

$$\mathrm{T}^*M_{\mathbb{C}} = \mathcal{A}_M^{1,0} \oplus \mathcal{A}_M^{0,1}$$

where $\mathcal{A}_M^{1,0}$ is the dual of $\mathrm{T}^{1,0}M$. We also have a corresponding decomposition for $\overset{p}{\bigwedge}(\mathrm{T}^*M)$.

Write $\mathcal{A}_M^m$ for the sheaf of real smooth m-forms on M and $\mathcal{A}_{M,\mathbb{C}}^m = \mathcal{A}_M^m \otimes_{\mathbb{R}} \mathbb{C}$. Then we have decomposition $\mathcal{A}_{M,\mathbb{C}}^m = \oplus_{p+q=m}\mathcal{A}_M^{p,q}$ where $\mathcal{A}_M^{p,q}$ is the sheaf of (p,q)-forms on M. Note that $\overline{\mathcal{A}_M^{p,q}} = \mathcal{A}_M^{q,p}$.

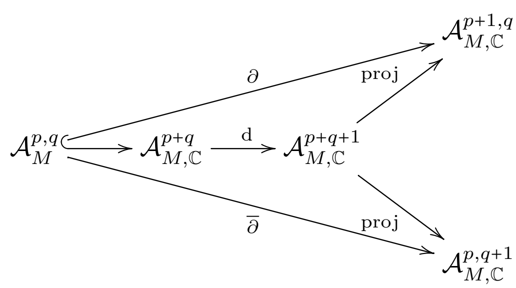
Note that $\mathcal{A}_M^{p,0}$ is the sheaf of smooth sections of a holomorphic vector bundle. Inside it we have $\mathcal{A}_M^{p,0} \supseteq \Omega^p$ the sheaf of holomorphic sections. Then Ω^1 is the sheaf of holomorphic sections of $\left(\mathrm{T}^{1,0}M\right)^*$.

Locally in a chart with coordinates $z_1, \dots, z_n$, $\mathcal{A}^{p,q}$ is the free $\mathscr{C}_{M,\mathbb{C}}^\infty$-module, with basis $\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J, |I| = p, |J| = q$ where if $I = i_1 < \cdots < i_p$,

$$\begin{aligned} \mathrm{d}z_I &= \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p} \\ \mathrm{d}\overline{z}_I &= \mathrm{d}\overline{z}_{i_1} \wedge \cdots \wedge \mathrm{d}\overline{z}_{i_p} \end{aligned}$$

The module Ω^p consists of basis $\mathrm{d}z$ with coefficients holomorphics functions, i.e. smooth functions only invovle z. Meanwhile $\mathcal{A}^{p,q}$ involves basis like above and coefficients of smooth functions in $z, \overline{z}$.

Recall that we have de Rham differentials $\mathrm{d} : \mathcal{A}_M^m \to \mathcal{A}_M^{m+1}$. Given p, q, consider following diagram



Then extend this linearly to $\mathcal{A}_{M,\mathbb{C}}^m$.

Proposition 13.1. *We have* $\mathrm{d} = \partial + \overline{\partial}$.

Proof. Let us compute ∂ and $\overline{\partial}$ locally. Consider the chart with coordinates $z_1, \dots, z_n$. Write

$$\begin{aligned} \omega &= f \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J \\ \mathrm{d}\omega &= \mathrm{d}f \wedge \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J \end{aligned}$$

where $|I| = p$ and $|J| = q$. Note that

$$\begin{aligned} \mathrm{d}f &= \sum_{i=1}^n \frac{\partial f}{\partial x_j}\mathrm{d}x_j + \sum_{j=1}^n \frac{\partial f}{\partial y_j}\mathrm{d}y_j \\ &= \underbrace{\sum_{i=1}^n \frac{\partial f}{\partial z_j}\mathrm{d}z_j}_{\partial f} + \underbrace{\sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j}\mathrm{d}\overline{z}_j}_{\overline{\partial} f} \end{aligned}$$

So we have

$$\partial\omega = \partial f \wedge \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J$$
$$\overline{\partial}\omega = \overline{\partial} f \wedge \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J$$

Hence $\mathrm{d}\omega = \partial\omega + \overline{\partial}\omega$. □

Corollary 13.2. *We have $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$.*

Proof. Use $\mathrm{d}^2 = 0$ and look at the corresponding graded pieces. □

Corollary 13.3. *Both $\partial, \overline{\partial}$ are derivations in the sense:*

$$\partial(\omega_1 \wedge \omega_2) = \partial\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge \partial\omega_2$$

and similarly for $\overline{\partial}$.

For every p, we have following complex:

$$0 \to \mathcal{A}_M^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}_M^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}_M^{p,n} \to 0$$

where $n = \dim X$ as a complex manifold.

Then the global sections $\Gamma(M, \mathcal{A}_M^{p,\bullet})$ are the pth *Dolbeault complex* of M and its cohomology is called *Dolbeault cohomology*.

$$\mathrm{H}^{p,q}(M) := \mathrm{H}^q\left(\Gamma(M, \mathcal{A}_M^{p,\bullet})\right).$$

Remark 13.4. The (sheaf) kernel $\mathscr{K}\!er(\mathcal{A}_M^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}_M^{p,1}) = \Omega_M^p$.

Proof of Remark 13.4. Locally, we have $\omega = \sum_I f_I \mathrm{d}z_I$. Then it is in the kernel if and only $\frac{\partial f_I}{\partial \overline{z}_j} = 0$ for any j, I, which is if and only if f_I is holomorphic for any I. □

Look at following diagram

$$\begin{array}{ccccccc}
\mathcal{A}_M^{p,0} & \xrightarrow{\overline{\partial}} & \mathcal{A}_M^{p,1} & \xrightarrow{\overline{\partial}} & \mathcal{A}_M^{p,2} & \xrightarrow{\overline{\partial}} & \cdots \\
\uparrow{\scriptstyle\partial} & & \uparrow{\scriptstyle\partial} & & \uparrow{\scriptstyle\partial} & & \\
\vdots & & \vdots & & \vdots & & \cdots \\
\uparrow{\scriptstyle\partial} & & \uparrow{\scriptstyle\partial} & & \uparrow{\scriptstyle\partial} & & \\
\mathcal{A}_M^{1,0} & \xrightarrow{\overline{\partial}} & \mathcal{A}_M^{1,1} & \xrightarrow{\overline{\partial}} & \mathcal{A}_M^{1,2} & \xrightarrow{\overline{\partial}} & \cdots \\
\uparrow{\scriptstyle\partial} & & \uparrow{\scriptstyle\partial} & & \uparrow{\scriptstyle\partial} & & \\
\mathcal{A}_M^{0,0} & \xrightarrow{\overline{\partial}} & \mathcal{A}_M^{0,1} & \xrightarrow{\overline{\partial}} & \mathcal{A}_M^{0,2} & \xrightarrow{\overline{\partial}} & \cdots
\end{array}$$

The kernel of the first two columns gives you the complex of Ω_M^p. The kernel of consective columns are the measure of "non-holomonicness". If the differential donesn't involve any $\mathrm{d}\overline{z}_i$, then the coefficient f necessarily satisfies $\frac{\partial f}{\partial \overline{z}_i} = 0$. So for the first first column, the kernel is the set of holomorphic differentials. Meanwhile for later columnes, we are asking being holomorphic according to the $\mathrm{d}\overline{z}_i$ in the differential.

More generally, suppose E is a holomorphic vector bundle with sheaf of smooth sections $\mathscr{E}$.

Claim 13.5. *We can define a canonical map:*

$$\overline{\partial}_{\mathscr{E}} : \mathcal{A}_M^{p,q} \otimes \mathscr{E} \to \mathcal{A}_M^{p,q+1} \otimes \mathscr{E}$$

such that $\overline{\partial}_{\mathscr{E}}^2 = 0$ and $\overline{\partial}_{\mathscr{E}}$ is a derivation.

Proof. Working locally on the chart U such that $\mathscr{E}|_U$ has a trivialization by *holomorphic* sections $s_1, \ldots, s_r$. Then any section ω of $\mathcal{A}_M^{p,q} \otimes \mathscr{E}$ can be written as $\omega = \sum_{i=1}^r \omega_i s_i$. So we have

$$\overline{\partial}_{\mathscr{E}}(\omega) = \sum_{i=1}^r \overline{\partial}(\omega_i) s_i$$

This is independent of choice of trivialization becase $\overline{\partial}$ is linear with respect to holomorphic functions. So the independence of choice implies that these glue to give $\overline{\partial}_{\mathscr{E}}$ on M. It's easy to check that $\overline{\partial}_{\mathscr{E}}^2 = 0$ and $\overline{\partial}_{\mathscr{E}}$ is a derivation. □

So we get a twisted Dolbeault complex:

$$0 \to \mathcal{A}^{p,0} \otimes \mathscr{E} \xrightarrow{\overline{\partial}_{\mathscr{E}}} \mathcal{A}^{p,1} \otimes \mathscr{E} \to \cdots$$

and

$$\mathrm{H}^{p,q}(M; \mathscr{E}) := \mathrm{H}^q(\Gamma(M, \mathcal{A}_M^{p,\bullet} \otimes \mathscr{E})).$$

Two things to do:

- $\mathcal{A}_M^{p,\bullet}$ is acyclic, i.e. $\mathscr{H}^q(\mathcal{A}_M^{p,\bullet}) = 0$ for $q \geq 1$. (Use $\mathscr{H}$ for cohomology of complexes.)
- We can use this complex to compute $\mathrm{H}^q(M, \Omega_M^p)$.

Proposition 13.6 ($\overline{\partial}$-lemma)**.** *If ω is a (p,q)-form on $U \subseteq M$ such that $\overline{\partial}\omega = 0, q \geq 1$, then locally we can find $\beta \in \mathcal{A}_M^{p,q-1}$ such that $\overline{\partial}\beta = \omega$, i.e. $\mathcal{A}_M^{p,\bullet}$ is acyclic.*

Proof. Working locally, we may assume that we have a chart with coordinates $z_1, \ldots, z_n$.

Step 1: Reduce to $p = 0$. Write

$$\omega = \sum_{I,J} f_{I,J} \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J,$$

and we know that $\overline{\partial}\omega = 0$.

For every I, consider $\omega_I = \sum_J f_{I,J} \mathrm{d}\overline{z}_J$. Then $\overline{\partial}\omega = 0 \Rightarrow \overline{\partial}\omega_I = 0$ for all I. If we know the case for $p = 0$, then locally $\omega_I = \overline{\partial}\beta_I$ for some $(0, q-1)$-form β_I. If $\beta = \sum_I (-1)^p \mathrm{d}z_I \wedge \beta_I$, then $\overline{\partial}\beta = \omega$. □

14. October 11, 2019

14.1. The Dolbeault complex (continued).

Remark 14.1. Remarks about $\mathrm{T}^{1,0}M$:

(1) As in the case of the tangent bundle to a smooth manifold, this can be described as the derivations on the local rings $\mathcal{O}_{M,x}$ for $x \in M$.
(2) If X is a smooth algebraic variety, and $M = X^{\mathsf{an}}$, then $\mathrm{T}^{1,0}M \cong (\mathrm{T}X)^{\mathsf{an}}$.

Proof of Proposition 13.6. **Step 2**: We may assume that $p = 0$. Working locally, we are in a chart with coordinates $z_1, \ldots, z_n$. Write $\omega = \sum_{|J|=q} f_J \mathrm{d}\overline{z}_J$. We proceed by induction on k being the largest index such that $\mathrm{d}\overline{z}_k$ shows up in some $\mathrm{d}\overline{z}_J$ with nonzero coefficients.

If $\omega \neq 0$, then the smallest index is q in z, and we have

$$\omega = f \mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_q$$

So we have

$$\overline{\partial}\omega = 0 \Leftrightarrow \frac{\partial f}{\partial \overline{z}_i} = 0 \text{ for } i > q$$

i.e. f is holomorphic with respect to $z_{q+1},\cdots,z_n$. By Theorem 3.7 (1-variable $\overline{\partial}$-lemma), locally we have some g smooth and holomorphic with respect to $z_{q+1},\cdots,z_n$ such that $\frac{\partial g}{\partial \overline{z}_1} = f$ and hence

$$\overline{\partial}\left(g\mathrm{d}\overline{z}_2 \wedge \cdots \wedge \mathrm{d}\overline{z}_q\right) = \omega$$

For induction step, write

$$\omega = \underbrace{\omega_1}_{(0,q)\text{-form}} + \underbrace{\omega_2}_{(0,q-1)\text{-form}} \wedge \mathrm{d}\overline{z}_k$$

such that ω_1 and ω_2 only involve $\mathrm{d}\overline{z}_1,\cdots,\mathrm{d}\overline{z}_{k-1}$. Then $\overline{\partial}\omega = 0$ implies that the coefficients of ω_1,ω_2 are holomorphic in $z_{k+1},\cdots,z_n$. Write $\omega_2 = \sum_{|J|=q-1} a_J \mathrm{d}\overline{z}_J$. Apply Theorem 3.7 (1-variable $\overline{\partial}$-lemma) again to find locally smooth functions b_J, holomorphic in $z_{k+1},\cdots,z_n$ such that $\frac{\partial b_J}{\partial \overline{z}_k} = a_J$. So

$$\overline{\partial}\underbrace{\left(\sum_{|J|=q-1} b_J \mathrm{d}\overline{z}_J\right)}_{\beta'} = \sum_{|J|=q-1} (-1)^{q-1} a_J \mathrm{d}\overline{z}_J \wedge \mathrm{d}\overline{z}_k + \text{stuff involving only } \mathrm{d}\overline{z}_1,\cdots,\mathrm{d}\overline{z}_{k-1}.$$

implies that $\omega - (-1)^{q-1}\overline{\partial}(\beta')$ only involves $\mathrm{d}\overline{z}_1,\cdots,\mathrm{d}\overline{z}_{k-1}$. By induction, this is $\overline{\partial}(\gamma) = 0$. So $\omega = \overline{\partial}(\gamma + (-1)^{q-1}\beta')$. □

Exercise 14.2. Repeat the whole argument when M is a smooth manifold to show that if ω is a p-form, $p \geq 1$, which is closed ($\mathrm{d}\omega = 0$), then ω is locally exact.

Corollary 14.3. *For every $p \geq 0$, we have an exact complex of sheaves on M,*

$$0 \to \Omega_M^p \hookrightarrow \mathcal{A}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1} \to \cdots \xrightarrow{\overline{\partial}} \mathcal{A}^{p,n} \to 0.$$

More generally, if E is the holomorphic vector bundle, with sheaf of smooth sections $\mathscr{E}$, sheaf of holomorphic sections $\mathscr{E}^{\mathrm{hol}}$, we have exact complex,

$$0 \to \Omega_M^p \otimes_{\mathcal{O}_M} \mathscr{E}^{\mathrm{hol}} \to \mathcal{A}^{p,0} \otimes_{\mathscr{C}^\infty_{M,\mathbb{C}}} \mathscr{E}^{\mathrm{hol}} \xrightarrow{\overline{\partial}_E} \mathcal{A}^{p,1} \otimes_{\mathscr{C}^\infty_{M,\mathbb{C}}} \mathscr{E}^{\mathrm{hol}} \to \cdots$$

Exactness follows since locally the complex is isomorphic to a direct sum of $r = \mathrm{rank}(E)$ copies of the Dolbeault complex.

In the case of smooth manifold, we have similar exact complex: De Rham complex

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}_M^0 \xrightarrow{\mathrm{d}} \mathcal{A}_M^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{A}_M^n \to 0$$

where $\dim_{\mathbb{R}} M = n$ and $\underline{\mathbb{R}}$ are locally constant functions.

14.2. **Soft sheaves on paracompact spaces.** Let X be a topological space and let $\mathscr{F}$ be the sheaf of abelian groups on X. If Z is a subset of X with $i : Z \hookrightarrow X$ an inclusion, then $\mathscr{F}(Z) = \Gamma(Z,\mathscr{F}) \coloneqq \Gamma(Z, i^{-1}\mathscr{F})$ is the set of maps $s : Z \to \coprod_{x\in Z} \mathscr{F}_x$ such that $\forall x \in Z$ we have $s(x) \in \mathscr{F}_x$ and there is an open neighbourhood U of x in X and $t \in \mathscr{F}(U)$ such that $s(y) = t_y, \forall y \in U \cap Z$.

Proposition 14.4. *If $\mathscr{F}$ as above, $Z_1,\ldots,Z_r$ closed subsets of X, then we have exact sequence*

$$0 \to \mathscr{F}(\bigcup_i Z_i) \to \prod_i \mathscr{F}(Z_i) \to \prod_{i,j} \mathscr{F}(Z_i \cap Z_j)$$

induced by restriction maps.

Remark 14.5. If $Z' \subseteq Z$, then we have natural restriction maps $\mathscr{F}(Z) \to \mathscr{F}(Z')$ which are functorial.

Proof of Proposition 14.4. Suppose $(s_i)_{1\leq i\leq r}$ where $s_i : Z_i \to \coprod_{x\in Z_i} \mathscr{F}_x$ such that $s_i(x) \in \mathscr{F}_x$ and are compatible with each other. We know that $s_i(x) = s_j(x), \forall x \in Z_i \cap Z_j$. Only need to check the compatibility condition. Fix $x \in X$, we can replace X by an open neighbourhood of x. Since Z_i's are closed, we may assume that $x \in Z_1 \cap \cdots \cap Z_r$. Moreover, may assume that for any i, there exists $t_i \in \mathscr{F}(X)$ such that $(t_i)_y = s_i(y)$ for any $y \in Z_i$. In particular, we have $(t_1)_x = \cdots = (t_r)_x$. Then further restrict to make sure that

$t_1 = \cdots = t_r = t$. Clearly $t_y = s_i(y)$ for any y and any suitable i. Then we can conclude that there exists a unique $s : \cup_i Z_i \to \coprod_{x \in \cup Z_i} \mathscr{F}_x$ such that $s(x) \in \mathscr{F}_x, \forall x \in X$ and $s|_{Z_i} = s_i$. □

Definition 14.6. X is *paracompact* if

- X is Hausdorff and
- every open cover admits a refinement which is locally finite.

Example 14.7. Some paracompact spaces

(1) Topological manifolds are paracompact
(2) CW complexes are paracompact

Remark 14.8. If Z is closed in X and X is paracompact, then Z is paracompact.

Fact 14.9. *If $X = \cup_{i \in I} U_i$ locally finite open cover and X is paracompact, then there exists an open cover $X = \cup_i V_i$ such that $\overline{V}_i \subseteq U_i$ for all i. e.g. If $A, B \subseteq X$ are closed subsets and $A \cap B = \varnothing$, then there exists U, V open such that $A \subseteq U, B \subseteq V, U \cap V = \varnothing$. (i.e. X is a normal space)*

Definition 14.10. Let $\mathscr{F}$ be a sheaf of abelian groups on X. Then $\mathscr{F}$ is *soft* if for any $Z \subseteq X$ closed, the restriction map $\mathscr{F}(X) \to \mathscr{F}(Z)$ is surjection.

Compare: $\mathscr{F}$ is flasque if for any $U \subseteq X$ open, $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective.

Fact 14.11. *Flasque sheaves are* acyclic *(i.e. the higher cohomology vanishes).*

So we can compute cohomology via flasque resolutions.

15. October 16, 2019

15.1. Soft sheaves on paracompact spaces (continued).

Let X be a paracompact topological space.

Proposition 15.1. *Let $\mathscr{F}$ be a sheaf of abelian groups on X. If $Z \subseteq X$ is closed, then for any $s \in \mathscr{F}(Z)$, there exists $U \supseteq Z$ open and $t \in \mathscr{F}(U)$ such that $t|_Z = s$.*

Proof. See [Mus, Lemma 2.3(ii)]. □

Corollary 15.2. *If $\mathscr{F}$ is flasque, then $\mathscr{F}$ is soft.*

Proposition 15.3. *If $0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$ is exact, and $\mathscr{F}'$ is soft, then $0 \to \mathscr{F}'(X) \to \mathscr{F}(X) \to \mathscr{F}''(X) \to 0$ is exact.*

Proof. Only need to show that if $s \in \mathscr{F}''(X)$, then there exists $\widetilde{s} \in \mathscr{F}(X)$ such that $\psi(\widetilde{s}) = s$. Since ψ is surjective, there exists open cover $X = \cup_{i \in I} U_i$, $\widetilde{s}_i \in \mathscr{F}(U_i)$ such that $\psi(\widetilde{s}_i) = s|_{U_i}$. After passing to some refinement, we may assume that this is a locally finite cover. Then there exists an open cover $X = \cup_i V_i$ with $\overline{V}_i \subseteq U_i$. For $J \subseteq I$, $Z_J := \cup_{i \in J} \overline{V}_i$ closed in X by local finiteness.

Consider the pair (J, t) with $J \subseteq I, t \in \mathscr{F}(Z_J)$ such that $\psi(t) = s|_{Z_J}$. We say $(J, t) \le (J', t')$ if $J \subseteq J'$ and $t'|_{Z_J} = t$. Then by Zorn's lemma, we can choose a maximal element (J, t). If $J \neq I$,then there exists $i \in I \setminus J$. Since $t \in \mathscr{F}(Z_J)$ and $\widetilde{s}_i|_{\overline{V}_i} \in \mathscr{F}(\overline{V}_i)$, we have

$$\psi\left(t|_{Z_J \cap \overline{V}_i}\right) = \psi\left(\widetilde{s}_i|_{Z_J \cap \overline{V}_i}\right)$$
$$\Rightarrow t|_{Z_J \cap \overline{V}_i} - \widetilde{s}_i|_{Z_J \cap V_i} = \varphi(w)$$

for some $w \in \mathscr{F}'(Z_J \cap \overline{V}_i)$. Since $\mathscr{F}'$ is soft, there exists $\widetilde{w} \in \mathscr{F}'(X)$ such that $\widetilde{w}|_{Z_J \cap \overline{V}_i} = w$. Replacing $\widetilde{s}_i|_{U_i}$ by $\widetilde{s}_i|_{U_i} + \varphi(\widetilde{w}|_{U_i})$, we may assume that $t|_{Z_J \cap \overline{V}_i} = \widetilde{s}_i|_{Z_J \cap \overline{V}_i}$. By Proposition 14.4, there exists $t' \in \mathscr{F}(Z_{J \cup \{i\}})$ such that $t'|_{Z_J} = t, t'|_{\overline{V}_i} = \widetilde{s}_i|_{\overline{V}_i}$. So $\psi(t') = s|_{Z_{J \cup \{i\}}}$, which is a contradiction!

Hence $J = I$ and the proof is finished. □

Corollary 15.4. *Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence. If $\mathscr{F}', \mathscr{F}$ are both soft, then so is $\mathscr{F}''$.*

Proof. $Z \subseteq X$ closed, so it is paracompact and $\mathscr{F}'|_Z$ is soft. We have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}'(X) & \longrightarrow & \mathscr{F}(X) & \longrightarrow & \mathscr{F}''(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{F}'(Z) & \longrightarrow & \mathscr{F}(Z) & \twoheadrightarrow & \mathscr{F}''(Z) & \longrightarrow & 0. \end{array}$$

The middle vertical map is surjective because $\mathscr{F}$ is soft. The right horizontal map is surjective by Proposition 15.3. So $\mathscr{F}''(X) \to \mathscr{F}''(Z)$ is surjective. ☐

Theorem 15.5. *We have*

(1) If $\mathscr{E}$ is soft sheaf on X, then $\mathrm{H}^i(X, \mathscr{E}) = 0, \forall i > 0$.
(2) If $\mathscr{F}$ has a resolution $0 \to \mathscr{F} \to \mathscr{E}^0 \to \mathscr{E}^1 \to \cdots$ with all $\mathscr{E}^i$ soft, then

$$\mathrm{H}^i(X, \mathscr{F}) = \mathscr{H}^i(\Gamma(X, \mathscr{E}^\bullet))$$

Proof. (2) follows from (1) by general arguments.

For (1), argue by induction on i. Consider an exact sequence

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$$

with $\mathscr{F}$ flasque (in particular, $\mathscr{F}$ is soft). By Corollary 15.4, $\mathscr{G}$ is soft.

Look at the long exact sequence in cohomology:

$$\Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G}) \to \mathrm{H}^1(X, \mathscr{E}) \to \underbrace{\mathrm{H}^1(X, \mathscr{F})}_{\text{is 0 because } \mathscr{F} \text{ is flasque}}$$

The first map is surjection by Proposition 15.3 since $\mathscr{E}$ is soft. Hence $\mathrm{H}^1(X, \mathscr{E}) = 0$.

For higher ones, we have $\mathrm{H}^{i+1}(X, \mathscr{E}) \cong \mathrm{H}^i(X, \mathscr{G})$ for $i \geq 1$. Since $\mathscr{G}$ is soft. The proof is finished by induction. ☐

Exercise 15.6. Suppose $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces with X, Y paracompact, and $\mathscr{F}$ sheaf on Y, $\mathscr{G}$ sheaf on X such that we have induced morphism $f^*\mathscr{F} \to \mathscr{G}$. This induces $\mathrm{H}^i(Y, \mathscr{F}) \to \mathrm{H}^i(X, f^*\mathscr{F}) \to \mathrm{H}^i(X, \mathscr{G})$. Show that if $\mathscr{F} \to \mathscr{E}^\bullet$ and $\mathscr{G} \to \mathscr{M}^\bullet$ are soft resolutions repesctively, and morphisms

$$\begin{array}{ccc} f^*\mathscr{F} & \longrightarrow & f^*\mathscr{E}^\bullet \\ \downarrow & & \downarrow \\ \mathscr{G} & \longrightarrow & \mathscr{M}^\bullet \end{array}$$

Then we have

$$\begin{array}{ccc} \mathrm{H}^i(Y, \mathscr{F}) & \xrightarrow{\cong} & \mathscr{H}^i\Gamma(Y, \mathscr{E}^\bullet) \\ \downarrow & & \downarrow \\ \mathrm{H}^i(X, \mathscr{G}) & \xrightarrow{\cong} & \mathscr{H}^i\Gamma(X, \mathscr{M}^\bullet) \end{array}$$

Proposition 15.7. *If M is a smooth real manifold, then any $\mathscr{C}^\infty_M$-module is soft.*

Proof. Let $\mathscr{F}$ be a $\mathscr{C}^\infty_M$-module, $Z \subseteq M$ is a closed subset. For $s \in \mathscr{F}(Z)$, Proposition 15.1 implies that there exists $U \supseteq Z$ open, $\widetilde{s} \in \mathscr{F}(U)$ such that $\widetilde{s}|_Z = s$.

Consider $Z \subseteq U$. There exists U_1 open such that $Z \subseteq U_1 \subseteq \overline{U}_1 \subseteq U$ and U_2 open such that $\overline{U}_1 \subseteq U_2 \subseteq \overline{U}_2 \subseteq U$.

By smooth version of Urysohn's lemma, there exists smooth function φ such that $\varphi = 1$ on $\overline{U}_1$ and $\varphi = 0$ on $M \setminus U_2$. Consider $\varphi|_U \cdot \widetilde{s}$ which is 0 on $U \setminus U_2$. There exists $s' \in \mathscr{F}(M)$ such that $s'|_{M\setminus \overline{U}_2} = 0$ and $s'|_U = \varphi|_U \cdot \widetilde{s}|_U$ since $\varphi = 0$ on $M \setminus \overline{U}_2$. And $s'|_{U_1} = \widetilde{s}|_{U_1}$ since $\varphi = 1$ on U_1. We conclude that $s'|_Z = s$. □

Next we discuss some applications of what we have proved.

(1) M is a smooth manifold of dimension n. We have a resolution of $\underline{\mathbb{R}}$ by

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0_M \xrightarrow{\mathrm{d}} \mathcal{A}^1_M \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{A}^n_M \to 0$$

By de Rham's theorem, $\mathrm{H}^p(X, \underline{\mathbb{R}}) \cong \mathrm{H}^p_{\mathsf{dR}}(X)$.

Fact 15.8. *Since M is paracompact and locally contractible, the singular cohomology $\mathrm{H}^p(M, \mathbb{R})$ is isomorphic to the sheaf cohomology $\mathrm{H}^p(M, \underline{\mathbb{R}})$.*

(2) If M is a complex manifold of dimension n. For any p, we have an exact complex

$$0 \to \Omega^p_M \to \mathcal{A}^{p,0}_M \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1}_M \to \cdots \to \mathcal{A}^{p,n}_M \to 0 \tag{15.1}$$

and

$$\mathrm{H}^{p,q}(X) \coloneqq \mathscr{H}^q(\Gamma(M, \mathcal{A}^{p,\bullet}_M)) \cong \mathrm{H}^q(M, \Omega^p_M)$$

More generally, if E is a holomorphic vector bundle with sheaf of holomorphic sections $\mathscr{E}$, then the sheaf section is $\mathscr{C}^\infty_M \otimes_{\mathcal{O}_M} \mathscr{E} = \mathscr{E}_{\mathrm{sm}}$. We have with (15.1) $\otimes_{\mathscr{C}^\infty_M} \mathscr{E}_{\mathrm{sm}}$ = (15.1) $\otimes_{\mathcal{O}_M} \mathscr{E}$, get complex

$$0 \to \Omega_M \otimes_{\mathcal{O}_X} \mathscr{E} \to \mathcal{A}^{p,0}_{M,\mathscr{E}} \xrightarrow{\overline{\partial}_{\mathscr{E}}} \cdots \xrightarrow{\overline{\partial}_{\mathscr{E}}} \mathcal{A}^{p,n}_{M,\mathscr{E}} \to 0,$$

which induces

$$\mathrm{H}^{p,q}(X, \mathscr{E}) \coloneqq \mathrm{H}^q\left(\Gamma\left(M, \mathcal{A}^{p,\bullet}_{M,\mathscr{E}}\right)\right) \cong \mathrm{H}^q\left(M, \Omega^p_M \otimes \mathscr{E}\right).$$

16. October 18, 2019

16.1. Hodge thoery on compact, oriented, Riemannian manifolds.

16.1.1. *Some linear algebra.* Let V be a finite dimensional vector space over $\mathbb{R}$.

Definition 16.1. A *scalar product* on V is a symmetric bilinear form $\langle -, - \rangle$ on $V \times V$ with values in $\mathbb{R}$, which is positive definite, i.e. $\langle v, v \rangle > 0$ for any $v \neq 0$.

Given a scalar product on V, have $V \cong V^*, v \mapsto \varphi_v = \langle v, - \rangle$ and we put a scalar product on V^* such that $\langle \varphi_v, \varphi_w \rangle = \langle v, w \rangle$ for any v, w. E.g. If $e_1, \ldots, e_n$ is an orthonormal basis of V, then $e_1^*, \ldots, e_n^*$ is the dual basis on V^*. The basis $e_1^*, \ldots, e_n^*$ is also orthonormal.

Exercise 16.2. Given a scalar product $\langle -, - \rangle$ on V, we have an induced scalar product on each $\bigwedge^p V$ such that $\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \det\left(\langle v_i, w_j \rangle\right)$.

We only need to check if this is positive definite. Consider a vector $a = \sum_I a_I e_I$ where I runs through all p-element subsets of $\{1, 2, \cdots, n\}$ and e_is are orthonormal basis. Then

$$\begin{aligned}
\langle a, a \rangle &= \sum_{I,J} a_I a_J \left\langle e_{i_1} \wedge \cdots \wedge e_{i_p}, e_{j_1} \wedge \cdots \wedge e_{j_p} \right\rangle \\
&= \sum_{I,J} a_I a_J \det\left(\langle e_{i_k}, e_{j_l} \rangle\right)_{k,l}
\end{aligned}$$

If, say e_{i_1}, is not in J, then the first row of the matrix will be zero and hence the determinant is zero. So the only possibility for the determinant to be nonzero is that $I = J$. Hence we have

$$\begin{aligned}
\langle a, a \rangle &= \sum_I a_I^2 \det\left(\langle e_{i_k}, e_{i_l} \rangle\right)_{k,l} \\
&= \sum_I a_I^2 > 0
\end{aligned}$$

which is positive unless a is a zero vector.

Under this definition, if $e_1, \dots, e_n$ is an orthonrmal basis, then $\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_p} \mid I = \{i_1 < \cdots < i_p\} \subseteq \{1, \cdots, n\}\}$ is an orthonormal basis for $\bigwedge^p V$. Suppose now, in addition, that on V we also have an orientation. In this case, we get canonical *volume element* $\mathsf{vol} \in \bigwedge^n V$ where $n = \dim V$. By choosing orthonormal basis $e_1, \dots, e_n$ such that $e_1 \wedge \cdots \wedge e_n$ positive, then $\mathsf{vol} = e_1 \wedge \cdots \wedge e_n$.

If $e'_1, \dots, e'_n$ is another such basis, then $e'_i = \sum_j a_{ij} e_j$. The transition matrix $A = (a_{ij})$ is a orthonormal matrix, i.e. $A \cdot A^t = I_n \Rightarrow (\det(A))^2 = 1$. Since $e'_1 \wedge \cdots \wedge e'_n = (\det A)\, e_1 \wedge \cdots \wedge e_n$, and both $e'_1 \wedge \cdots \wedge e'_n$ and $e_1 \wedge \cdots \wedge e_n$ are positive, we have $\det(A) = 1$.

16.1.2. *The $*$ operator.* Given $(V, \langle -, - \rangle, \text{orientation})$ and $n = \dim V$.

Proposition 16.3. *For every p, $0 \leqslant p \leqslant n$, there is a unique isomorphism*

$$\bigwedge^p V \xrightarrow{*} \bigwedge^{n-p} V$$

such that $v \wedge (*w) = \langle v, w \rangle \mathsf{vol}$ in $\bigwedge^n V$ for all $v, w \in \bigwedge^p V$.

Proof. Recall that we have a nondegenerate bilinear map $\bigwedge^p V \times \bigwedge^{n-p} V \to \bigwedge^n V \cong \mathbb{R}$. Hence every linear map $\bigwedge^p V \to \bigwedge^n V$ comes from some element in $\bigwedge^{n-p} V$. Given $w \in \bigwedge^p V$, we have a map $\langle -, w \rangle \mathsf{vol} : \bigwedge^p V \to \bigwedge^n V \cong \mathbb{R}$. So this map corresponds to some element in $\bigwedge^{n-p} V$, call it $*w$. Then we get a linear map $\bigwedge^p V \xrightarrow{*} \bigwedge^{n-p} V, w \mapsto \langle -, w \rangle \mathsf{vol} \mapsto *w$.

It is clearly injective. Since if $*w = 0$, then for any v, we have $\langle v, w \rangle = 0 \Rightarrow w = 0$. Then by dimension constrain, $*$ is an isomorphism. □

We describe $*$ via orthonormal basis $e_1, \dots, e_n$. The map $* : \bigwedge^p V \to \bigwedge^{n-p} V$ is such that $e_J \wedge (*e_I) = \langle e_J, e_I \rangle \mathsf{vol}$. So $*e_I = \varepsilon(I, \overline{I}) e_{\overline{I}}$ where $\overline{I} = \{1, \cdots, n\} \setminus I$ and $\varepsilon(I, \overline{I})$ is the signature of the permutation $(I, \overline{I})$.

Proposition 16.4. *We have*

(1) $*\mathsf{vol} = 1$
(2) $** = (-1)^{p(n-p)} : \bigwedge^p V \to \bigwedge^p V$

Proof. Both follows from the calculation above. (2) uses

$$(a_1 \wedge \cdots \wedge a_p) \wedge (b_1 \wedge \cdots \wedge b_{n-p}) = (-1)^{p(n-p)} (b_1 \wedge \cdots \wedge b_{n-p}) \wedge (a_1 \wedge \cdots \wedge a_p)$$

For (2), let us calculate e_I:

$$\begin{aligned} * * e_I &= *\varepsilon(I, \overline{I}) e_{\overline{I}} \\ &= \varepsilon(I, \overline{I}) \varepsilon(\overline{I}, I) e_I \end{aligned}$$

Write $I = \{i_1, \cdots, i_p\}$ and $\overline{I} = \{j_1, \cdots, j_{n-p}\}$. Then for each, say i_1, if there are k elements larger than i_1 in $\overline{I}$, then there are $n-p-k$ elements smaller than i_1 in $\overline{I}$. Therefore the total contribution for the sign is $(-1)^{k+n-p-k} = (-1)^{n-p}$. Hence the sign should be $((-1)^{n-p})^p = (-1)^{p(n-p)}$. □

16.1.3. *Global situation.* Let M be a smooth manifold, and E a smooth real vector bundle on M of rank n. Let $\mathscr{E}$ be the sheaf of sections.

Definition 16.5. A *metric* (or *scalar product*) on E is a smoothly varying family of scalar product on the fibers of E, i.e. $\forall p \in M$, we have a scalar product $\langle -, - \rangle$ on E_p such that for sections $s, t \in \mathscr{E}(U)$, $U \ni p \mapsto \langle s(p), t(p) \rangle \in \mathbb{R}$ is a smooth function.

It is enough to check this for $s_1, \dots, s_n$ which trivialize $\mathscr{E}$ over open subsets.

Example 16.6. If $E = M \times \mathbb{R}^n$, the standard scalar product on $\mathbb{R}^n$ gives a scalar product on each fiber. This gives a metric on E.

In particular, we always have such metrics on E locally on M. By using partitions of unity, we get a metric on E globally. Hence on every E, we have such a metric. If, in addition, we have an *orientation* of E, then we get an element $\mathsf{vol} \in \Gamma(M, \bigwedge^n \mathscr{E})$ which is everywhere nonzero, belonging to the orientation. We also get the $*$ operator $* : \bigwedge^p \mathscr{E} \xrightarrow{\sim} \bigwedge^{n-p} \mathscr{E}$ globalizing the one on each fiber.

Recall following definition.

Definition 16.7. A *Riemannian metric* on M is a metric on $\mathrm{T}M$.

A Riemannian metric induces a metric on T^*M and $\bigwedge^p \mathrm{T}^*M$. If M is oriented, i.e. we have an orientation on $\mathrm{T}M$ ($\Leftrightarrow$ on T^*M), we can apply previous construction. In particular, we have n-form $\mathrm{d}V$ ($n = \dim M$) called "volume element" which is everywhere nonzero, positive oriented. We get $* : \mathcal{A}^p_M \xrightarrow{\sim} \mathcal{A}^{n-p}_M$ such that $\omega \wedge (*\eta) = \langle \omega, \eta \rangle \, \mathrm{d}V$.

From now on, assume M to be compact manifold with orientation and Riemannian structure. Compactness allows us to define a "scalar product" on $\mathcal{A}^p(M)$ such that $\langle\!\langle \omega, \eta \rangle\!\rangle := \int_M \langle \omega, \eta \rangle \, \mathrm{d}V$. It is clear that this is a bilinear, symmetric and "positive definite", i.e. if $\omega \neq 0$, then $\langle\!\langle \omega, \omega \rangle\!\rangle > 0$.

Caveat is that $\mathcal{A}^p(M)$ is not a finite-dimensional vector space. In fact, it is not even *complete* with respect to the metric induced by $\langle\!\langle -, - \rangle\!\rangle$.

17. October 21, 2019

17.1. **Laplace-Beltrami operator.** Let M be a compact manifold with orientation and Riemannian structure of dimension n.

Definition 17.1. Let $\mathrm{d}^* : \mathcal{A}^p_M \to \mathcal{A}^{p-1}_M$ be the composition $(-1)^{n(p-1)+1}(*)\mathrm{d}(*) = (-1)^p(*^{-1})\mathrm{d}(*)$.

The equality is because of $** = (-1)^{p(n-p)}$. The most outside $(-1)^{n(p-1)+1}*$ acts on $\mathcal{A}^{p-1}_M$, so we have $* = (-1)^{(p-1)(n-p+1)} *^{-1}$. Therefore $(-1)^{n(p-1)+1} * = (-1)^{n(p-1)+1+(p-1)(n-p+1)} *^{-1} = (-1)^{(2n-p+1)(p-1)+1} *^{-1} = (-1)^p *^{-1}$.

In some sense, what d^* does is that we take derivative of each wedge according to the $\mathrm{d}z_i$s occur, e.g. if $\omega = f \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_k$, then $*\omega = \sum_{i=1}^k (-1)^{\text{some power}} \frac{\partial f}{\partial z_i} z_1 \wedge \cdots \wedge \not{\mathrm{d}z_i} \wedge \cdots \wedge \mathrm{d}z_k$

Proposition 17.2. *For every p, $\mathrm{d}^* : \mathcal{A}^{p+1}(M) \to \mathcal{A}^p(M)$ is the* formal adjoint *of $\mathrm{d} : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$ with respect to $\langle\!\langle -, - \rangle\!\rangle$, i.e.*

$$\langle\!\langle \mathrm{d}\omega, \eta \rangle\!\rangle = \langle\!\langle \omega, \mathrm{d}^*\eta \rangle\!\rangle$$

for all $\eta \in \mathcal{A}^{p+1}(M), \omega \in \mathcal{A}^p(M)$.

Remark 17.3. Such a formal adjoint is unique if it exists. If $\overline{\mathrm{d}^*}$ is another such adjoint operator, then $\langle\!\langle \omega, \mathrm{d}^*\eta \rangle\!\rangle = \langle\!\langle \omega, \overline{\mathrm{d}^*}\eta \rangle\!\rangle$. Set $\omega = \mathrm{d}^*\eta - \overline{\mathrm{d}^*}\eta$. Then $\langle\!\langle \omega, \omega \rangle\!\rangle = 0 \Rightarrow \omega = 0$.

Proof of Proposition 17.2. For any $\eta \in \mathcal{A}^{p+1}(M), \omega \in \mathcal{A}^p(M)$, we have

$$\begin{aligned}
\langle\!\langle \omega, \mathrm{d}^*\eta \rangle\!\rangle &= \int_M \left\langle \omega, (-1)^{p+1} *^{-1} \mathrm{d} * \eta \right\rangle \mathrm{d}V \\
&= (-1)^{p+1} \int_M \underbrace{\omega \wedge \mathrm{d} * \eta}_{(-1)^p(\mathrm{d}(\omega\wedge *\eta) - \mathrm{d}\omega \wedge (*\eta))} \\
&= \underbrace{-\int_M \mathrm{d}(\omega \wedge *\eta)}_{0 \text{ by Stokes}} + \int_M \mathrm{d}\omega \wedge *\eta \\
&= \int_M \langle \mathrm{d}\omega, \eta \rangle \, \mathrm{d}V \\
&= \langle\!\langle \mathrm{d}\omega, \eta \rangle\!\rangle
\end{aligned}$$

□

Definition 17.4. The *Laplace-Beltrami operator* is defined to be

$$\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} : \mathcal{A}^p(M) \to \mathcal{A}^p(M)$$

Proposition 17.5. Δ *is formally self-adjoint.*

Proof. For any $\omega, \eta \in \mathcal{A}^p(M)$, we have

$$\langle\!\langle \Delta\omega, \eta \rangle\!\rangle = \langle\!\langle \mathrm{d}\mathrm{d}^*\omega + \mathrm{d}^*\mathrm{d}\omega, \eta \rangle\!\rangle \overset{\text{Proposition 17.2}}{=} \langle\!\langle \mathrm{d}^*\omega, \mathrm{d}^*\eta \rangle\!\rangle + \langle\!\langle \mathrm{d}\omega, \mathrm{d}\eta \rangle\!\rangle \tag{17.1}$$

By symmetry, this is $\langle\!\langle \omega, \Delta\eta \rangle\!\rangle$. □

Definition 17.6. An element $\omega \in \mathcal{A}^p(M)$ is *harmonic* if $\Delta\omega = 0$.

Proposition 17.7. *An element* $\omega \in \mathcal{A}^p(M)$ *is harmonic if and only if* $\mathrm{d}\omega = \mathrm{d}^*\omega = 0$.

Proof. The direction $\Leftarrow$ is clear by definition. For the convere direction $\Rightarrow$, let $\eta = \omega$ in the formula (17.1) above. We get $0 = \|\mathrm{d}^*\omega\|^2 + \|\mathrm{d}\omega\|^2$. So both of them are zero. □

Next, we want to show that in every de Rham chomology class, there exists a unique harmonic representative.

Remark 17.8. For a given de Rham chomology class, a representative ω is harmonic if and only if $\|\omega\|$ is minimal.

Proof of Remark 17.8. Given a p-form ω with $\mathrm{d}\omega = 0$, consider $\omega + \mathrm{d}\eta$ for all $p-1$-form η. We have

$$\begin{aligned}
\|\omega + \mathrm{d}\eta\|^2 &= \|\omega\|^2 + \|\mathrm{d}\eta\|^2 + 2\langle\!\langle \omega, \mathrm{d}\eta \rangle\!\rangle \\
&= \|\omega\|^2 + \|\mathrm{d}\eta\|^2 + 2\langle\!\langle \mathrm{d}^*\omega, \eta \rangle\!\rangle
\end{aligned}$$

Since $\mathrm{d}\omega = 0$, ω is harmonic if and only if $\mathrm{d}^*\omega = 0$.

If this holds, we have $\|\omega + \mathrm{d}\eta\|^2 = \|\omega\|^2 + \|\mathrm{d}\eta\|^2 \geqslant \|\omega\|^2$ for all η.

Conversely, if $\|\omega\|^2$ is minimal among all $\|\omega + \mathrm{d}\eta\|^2$, then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|\omega + t\mathrm{d}\eta\|^2 \bigg|_{t=0} &= 0 \\
\Rightarrow 2\|\omega, \mathrm{d}\eta\| &= 0 \\
\text{Take } \eta = \mathrm{d}^*\omega \Rightarrow \langle\!\langle \omega, \mathrm{d}\mathrm{d}^*\omega \rangle\!\rangle &= 0 \\
\Rightarrow \langle\!\langle \mathrm{d}^*\omega, \mathrm{d}^*\omega \rangle\!\rangle &= 0 \\
\Rightarrow \mathrm{d}^*\omega &= 0
\end{aligned}$$

□

Note that if ω, ω' are both harmonic in the same cohomology class, then $\|\omega\|^2 = \|\omega'\|^2$. Write $\omega = \omega' + \mathrm{d}\eta$. Then we have $\|\mathrm{d}\eta\|^2 = 0 \Rightarrow \omega = \omega'$.

Proposition 17.9. *We have*

$$*\Delta = \Delta *$$

Corollary 17.10. *ω is harmonic if and only if $*\omega$ is harmonic.*

Proof of Proposition 17.9. Working on p-forms, we have

$$\begin{aligned} *\Delta &= *\left(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}\right) \\ &= (-1)^{n(p-1)+1} * \mathrm{d} * \mathrm{d} * + (-1)^{p+1} \mathrm{d} * \mathrm{d} \\ \Delta * &= \left(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}\right) * \\ &= (-1)^{n(n-p-1)+1} \mathrm{d} * \mathrm{d} \underbrace{**}_{(-1)^{p(n-p)}} + (-1)^{n(n-p)+1} * \mathrm{d} * \mathrm{d} * \end{aligned}$$

It is not hard to check that the sign matches and we win. □

Note that if the dimension n is even, then $\mathrm{d}^* = - * \mathrm{d} *$ and the proof is much simpler.

We have a formally self-adjoint operator

$$\Delta : \mathcal{A}^p(M) \to \mathcal{A}^p(M).$$

If we have a self-adjoint linear map $T : V \to V$ where $(V, \langle -,- \rangle)$ is a finite-dimensional normed vector space, then we have a orthogonal decomposition $V = \mathrm{Ker}(T) \oplus \mathrm{Im}(T)$. Because for any $u \in \mathrm{Ker}(T), Tv \in \mathrm{Im}(T)$, we have $\langle u, Tv \rangle = \langle Tu, v \rangle = 0$. The same conclusion holds if T is a linear operator on a Hilbert space.

17.2. **Differential operators.** Let M be a smooth manifold and let $\mathscr{C}_M^\infty$ be the sheaf of real-valued smooth functions on M. Then $\mathscr{D}_M \subseteq \mathscr{E}nd_{\mathbb{R}}(\mathscr{C}_M^\infty)$ is generated as a sheaf of rings by $\mathscr{C}_M^\infty$ (acting by homotheties) and $\mathscr{D}er_{\mathbb{R}}(\mathscr{C}_M^\infty)$. (this is a sheaf of non-commutative rings)

If $U \subseteq M$ is a chart with coordinates $x_1, \ldots, x_n$, then $\mathscr{D}er(\mathscr{C}_M^\infty)$ is generated over $\mathscr{C}_M^\infty$ by $\partial_1, \ldots, \partial_n$ where $\partial_i = \frac{\partial}{\partial x_i}$. So $\mathscr{D}_U$ is free over $\mathscr{C}_M^\infty$ (both as a left or as a right module) with basis given by $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$.

It has a filtration $\mathrm{F}_k\mathscr{D}_M \subseteq \mathscr{D}_M$, which is the subsheaf locally generated (in charts as above) as left $\mathscr{C}_M^\infty$-module by ∂^α where $|\alpha| = \alpha_1 + \cdots + \alpha_n \leqslant k$. For example, $\mathrm{F}_0\mathscr{D}_M = \mathscr{C}_M^\infty$ and $\mathrm{F}_1\mathscr{D}_M = \mathscr{C}_M^\infty + \mathscr{D}er_{\mathbb{R}}(\mathscr{C}_M^\infty)$.

18. October 23, 2019

18.1. **Differential operators between vector bundles.** Let M be a smooth manifold of dimension m. Let $\mathscr{D}_M \subseteq \mathscr{E}nd_R(\mathscr{C}_M^\infty)$ be the sheaf of differential operators, locally generated by $\mathscr{C}_M^\infty$ and $\mathscr{T}_M = \mathscr{D}er_R(\mathscr{C}_M^\infty)$. We have a filtration $\mathrm{F}_\bullet\mathscr{D}_M$ where $\mathrm{F}_k\mathscr{D}_M$ is the differential operators "of order k". So $\mathrm{F}_k\mathscr{D}_M$ are the operators that locally generated by $\sum_{|\alpha| \leq k} f_\alpha \partial^\alpha$.

1. $\mathrm{F}_k\mathscr{D}_M \cdot \mathrm{F}_l\mathscr{D}_M \subseteq \mathrm{F}_{k+l}\mathscr{D}_M$ (use the relation $[\partial_k, g] = \frac{\partial g}{\partial x_j}$). This implies that $\mathrm{gr}_{\mathrm{F}}\, \mathscr{D}_M = \oplus_{k \geqslant 0} \mathrm{F}_k\mathscr{D}_M / \mathrm{F}_{k-1}\mathscr{D}_M$ has an induced graded ring structure.
2. $[\mathrm{F}_k\mathscr{D}_M, \mathrm{F}_l\mathscr{D}_M] \subseteq \mathrm{F}_{k+l-1}\mathscr{D}_M$, so $\mathrm{gr}_{\mathrm{F}}\, \mathscr{D}_M$ is a sheaf of commutative ring.
3. We have $\mathrm{gr}_{\mathrm{F}}\, \mathscr{D}_M = \mathscr{C}_M^\infty \oplus \mathscr{T}_M \oplus \cdots$. By the universal property of the symmetric algebra, we have a morphism of sheaves of graded commutative $\mathscr{C}_m^\infty$-algebras $\mathscr{S}ym_{\mathscr{C}_M^\infty}(\mathscr{T}_M) \to \mathrm{gr}_{\mathrm{F}}\, \mathscr{D}_M$. By using the local description of $\mathscr{D}_M$ in a chart, we see that this is an isomorphism.

Given an operator $P \in \Gamma(M, \mathscr{D}_M)$ of order k (order $\leqslant k$, but not $\leqslant k-1$). The "symbol of P" is $\sigma_k(P) \in \Gamma(M, \mathscr{S}ym^k(\mathscr{T}_M))$.

More generally, suppose E, F are smooth (real) vector bundles on M, with corresponding sheaves $\mathscr{E}, \mathscr{F}$. The set of differential operators $\mathrm{Diff}_k(\mathscr{E}, \mathscr{F})$ is the set $\{P \in \mathscr{E}nd_{\mathbb{R}}(\mathscr{E}, \mathscr{F})\}$ such that locally on an open subset $U \subseteq X$, we have $\mathscr{E}|_U \cong \mathcal{O}_U^{\oplus r}, \mathscr{F}|_U \cong \mathcal{O}_U^{\oplus s}$, P is given by (P_{ij}) with each P_{ij} a differential operator of order $\leqslant k$.

Example 18.1. The differential $\mathrm{d} : \mathcal{A}_M^p \to \mathcal{A}_M^{p+1}$ is a differential operator of order 1.

If P is a differential operator of order $\leqslant k$, we want to define $\sigma_k(P)$. Locally on $U \subseteq X$ as before, $\mathscr{E}|_U \cong \mathcal{O}_U^{\oplus r}, \mathscr{F}|_U \cong \mathcal{O}_U^{\oplus s}$. If P is given by $(P_{ij})_{i,j}$, then we can consider $(\sigma_k(P_{ij}))_{i,j}$ where $\sigma_k(P_{ij}) \in \Gamma(U, \mathscr{S}ym^k(\mathscr{T}_M))$. These glue together to give $\sigma_k(P) \in \Gamma\left(M, \mathscr{S}ym^k(\mathscr{T}_M) \otimes \mathscr{H}om(\mathscr{E}, \mathscr{F})\right)$. Given $x \in M$, we get a homogeneous polynomial of degree k as a map $\mathrm{T}_x^* M \to \mathrm{Hom}_{\mathbb{R}}(E_{(x)}, F_{(x)})$.

Definition 18.2. Given $P \in \mathrm{Diff}_k(E, F)$, with $\mathrm{rank}(E) = \mathrm{rank}(F)$. It is *elliptic* if $\forall x \in M, \forall v \in \mathrm{T}_x^*(M)$ nonzero, the map $\sigma_k(P)_x(v)$ is an isomorphism.

Example 18.3 (Main Example)**.** The Laplace-Beltrami operator $\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$, where $\mathrm{d}^* = \pm * \mathrm{d} *$ and $*$ has order 0 and d has order 1, is a differential operator of order $\leqslant 2$.

The goal is to compute following

- Δ on $\mathbb{R}^n$ with usual metric orientation.
- $\sigma_2(\Delta)$ in general and show that Δ is elliptic.

Recall that if M is a smooth manifold, X a vector field on M and ω a p-form, then $i_X(\omega)$ "the contraction of ω with respect to X" is given by following description: if $X_1, \ldots, X_{p-1}$ are vector fields, then

$$(i_X(\omega))(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1}).$$

For example, $i_X(\mathrm{d}f) = X(f)$.

Fact 18.4. i_X *"behaves well" with respect to* $\wedge$*:*

$$i_X(\alpha \wedge \beta) = i_X(\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i_X(\beta)$$

If $\xi_1, \ldots, \xi_n$ trivializes $\mathrm{T}M$ on U and $\xi_1^*, \ldots, \xi_n^*$ dual basis of T^*M, write $\xi_I^* := \xi_{i_1}^* \wedge \cdots \wedge \xi_{i_p}^*$ for $I = i_1 < \cdots < i_p$, then

$$i_{\xi_j}(\xi_I^*) = \begin{cases} 0 & \text{if } j \notin I \\ (-1)^{k-1} \xi_{I \smallsetminus \{j\}}^* & \text{if } j = i_k \end{cases}$$

Lemma 18.5. *Let M be an oriented Riemannian manifold, $\xi_1, \ldots, \xi_n$ orthonormal positive-oriented local basis for $\mathrm{T}M$. Then for any I with $|I| = p$,*

$$*\left(\xi_j^* \wedge *\xi_I^*\right) = (-1)^{n(p-1)} i_{\xi_j}(\xi_I^*)$$

Proof. Note if $j \in I$, then we write $I = \{a_1, \ldots, a_p\}, j = a_l, \overline{I} = \{b_1, \ldots, b_{n-p}\}$ and suppose that $b_k < a_l = j < b_{k+1}$. Then $*\xi_I^* = \varepsilon(I, \overline{I}) \xi_{\overline{I}}^*$. Write $\overline{I} \cup \{j\}$ for the *ordered* sequence, then

$$\begin{aligned} \mathrm{LHS} &= *\left(\xi_j^* \wedge \varepsilon(I, \overline{I}) \xi_{\overline{I}}^*\right) \\ &= \varepsilon(I, \overline{I})(-1)^k * \xi_{\overline{I} \cup \{j\}}^* \\ &= \varepsilon(I, \overline{I}) \varepsilon(\overline{I} \cup \{j\}, I \smallsetminus \{j\})(-1)^k \xi_{I \smallsetminus \{j\}}^* \end{aligned}$$

Now we have following two tables

$$\varepsilon(I, \overline{I}) : \begin{matrix} a_1 & \cdots & a_{l-1} & a_l & a_{l+1} & \cdots & a_p \\ b_1 & \cdots & b_k & & b_{k+1} & \cdots & b_{n-p} \end{matrix}$$

$$\varepsilon(\overline{I} \cup \{j\}, I \smallsetminus \{j\}) : \begin{matrix} b_1 & \cdots & b_k & a_l & b_{k+1} & \cdots & b_{n-p} \\ a_1 & \cdots & a_{l-1} & & a_{l+1} & \cdots & a_p \end{matrix}$$

Note that for $a_1, \dots, a_{l-1}$, every element in b will contribute once together with a_l to the sign, so the total contribution is $(-1)^{n-p+1}$. There are $l-1$ of them. So $((-1)^{n-p+1})^{l-1}$.

For a_l, it has k b's smaller than it and $l-1$ a's smaller than it. So the contribution is $(-1)^{k+l-1}$.

For $a_{l+1}, \cdots, a_p$, the contribution is unaffected by removing a_l, so the contribution is $((-1)^{n-p})^{p-l}$.

Therefore the total sign is -1 raising to $(n-p+1)(l-1)+(k+l-1)+(n-p)(p-l)+k \equiv (n-p)(p-1) \equiv n(p-1)$ mod 2. $\square$

Now let us compute $\sigma_2(\Delta)$ in general and Δ if $M = \mathbb{R}^n$ with standard metric, orientation:

$$
\begin{aligned}
\omega &= \sum_{|I|=p} f_I \xi_I^* \\
\mathrm{d}^*\omega &= (-1)^{n(p-1)+1} * \mathrm{d} * (\omega) \\
&= (-1)^{n(p-1)+1} * \mathrm{d}\left(\sum_{|I|=p} f_I * \xi_I^*\right) \\
&= (-1)^{n(p-1)+1} * \sum_{|I|=p} \sum_{k=1}^{n} \xi_k(f_I)(\xi_k^* \wedge *\xi_I^*) + (-1)^{n(p-1)+1} * \sum_{|I|=p} f_I \mathrm{d}(*\xi_I^*)
\end{aligned}
$$

The second term is 0 in $\mathbb{R}^n$ with $\xi_i = \frac{\partial}{\partial x_i}$, and can be ignored in general for computation of $\sigma_2(\Delta)$.

$$
\begin{aligned}
\Rightarrow \mathrm{d}^*\omega &= (-1)^{n(p-1)+1} \sum_{|I|=p} \sum_{k=1}^{n} \xi_k(f_I) \underbrace{*(\xi_k^* \wedge *\xi_I^*)}_{(-1)^{n(p-1)} i_{\xi_k}(\xi_I^*)} + \cdots \\
\mathrm{d}^*\omega &= -\sum_I \sum_k \xi_k(f_I) i_{\xi_k}(\xi_I^*) + \text{stuff}
\end{aligned}
$$

where stuff is either 0 in $\mathbb{R}^n$ or can be ignored when computing σ_2.

19. October 25, 2019

19.1. **The symbol of the Laplace-Beltrami operator.** Recall: $P \in \mathrm{Diff}_k(\mathscr{E}, \mathscr{F})$ and $\sigma_k(P) \in \Gamma(M, \mathscr{Sym}^k(\mathscr{T}_M) \otimes \mathscr{Hom}(\mathscr{E}, \mathscr{F}))$. For $x \in M, v \in \mathrm{T}_x^* M$, we have $\sigma_k(P)_x(v) \in \mathrm{Hom}(E_{(x)}, F_{(x)})$.

Let M be a compact oriented manifold with Riemannian metric. Then $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. Let $U \subseteq M$ be a chart and let $\xi_1, \dots, \xi_n$ positive oriented orthonormal basis of $\mathrm{T}M|_U$ and $\xi_1^*, \dots, \xi_n^*$ the dual basis.

For any ω a p-form, write $\omega = \sum_{|I|=p} f_I \xi_I^* \Rightarrow \mathrm{d}^*\omega = -\sum_I \sum_k \xi_k(f_I) i_{\xi_k}(\xi_I^*) +$ (linear operator in w). The last term is zero if $U = \mathbb{R}^n$ with standard metric and basis.

So

$$\begin{aligned}
\mathrm{d}\omega &= \sum_I \sum_j \xi_j(f_I)\xi_j^* \wedge \xi_I^* \\
\mathrm{d}\mathrm{d}^*\omega &= -\sum_I \sum_{j,k} \xi_j\xi_k(f_I)\xi_j^* \wedge i_{\xi_k}(\xi_I^*) + \text{ operators of order } \leqslant 1 \text{ in } \omega \text{ (this is 0 in } \mathbb{R}^n) \\
\mathrm{d}^*\mathrm{d}\omega &= -\sum_I \sum_{j,k} \xi_k\xi_j(f_I) i_{\xi_k}(\xi_j^* \wedge \xi_I^*) + \text{operator of order } \leqslant 1 \\
\Delta\omega &= -\sum_I \sum_{j,k} \xi_k\xi_j(f_I)\left(\xi_j^* \wedge i_{\xi_k}(\xi_I^*) + i_{\xi_k}(\xi_j^* \wedge \xi_I^*)\right) + \text{ operator of order } \leqslant 1 \text{ in } \omega\text{, which is 0 in } \mathbb{R}^n
\end{aligned}$$

Use the fact that $[\xi_k, \xi_j]$ has operator order $\leqslant 1$

$$\begin{aligned}
&= -\sum_I \sum_{j,k} \xi_k\xi_j(f_I) i_{\xi_k}(\xi_j^* \wedge \xi_I^*) + \cdots \\
&= -\sum_I \sum_k \xi_k^2(f_I)\xi_I^* + \cdots
\end{aligned}$$

Conclusion is that

(1) If $M = \mathbb{R}^n$, then

$$\Delta\left(\sum_I f_I \mathrm{d}x_I\right) = -\sum_I \left(\sum_{k=1}^n \frac{\partial^2 f}{\partial x_k^2}\right)\mathrm{d}x_I$$

(2) In general, $\sigma_2(\Delta)$ is given by

$$\sigma_2(\Delta)\left(\sum_{k=1}^n v_k \xi_k^*(x)\right) = -\left(\sum_{k=1}^n v_k^2\right)\mathrm{id}$$

and

$$\sigma_2(\Delta)_x(v) = -\|v\|^2 \cdot \mathrm{id}\,.$$

Hence this is an isomorphism if $v \neq 0$.

Supppose that M is a compact oriented manifold. If we have a metric on $\mathscr{E}$ and a *volume element* $\mathrm{d}V$, then we can define a scalar product on $\mathscr{E}(M)$:

$$\langle\!\langle s, t\rangle\!\rangle = \int_M \langle s, t\rangle \,\mathrm{d}V$$

Given $P \in \mathrm{Diff}_k(\mathscr{E}, \mathscr{F})$ such that both $\mathscr{E}, \mathscr{F}$ carry metrics, there exists a *formal adjoint* $P^* \in \mathrm{Diff}_k(\mathscr{F}, \mathscr{E})$ such that

$$\langle\!\langle Ps, t\rangle\!\rangle = \langle\!\langle s, P^*t\rangle\!\rangle$$

for any $s \in \mathscr{E}(M), t \in \mathscr{F}(M)$.

Moreover, for any $x \in M, v \in \mathrm{T}_x^*M$, we have $\sigma_k(P^*)_x(v) = (\sigma_k(P)_x(v))^*$ where $(-)^*$ is adjoint with respect to scalar product on fibers. In particular, if $\mathrm{rank}(E) = \mathrm{rank}(F)$, then P is elliptic if and only if P^* is.

Theorem 19.1 (Fundamental Theorem)**.** *Let M be a compact oriented manifold with E, F smooth vector bundles of the same rank, with metrics, have volume element* $\mathrm{d}V$*, and* $P \in \mathrm{Diff}_k(E, F)$ elliptic*, then*

(1) $\dim_{\mathbb{R}} \mathrm{Ker}(P) < \infty$.

(2) $\mathscr{E}(M) = \mathrm{Ker}(P) \overset{\perp}{\oplus} \mathrm{Im}(P^*)$. *(Note:* $\mathrm{Ker}(P) \perp \mathrm{Im}(P^*)$ *by adjointness)*

In particular, if P is self-adjoint, then $\mathscr{E}(M) = \mathrm{Ker}\, P \overset{\perp}{\oplus} \mathrm{Im}(P)$.

Subtlety: $\mathscr{E}(M), \mathscr{F}(M)$ are not complete with repsect to $\langle\!\langle -, -\rangle\!\rangle$. Have to enlarge the space to suitable spaces with distribution. The hard part is to show that given P^* elliptic and s the section of $\mathscr{E}$ with coefficients in distribution, if P^*s is smooth, then so is s.

Apply this to Δ. Since Δ is elliptic and self-adjoint, we have $\mathcal{A}^p(M) = \mathrm{Ker}(\Delta) \oplus \mathrm{Im}(\Delta)$ where the kernel is $\mathscr{H}^p(M,\mathbb{R}) = \{\omega \in \mathcal{A}^p(M) \mid \Delta\omega = 0\}$. The image is $\Delta(\mathcal{A}^p(M)) \subseteq \mathrm{d}\left(\mathcal{A}^{p-1}(M)\right) + \mathrm{d}^*\left(\mathcal{A}^{p+1}(M)\right)$ since $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$.

Note

(1) $\mathrm{d}\left(\mathcal{A}^{p-1}(M)\right) \perp \mathrm{d}^*\left(\mathcal{A}^{p+1}(M)\right)$ as $\langle\langle \mathrm{d}\omega, \mathrm{d}^*\eta\rangle\rangle = \langle\langle \mathrm{d}^2\omega, \eta\rangle\rangle = 0$.
(2) $\mathscr{H}^p(M,\mathbb{R})$ orthogonal to $\mathrm{d}\left(\mathcal{A}^{p-1}(M)\right) + \mathrm{d}^*\left(\mathcal{A}^{p+1}(M)\right)$. Since ω is harmonic, we have $\mathrm{d}\omega = \mathrm{d}^*\omega = 0$. So $\langle\langle \omega, \mathrm{d}\eta\rangle\rangle = \langle\langle \mathrm{d}^*\omega, \$\eta\rangle\rangle = 0$. Similarly, $\langle\langle \omega, \mathrm{d}^*\theta\rangle\rangle = 0$.

The orthogonal decomposition together with the inclusion, (2) implies that

$$\Delta(\mathcal{A}^p(M)) = \mathrm{d}\left(\mathcal{A}^{p-1}(M)\right) \overset{\perp}{\oplus} \mathrm{d}^*\left(\mathcal{A}^{p+1}(M)\right)$$

What is the kernel $\mathrm{Ker}(\mathrm{d} : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M))$? This contains $\mathscr{H}^p(M,\mathbb{R}), \mathrm{d}\left(\mathcal{A}^{p-1}(M)\right)$.

Claim. $\mathrm{Ker}(\mathrm{d} : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)) = \mathscr{H}^p(M,\mathbb{R}) \overset{\perp}{\oplus} \mathrm{d}\left(\mathcal{A}^{p-1}(M)\right)$.

Proof. Enought to show that if $\mathrm{d}^*\eta \in \mathrm{Ker}(\mathrm{d})$, then $\mathrm{d}^*(\eta) = 0$. Since $\mathrm{d}\mathrm{d}^*\eta = 0$, we have $\langle\langle \mathrm{d}^*\eta, \mathrm{d}^*\eta\rangle\rangle = \langle\langle \eta, \mathrm{d}\mathrm{d}^*\eta\rangle\rangle = 0 \Rightarrow \mathrm{d}^*\eta = 0$. □

We have decomposition

$$\mathcal{A}^p(M) = \underbrace{\mathscr{H}^p(M,\mathbb{R}) \overset{\perp}{\oplus} \mathrm{d}\left(\mathcal{A}^{p-1}(M)\right)}_{\mathrm{Ker}(\mathrm{d})} \overset{\perp}{\oplus} \mathrm{d}^*\left(\mathcal{A}^{p+1}(M)\right)$$

Corollary 19.2. *Let M be a compact, oriented Riemannian manifold.*

(1) We have a canonical isomorphism:

$$\mathrm{H}^p_{\mathsf{dR}}(M,\mathbb{R}) \cong \mathscr{H}^p(M,\mathbb{R})$$

(2) $\dim_{\mathbb{R}} \mathrm{H}^p_{\mathsf{dR}}(M,\mathbb{R}) < \infty$ *(Use (1) in the theorem)*

An elementary application is that if M is compact and orientable, then we have Poincaré duality.

Put a metric on M and choose an orientation. The operation $*$ gives an isomorphism $\mathscr{H}^p(M,\mathbb{R}) \cong \mathscr{H}^{n-p}(M,\mathbb{R})$. However, this depends on the choice of metric. A better statement uses is following: the pairing

$$\mathrm{H}^p_{\mathsf{dR}}(M,\mathbb{R}) \times \mathrm{H}^{n-p}_{\mathsf{dR}}(M,\mathbb{R}) \to \mathbb{R}$$
$$(\omega,\eta) \mapsto \int_M \omega \wedge \eta$$

is nondegenerate. To see this, enough to show that for every p and every $\alpha \in \mathrm{H}^p_{\mathsf{dR}}(M,\mathbb{R})$, there exists some $\beta \in \mathrm{H}^{n-p}_{\mathsf{dR}}(M,\mathbb{R})$ such that $\int \alpha \wedge \beta \neq 0$. For this, we choose metric and $\omega \in \mathscr{H}^p(M,\mathbb{R})$ harmonic such that $[\omega] = \alpha$. Take $\beta = *\omega$. Then $\int_M \underbrace{\omega \wedge *\omega}_{\langle\omega,\omega\rangle \mathrm{d}V} = \langle\langle \omega,\omega\rangle\rangle > 0$ if $\omega \neq 0$.

20. October 28, 2019

20.1. Hodge theory of complex manifolds.

20.1.1. *Some linear algebra.* Let V be a finite dimensional complex vector space, then $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = V' \oplus V''$.

Definition 20.1. A *Hermitian form* on V is a bilinear map $h : V \times \overline{V} \to \mathbb{C}$ such that $h(v,w) = \overline{h(w,v)}$.

Given such h, write it as $h = S + \mathrm{i}A$ where $S, A : V \times \overline{V} \to \mathbb{R}$. Then the condition on h implies that S is symmetric and A is skew-symmetric, and both are bilinear on $\mathbb{R}$. Also we have

$$S(\mathrm{i}v,w) + \mathrm{i}A(\mathrm{i}v,w) = \mathrm{i}\left(S(v,w) + \mathrm{i}A(v,w)\right) \tag{20.1}$$
$$\Rightarrow S(\mathrm{i}v,w) = -A(v,w) \tag{20.2}$$
$$\text{and } A(\mathrm{i}v,w) = S(v,w) \tag{20.3}$$

Easy to see that given such h,

- is equivalent to give a symmetric bilinear form $S : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ such that $S(\mathrm{i}v,\mathrm{i}w) = S(v,w)$ for any $v,w \in V$. In this case A is given by (20.2). We have $S(\mathrm{i}v,\mathrm{i}w) = -A(v,\mathrm{i}w) = A(\mathrm{i}w,v) = S(w,v) = S(v,w)$.
- is equivalent to give a skew-symmetric bilinear form $A : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ such that $A(\mathrm{i}v,\mathrm{i}w) = A(v,w)$ and S is given by (20.3). We have $A(\mathrm{i}v,\mathrm{i}w) = S(v,\mathrm{i}w) = S(\mathrm{i}w,v) = -A(w,v) = A(v,w)$.

Definition 20.2. A Hermitian form h is a *Hermitian metric* if it satisfies $h(v,v) > 0$ for all nonzero $v \in V$.

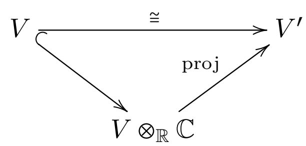
Because A is skew-symmetric, $h(v,v) = S(v,v) + \mathrm{i}A(v,v) = S(v,v)$ is a real number.

If h is a Hermitian metric, then real part S_h of h gives a scalar product on $V_{\mathbb{R}}$. Given a metric S, it can be uniquely extended to a Hermitian metric S_h on $V_{\mathbb{C}}$ by requring $S_h(\mathrm{i}v,w) = \mathrm{i}S(v,w)$ and $S_h(v,\mathrm{i}w) = -\mathrm{i}S(v,w)$. Then we have

$$S_h(v+\mathrm{i}w, v+\mathrm{i}w) = S(v,v) + S(w,w) + \mathrm{i}\cancel{(S(v,w) - S(w,v))} > 0$$

if $v + \mathrm{i}w \neq 0$. So h_S is a Hermitian metric on V.

Lemma 20.3. *The canonical isomorphism*



is compatible with the Hermitian forms (up to a constant scalar factor). It is saying that if we have a Hermitian metric h on V, take S to be its real part. Then we can talk about the extension of S to $V_{\mathbb{C}}$ and restrict back to V'. The lemma is saying that the new Hermitian metric we get is equivalent to the original one up to a constant scalar.

Proof. Recall that if $J : V \to V$ is the complex structure, then $V \to V', v \mapsto \frac{1}{2}(v - \mathrm{i}Jv)$. And we have $S_h(Jv,Jw) = S(v,w)$. So we have

$$\begin{aligned} S_h\left(\frac{1}{2}(v-\mathrm{i}Jv), \frac{1}{2}(w-\mathrm{i}Jw)\right) &= \frac{1}{4}((S(v,w) + \underbrace{S(Jv,Jw)}_{S(v,w)}) - \mathrm{i}(S(Jv,w) - \underbrace{S(v,Jw)}_{-S(Jv,w)})) \\ &= \frac{1}{2}(S(v,w) - \mathrm{i}\underbrace{S(Jv,w)}_{-A(v,w)}) \\ &= \frac{1}{2}h(v,w) \end{aligned}$$

□

Remark 20.4. We have

(1) If $h : V \times \overline{V} \to \mathbb{C}$ is a Hermitian metric, then we get an isomorphism $\overline{V} \to V^*, v \mapsto h_v = h(-, v)$. Also, the conjugate $\overline{h} : \overline{V} \times V \to \mathbb{C}$ gives a Hermitian metric on $\overline{V}$. Combine these, we get Hermitian metric h^* on V^* such that $h^*(h_v, h_w) = \overline{h(v,w)}$.

(2) Given Hermitian metric h on V, we get a Hermitian metrics on all $\bigwedge^p V$ such that

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \det \left(h(v_i, w_j) \right)$$

Given a Hermtitian metric h on V, we can consider the Hermitian metric S_h on $V_{\mathbb{C}}$ and use it to put Hermitian metric on all $\bigwedge^p (V_{\mathbb{C}})^*$.

Exercise 20.5. This is the same using the scalar product S on V to get scalar product on $\bigwedge^p V^*$ and then extend this by linearity to Hermitian metric on $\left(\bigwedge^p V^* \right)_{\mathbb{C}} \cong \bigwedge^p (V_{\mathbb{C}}^*)$.

Lemma 20.6. *The decomposition*

$$\bigwedge^p V_{\mathbb{C}}^* \cong \bigoplus_{i+j=p} \left(\bigwedge^i (V')^* \oplus \bigwedge^j (V'')^* \right)$$

is orthogonal with respect to the Hermitian metric.

Proof. To show this, it is enough to check that the decomposition of $V_{\mathbb{C}} = V' \oplus V''$ is orthogonal with respect to S_h.

$$S_h \left(\frac{1}{2}(v - \mathrm{i}Jv), \frac{1}{2}(w + \mathrm{i}Jw) \right) = \frac{1}{4} \left(S(v,w) - S(Jv, Jw) \right) - \mathrm{i} \left(S(Jv, w) + S(v, Jw) \right) = 0$$

$\square$

Suppose that (V, h) is a finite dimensional complex vector space equipped with a Hermitian metric h. Then we have scalar product S_h on V. Since V is a complex vector space, it has a canonical orientation on V. Let $n = \dim_{\mathbb{C}} V$. Then we have a canonical "volume element" $\mathrm{d}V \in \bigwedge^{2n} V^* \subseteq \bigwedge^{2n} V_{\mathbb{C}}^*$.

We have the Hodge operator $* : \bigwedge^p V^* \xrightarrow{\sim} \bigwedge^{2n-p} V^*$ which is an isomorphism over $\mathbb{R}$. Extend this to over $\mathbb{C}$ we get $* : \bigwedge^p V_{\mathbb{C}}^* \xrightarrow{\sim} \bigwedge^{2n-p} V_{\mathbb{C}}^*$.

Lemma 20.7. *For every $\omega, \eta \in \bigwedge^p V_{\mathbb{C}}^*$, we have*

$$\omega \wedge \overline{*\eta} = \langle \omega, \eta \rangle \, \mathrm{d}V$$

where $\langle -, - \rangle$ is the Hermitian metric induced by S_h.

Exercise 20.8. Check Lemma 20.7 using the fact that we know this if $\omega, \eta \in \bigwedge^p V^*$.

If $e_1, \ldots, e_{2n}$ is an orthonormal basis, then assume that $\omega = e_{i_1} \wedge \cdots \wedge e_{i_p}$ and $\eta = e_{j_1} \wedge \cdots \wedge e_{j_p}$. If any i_k is not in $\{j_1, \cdots, j_p\}$, then the Hermitian of ω and η is zero. Meanwhile, e_{i_k} will appear in $\overline{*\eta}$, which will result in zero for the wedge product. The possibility of being nonzero is that $\{i_1, \cdots, i_p\} = \{j_1, \cdots, j_p\}$. In which case, both sides are the sign of the permutation $\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix}$. So the formula is true by linearity.

Corollary 20.9. *Write $\bigwedge^{p,q} V_{\mathbb{C}}^* := \bigwedge^p (V')^* \oplus \bigwedge^q (V'')^*$. Then $*$ maps $\bigwedge^{p,q} V_{\mathbb{C}}^*$ to $\bigwedge^{n-p,n-q} V_{\mathbb{C}}^*$.*

Recall that if $w : \bigwedge^\bullet V_{\mathbb{C}}^* \to \bigwedge^\bullet V_{\mathbb{C}}^*$ is the de Rham operator acting by multiplication with $(-1)^m$ on $\bigwedge^m$, then $** = w$.

Note that $*$ defined by scalar extension from $\mathbb{R}$, so it is a real operator, i.e. $\overline{*\omega} = *\overline{\omega}$.

20.1.2. *Globalization.* Given a complex manifold M, a *Hermitian metric* on M is given by a smoothly varying family of Hermitian metrics on each $\mathrm{T}_pM, p \in M$ with the canonical complex vector space structure. For every $p \in M$, we have a scalar product on T_pM, so we get a Riemannian metric. We also have a canonical orientation. Hence we have a "volumne element" $\mathrm{d}V \in \Gamma(M, \mathcal{A}^{2n})$, which is an (n,n)-form.

21. October 30, 2019

21.1. **Hodge theory of complex manifolds (continued).** Let M be an n-dimensional complex manifold. We can always choose on M a Hermitian metric. The key point is that the finite linear combination of Hermitian metric with real function coefficients is still a Hermitian metric. Then we can construct such metrics locally and glue using the partition of unity.

Fix such a metric $h = \langle -,- \rangle$, $S = \mathrm{Re}(h)$ is a Riemannian metric on M with the standard orientation and we get the volumne element $\mathrm{d}V$, which is a real (n,n)-form on M. So we get the star operation $* : \mathcal{A}_M^{p,q} \xrightarrow{\sim} \mathcal{A}_M^{n-q,n-p}$. S also induces Hermitian metrics on all $\mathcal{A}_{M,\mathbb{C}}^m$ such that the (p,q)-components are orthogonal.

From now on we assume that M is compact. We get Hermitian metrics on each $\mathcal{A}^{p,q}(M)$ by

$$\langle\!\langle \omega, \eta \rangle\!\rangle := \int_M \langle \omega, \eta \rangle \,\mathrm{d}V = \int_M \omega \wedge \overline{*\eta}.$$

Note that $\langle\!\langle \omega, \omega \rangle\!\rangle > 0$ unless $\omega = 0$.

It is easy to verify that the induced Hermitian metric on $\mathcal{A}_{M,\mathbb{C}}^m(M) = \mathcal{A}_M^m(M) \otimes_{\mathbb{R}} \mathbb{C}$ is the one induced by extending to complexification, the one we associated before to the Riemannian structure.

21.2. **The operators ∂^* and $\overline{\partial}^*$.** Recall that $\partial : \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p+1,q}$ and $\overline{\partial} : \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p,q+1}$ and the de Rham operator is that $\mathrm{d} = \partial + \overline{\partial}$.

Definition 21.1. We define

$$\begin{aligned} \partial^* &:= - * \overline{\partial} * : \mathcal{A}_M^{p+1,q} \to \mathcal{A}_M^{p,q} \\ \overline{\partial}^* &:= - * \partial * : \mathcal{A}_M^{p,q+1} \to \mathcal{A}_M^{p,q} \end{aligned}$$

Proposition 21.2. *The pairs (∂, ∂^*) and $(\overline{\partial}, \overline{\partial}^*)$ are formal adjoint pairs.*

Proof. Let $u \in \mathcal{A}^{p,q}(M)$ and $v \in \mathcal{A}^{p+1,q}(M)$. Then

$$\begin{aligned} \langle\!\langle u, \partial^* v \rangle\!\rangle &= \int_M u \wedge \overline{*\partial^* v} \\ &= -\int_M u \wedge \overline{\underbrace{**}_{(-1)^{p+q}} \overline{\partial} * v} \\ &= (-1)^{p+q+1} \int_M \underbrace{u \wedge \partial * \overline{v}}_{(-1)^{p+q}(\partial(u \wedge * \overline{v}) - \partial u \wedge * \overline{v})} \\ &= -\int_M \underbrace{\partial(u \wedge * \overline{v})}_{\mathrm{d}(u \wedge * \overline{v}) \text{ because } \overline{\partial} \text{ acts via } 0} + \int_M \partial u \wedge * \overline{v} \end{aligned}$$

By Stokes, the first term is zero. The second term is $\langle\!\langle \partial u, v \rangle\!\rangle$. The proof for the other pair is similar. ☐

Definition 21.3. Let $\Delta' := \partial\partial^* + \partial^*\partial$ and $\Delta'' := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$.

As in the case of the Laplace-Baltrimi operator Δ, Proposition 21.2 implies that Δ' and Δ'' are formally self-adjoint.

Definition 21.4. A form ω is a ∂-harmonic form if $\Delta'\omega = 0$. A form ω is a $\overline{\partial}$-harmonic form if $\Delta''\omega = 0$.

Write $\mathscr{H}^{p,q}_{\Delta'} = \{\partial\text{-harmonic } (p,q)\text{-forms}\}$ and $\mathscr{H}^{p,q}_{\Delta''} = \{\overline{\partial}\text{-harmonic } (p,q)\text{-forms}\}$. As in the case of usual harmonic forms, we can prove

- ω is ∂ ($\overline{\partial}$)-harmonic if and only if $\partial\omega = 0, \partial^*\omega = 0$ ($\overline{\partial}\omega = 0, \overline{\partial}^*\omega = 0$).
- Since $\overline{\partial^*\omega} = -\overline{*\overline{\partial}*\omega} = \overline{\partial}^*\overline{\omega}$, we have $\overline{\Delta'\omega} = \Delta''\overline{\omega}$. Hence $\omega \mapsto \overline{\omega}$ gives an isomorphism $\mathscr{H}^{p,q}_{\Delta'}(M) \xrightarrow{\sim} \mathscr{H}^{q,p}_{\Delta''}(M)$.
- $*\Delta' = \Delta''*$ and $*\Delta'' = \Delta'*$. Because if both sides acts (p,q)-forms, then we have

$$\begin{aligned} *\Delta' &= *\left(\partial\partial^* + \partial^*\partial\right) \\ &= -*\left(\partial*\overline{\partial}* + *\overline{\partial}*\partial\right) \\ &= -*\partial*\overline{\partial}* + (-1)^{p+q+1}\overline{\partial}*\partial \\ \Delta''* &= \left(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}\right)* \\ &= -\left(\overline{\partial}*\partial* + *\partial*\overline{\partial}\right)* \\ &= (-1)^{p+q+1}\overline{\partial}*\partial - *\partial*\overline{\partial}* \end{aligned}$$

So we get isomorphisms

$$\begin{array}{ccc} \mathscr{H}^{p,q}_{\Delta'}(M) & \xrightarrow[\sim]{*} & \mathscr{H}^{n-q,n-p}_{\Delta''}(M) \\ & & \Big\downarrow{\scriptstyle \text{conj}\ \wr} \\ & & \mathscr{H}^{n-p,n-q}_{\Delta'}(M) \end{array}$$

- The same computation we've done for Δ implies that $\sigma_2(\Delta')_x(v) = -\frac{1}{2}\|v\|^2 \cdot \mathrm{id} = \sigma_2(\Delta'')_x(v)$. Hence, like Δ, the operators Δ' and Δ'' are elliptic operators. So $\mathscr{H}^{p,q}_{\Delta'}(M), \mathscr{H}^{p,q}_{\Delta''}(M)$ are finite dimensional vector spaces over $\mathbb{C}$. We have

$$\mathcal{A}^{p,q}(M) = \mathscr{H}^{p,q}_{\Delta''}(M) \overset{\perp}{\oplus} \underbrace{\mathrm{Im}\left(\Delta'' : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q}(M)\right)}_{\overline{\partial}(\mathcal{A}^{p,q-1}(M)) \overset{\perp}{\oplus} \overline{\partial}^*(\mathcal{A}^{p,q+1}(M))}$$

and the kernel is

$$\mathrm{Ker}\left(\overline{\partial} : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M)\right) = \mathscr{H}^{p,q}_{\Delta''}(M) \overset{\perp}{\oplus} \overline{\partial}\left(\mathcal{A}^{p,q-1}(M)\right).$$

The conclusion is that the Dolbeault cohomology $\mathrm{H}^{p,q}(M) := \mathscr{H}^q\left(\mathcal{A}^{p,\bullet}(M), \overline{\partial}\right) \cong \mathrm{H}^q(M, \Omega^p) \cong \mathscr{H}^{p,q}_{\Delta''}(M)$.

Again we have application to Poincaré duality. Consider following pairing,

$$\begin{aligned} \mathrm{H}^q(M,\Omega^p) \times \mathrm{H}^{n-q}(M,\Omega^{n-p}) &\to \mathbb{C} \\ ([\alpha],[\beta]) &\mapsto \int_M \alpha \wedge \beta \end{aligned}$$

where α is a (p,q)-form with $\overline{\partial}\alpha = 0$ and β is a $(n-p,n-q)$-form with $\overline{\partial}\beta = 0$. The claim is that this is a non-degenerate pairing.

To check this, we put a metric h on M. Given nonzero element in $\mathrm{H}^2(M,\Omega^p)$, choose $\overline{\partial}$-harmonic representative α. If $\beta = \overline{*\alpha}$, then β is harmonic ($\overline{\partial}\beta = 0$) and $\int_M \alpha \wedge \beta = \langle\!\langle \alpha, \alpha \rangle\!\rangle > 0$.

22. November 01, 2019

22.1. **Kähler manifolds.** Let V be a finite dimensional vector space over $\mathbb{C}$, let J be the multiplication by i. Then $V_{\mathbb{C}} = V' \oplus V''$. A Hermitian form h on V is equivalent to a bilinear alternating form $A : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ such that $A(u,v) = A(Ju,Jv), \forall u,v \in V_{\mathbb{R}}$. ($A$ will be the imaginary part of h and we have $h = S + \mathrm{i}A$ where $S(u,v) = A(Ju,v)$).

For any $A \in \bigwedge^2 V_{\mathbb{R}}^*$, let $\widetilde{A} \in \bigwedge^2 V_{\mathbb{C}}^*$ be the corresponding alternating bilinear form on $V_{\mathbb{C}}$. Then $\widetilde{A}$ is *real*.

Claim. *$A(Ju,Jv) = A(u,v), \forall u,v$ if and only if $\widetilde{A}$ is a $(1,1)$-form.*

Proof. $\widetilde{A}$ is a $(1,1)$-form iff $\widetilde{A}(V'\times V') = 0$ and $\widetilde{A}(V''\times V'') = 0$. Since $\widetilde{A}(\overline{u},\overline{v}) = \overline{\widetilde{A}(u,v)}$, it is enough to check the first condition. Since $V' = \{u - \mathrm{i}Ju \mid u \in V\}$. We have

$$\widetilde{A}(u-\mathrm{i}Ju, v-\mathrm{i}Jv) = (A(u,v) - A(Ju,Jv)) - \mathrm{i}(A(Ju,v) + \underbrace{A(u,Jv)}_{-A(J(Ju),Jv)})$$

So the direction $\Rightarrow$ is clear and the converse direction $\Leftarrow$ follows because A takes real values. $\square$

Definition 22.1. The *fundamental form* of the Hermitian metric h is the real $(1,1)$-form $\omega_h = -\widetilde{A}$.

Next we want to describe via basis. Choose $x_1,\dots,x_n$ basis of $V/\mathbb{C}$ and $y_j = Jx_j$. Then $x_1,\dots,x_n,y_1,\dots,y_n$ form a basis of $V_{\mathbb{R}}$. Write $e_j = \frac{1}{2}(x_j - \mathrm{i}y_j)$. Then $e_1,\dots,e_n$ form a basis for V' and $\overline{e}_1,\dots,\overline{e}_n$ form a basis for V''. Given Hermitian form $h : h_{ij} = h(x_i,x_j)$. We can write $\omega_h = \sum_{j<k} \lambda_{jk} e_j^* \wedge \overline{e}_k^*$. Then we have

$$\begin{aligned}
\lambda_{jk} &= \omega_h(e_j,\overline{e}_k)\\
&= -\frac{1}{4}\widetilde{A}(x_j - \mathrm{i}y_j, x_k + \mathrm{i}y_k)\\
&= -\frac{1}{4}((A(x_j,x_k) + \underbrace{A(y_j,y_k)}_{A(x_j,x_k)} + \mathrm{i}(\underbrace{A(x_j,y_k)}_{-S(x_k,x_j)} - \underbrace{A(y_j,x_k)}_{A(Jx_j,x_k)=S(x_j,x_k)}))\\
\Rightarrow \lambda_{jk} &= -\frac{1}{4}\cdot 2\,(A(x_j,x_k) - \mathrm{i}S(x_j,x_k))\\
&= \frac{\mathrm{i}}{2}\,(S(x_j,x_k) + \mathrm{i}A(x_j,x_k))\\
&= \frac{\mathrm{i}}{2}h_{jk}
\end{aligned}$$

So we conclude that $\omega_h = \frac{\mathrm{i}}{2}\sum_{j<k} h_{jk} e_j^* \wedge \overline{e}_k^*$.

Remark 22.2. This tells us that

(1) h is a metric iff $-\mathrm{i}\omega_h(v,v) > 0$ for all $v \neq 0$. Hence, a Hermitian metric on V is equivalent to a real $(1,1)$-form ω with $-\mathrm{i}\omega(v,v) > 0, \forall v$.
(2) Suppose $x_1,\dots,x_n$ is an orthonormal basis of V. Then $h_{jk} = \delta_{jk}$, and $\omega_h = \frac{\mathrm{i}}{2}\sum_{k=1}^n e_k^* \wedge \overline{e}_k^*$.

Recall that the bilinear symmetric form S gives a top form $\mathrm{d}V$. Suppose that $x_1, y_1, \cdots, x_n, y_n$ is a positive, orthonormal basis for S. Then

$$\begin{aligned}
\mathrm{d}V &= x_1^* \wedge y_1^* \wedge \cdots \wedge x_n^* \wedge y_n^*\\
\bigwedge^n \omega_h &= \left(\frac{\mathrm{i}}{2}\right)^n n! \cdot e_1^* \wedge \overline{e}_1^* \wedge \cdots \wedge e_n^* \wedge \overline{e}_n^*
\end{aligned}$$

where $e_j^* \wedge \overline{e}_j^* = \left(x_j^* + \mathrm{i}y_j^*\right) \wedge \left(x_j^* - \mathrm{i}y_j^*\right) = -2\mathrm{i}x_j^* \wedge y_j^*$. So we conclude that $\bigwedge^n \omega_h = n!\mathrm{d}V$.

Definition 22.3. Let M be a complex manifold. A Hermitian metric h on M is Kähler if the real $(1,1)$-form $\omega = \omega_h$ is closed, i.e. $\mathrm{d}\omega = 0$.

Example 22.4 (Trivial example)**.** Consider $(\mathbb{C}^n, \text{standard metric})$. It is Kähler with respect to the standard basis which is orthonormal for h. So $\omega_h = \frac{\mathrm{i}}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j$. This is clearly closed.

Note that the existence of a Kähler metric is a *global* property. The issue is that we cannot glue such metrics using partition of unity any more. Because if h is a Kähler metric with form ω and f is smooth and everywhere positive, then fh is a metric and $\omega_{fh} = f\omega_h$. Then

$$\mathrm{d}(f\omega_h) = \underbrace{\mathrm{d}f\wedge\omega_h}_{\text{not necessarily }0} + \underbrace{f\mathrm{d}\omega_h}_{0}$$

Example 22.5 (Important Example, the Fubini-Study metric on $\mathbb{P}^n$)**.** Let $z_0,\cdots,z_n$ be homogeneous coordinates on $\mathbb{P}^n$, $U_j = \{z_j \neq 0\}$. Let

$$\omega_j = \frac{\mathrm{i}}{2}\partial\overline{\partial}\log\left(\sum_{k=0}^{n}|\frac{z_k}{z_j}|^2\right) \in \mathcal{A}^{1,1}(U_j).$$

These glue to a global $(1,1)$-form because on $U_j \cap U_l$,

$$\sum_k |\frac{z_k}{z_j}|^2 = \left(\sum_k |\frac{z_k}{z_l}|^2\right)\cdot|\frac{z_l}{z_j}|^2$$
$$\Rightarrow \log\left(\sum_k |\frac{z_k}{z_j}|^2\right) = \log\left(\sum_k |\frac{z_k}{z_l}|^2\right) + \log|\frac{z_l}{z_j}|^2.$$

So it is enough to note that $\partial\overline{\partial}\log|w_j|^2 = 0$ if $w_1,\ldots,w_n$ are coordinates on $\mathbb{C}^n$. This is because $|w_j|^2 = w_j\cdot\overline{w}_j \Rightarrow \log|w_j|^2 = \log w_j + \log\overline{w}_j \Rightarrow \partial\overline{\partial}\log|w_j|^2 = 0$ on $\mathbb{C}^n \smallsetminus \{w_j = 0\}$.

So all these ω_j glue together to give an $\omega \in \mathcal{A}^{1,1}(\mathbb{P}^n)$.

- ω is closed. Since $\partial^2 = 0, \overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$ and $\mathrm{d} = \partial + \overline{\partial}$, we have $\partial\omega = 0, \overline{\partial}\omega = 0 \Rightarrow \mathrm{d}\omega = 0$.
- $\overline{\omega}_j = -\frac{\mathrm{i}}{2}\overline{\partial}\partial(\cdots) = \frac{\mathrm{i}}{2}\partial\overline{\partial}(\cdots)$

We have to check that $-\mathrm{i}\omega(v,v) > 0$ if $v \neq 0$. Work on $U_j \cong \mathbb{C}^n$ with $w_k = \frac{z_k}{z_j}$. Then

$$\omega_j = \frac{\mathrm{i}}{2}\partial\overline{\partial}\log\left(1 + \sum_{k=1}^{n}|w_k|^2\right)$$

$$\overline{\partial}\log\left(1 + \sum\underbrace{|w_k|^2}_{w_k\overline{w}_k}\right) = \sum_{k=1}^{n}\frac{w_k\mathrm{d}\overline{w}_k}{1+\sum_{l=1}^{n}|w_l|^2}$$

$$\frac{\partial}{\partial w_j}\left(\frac{w_k}{1+\sum_l|w_l|^2}\right) = \frac{\delta_{jk}}{1+\sum_l|w_l|^2} - \frac{w_k\overline{w}_j}{(1+\sum_l|w_l|^2)^2}$$

$$\omega_j = \frac{\mathrm{i}}{2}\left(\sum_k\frac{\mathrm{d}w_k\wedge\mathrm{d}\overline{w}_k}{1+\sum_l|w_l|^2} - \sum_{k,j}\frac{w_k\overline{w}_j\mathrm{d}w_j\wedge\mathrm{d}\overline{w}_k}{(1+\sum_l|w_l|^2)^2}\right)$$

So $a_{jk} = \left(1+\sum_l|w_l|^2\right)\delta_{jk} - w_k\overline{w}_j$. We need that if $v = (v_1,\ldots,v_n) \neq 0$, then $v\cdot(a_{jk})\overline{v}^t > 0$. Write $(-,-)$ for the standard Hermitian metric on $\mathbb{C}^n$. Then

$$v\cdot(a_{jk})\overline{v}^t = (1+(w,w))(v,v) - \sum_{j,k}v_jw_k\overline{w}_j\overline{v}_k$$
$$= \underbrace{(v,v)}_{>0} + \underbrace{\left((w,w)\cdot(v,v) - |(v,w)|^2\right)}_{\geqslant 0\text{ by Schwarz}}$$

So it is positive.

23. November 04, 2019

23.1. **Kähler manifolds (continued). Last time**: On $\mathbb{P}^n$, we have a Kähler metric. Recall that if M is a n-dimensional complex manifold and h is a Hermitian form, then $\omega_h = -\operatorname{Im}(h)$ is a real $(1,1)$-form on M. Then h is a metric if and only if $\mathrm{i}\omega_h(v,v) > 0, \forall v \neq 0$. We also have h or ω is Kähler if $\mathrm{d}\omega = 0 \Leftrightarrow \partial\omega = 0, \overline{\partial}\omega = 0$.

Remark 23.1. If h is a Hermitian metric on M, and $M' \hookrightarrow M$ is a submanifold, the restriction h' of h to $\mathrm{T}M'$ is a Hermitian metric on M', and $\omega_{h'} = \omega_h|_{M'}$. In particular, h Kähler $\Rightarrow h'$ Kähler.

The upshot is that if X is smooth, quasi-projective complex algebraic variety, then we have a locally closed immersion $X \hookrightarrow \mathbb{P}^n$ such that X^{an} is a submanifold of $(\mathbb{P}^n)^{\mathsf{an}}$. By restrcting the Fubini-Study metric to X^{an}, X^{an} has a Kähler metric.

Example 23.2 (Complex Tori)**.** $M = V/\Lambda$ where V is a n-dimensional complex vector space and $\Lambda \subseteq \mathbb{C}^n$ is a lattice ($\Lambda \cong \mathbb{Z}^{2n}$) such that $\Lambda \otimes \mathbb{R} = \mathbb{C}^n$.

If h is the standard Hermitian metric on $\mathbb{C}^n$, with form $\omega = \frac{\mathrm{i}}{2}\sum_{k=1}^n z_k \wedge \overline{z}_k$ and $\gamma_\lambda : V \to V$ is the translation by some $\lambda \in \Lambda$, then $\gamma_\lambda^*(\omega) = \omega$. So h induces a metric on the quotient M, which is again Kähler.

Remark 23.3. Note that for Λ general, M is not algebraic.

The goal is to show that Kähler metrics are not far from the standard one. Suppose that $p \in M$, coordinates $z_1, \dots, z_n$ in a chart around p such that $z_i(p) = 0, \forall i$. Suppose h is a Hermitian metric on M, with fundamental form $\omega = \frac{\mathrm{i}}{2}\sum_{j,k} h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$, we will say that ω osculates to order 2 to the standard metric at p (in these coordinates) if $h_{jk} = \delta_{jk}, \forall j,k$ and $\frac{\partial h_{jk}}{\partial z_l} = 0, \frac{\partial h_{jk}}{\partial \overline{z}_l} = 0, \forall j,k,l$.

Proposition 23.4. *Given a Hermitian metric h with fundamental forms ω, h is Kähler if and only if $\forall p \in M$, there exists a chart as above such that ω osculates to order 2 to the standard metric.*

Proof. Given $\omega = \frac{\mathrm{i}}{2}\sum_{j,k} h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$, we have

$$\begin{aligned} \omega &= \frac{\mathrm{i}}{2}\sum_{j,k} h_{jk}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k \\ \mathrm{d}\omega &= \frac{\mathrm{i}}{2}\sum_{j,k,l} \frac{\partial h_{jk}}{\partial z_l}\mathrm{d}z_l \wedge \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k - \frac{\mathrm{i}}{2}\sum_{j,k,l} \frac{\partial h_{jk}}{\partial \overline{z}_l}\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_l \wedge \mathrm{d}\overline{z}_k. \end{aligned}$$

Clearly if $\frac{\partial h_{jk}}{\partial z_l}(p) = \frac{\partial h_{jk}}{\partial \overline{z}_l}(p) = 0$, then $\mathrm{d}\omega(p) = 0$. Conversely, suppose that $\mathrm{d}\omega = 0$. Easy to see that there is a linear change of variables such that $h_{jk}(p) = \delta_{jk}$. Let $a_{jkl} = \frac{\partial h_{jk}}{\partial z_l}(p), a'_{jkl} = \frac{\partial h_{jk}}{\partial \overline{z}_l}(p)$.

The condition that $\mathrm{d}\omega(p) = 0$ and ω being real (i.e. $h_{jk} = \overline{h}_{kj}$) implying following repsectively,

$$a_{jkl} = a_{lkj}, a'_{jkl} = a'_{jlk} \tag{23.1}$$

$$\overline{a}_{jkl} = a'_{kjl} \tag{23.2}$$

We do the change of variables $w_j = z_j + \frac{1}{2}\sum_{k,l=1}^n a_{kjl}z_k z_l$. Compare ω with $\frac{\mathrm{i}}{2}\sum_{j=1}^n \mathrm{d}w_j \wedge \mathrm{d}\overline{w}_j$. We have

$$\begin{aligned} \mathrm{d}w_j &= \mathrm{d}z_j + \frac{1}{2}\sum_{k,l=1}^n a_{kjl}\left(z_k \mathrm{d}z_l + z_l \mathrm{d}z_k\right) \\ &\overset{(23.1)}{=} \mathrm{d}z_j + \sum_{k \leqslant l} a_{kjl} z_k \mathrm{d}z_l \\ \Rightarrow \mathrm{d}\overline{w}_j &\overset{(23.2)}{=} \mathrm{d}\overline{z}_j + \sum_{k \leqslant l} a'_{jkl}\overline{z}_k \mathrm{d}\overline{z}_l \\ \Rightarrow \frac{\mathrm{i}}{2}\sum_{j=1}^n \mathrm{d}w_j \wedge \mathrm{d}\overline{w}_j &= \frac{\mathrm{i}}{2}\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j + \frac{\mathrm{i}}{2}\sum_{k \leqslant l, j} \underbrace{a'_{jkl}}_{= a'_{jlk} = \frac{\partial h_{jl}}{\partial \overline{z}_k}(p)} \overline{z}_k \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_l \\ &\quad + \frac{\mathrm{i}}{2}\sum_{k \leqslant l, j} \underbrace{a_{kjl}}_{= a_{ljk} = \frac{\partial h_{lj}}{\partial z_k}(p)} z_k \mathrm{d}z_l \wedge \mathrm{d}\overline{z}_j + \text{terms vanishing at } p \text{ with order} \geqslant 2. \end{aligned}$$

Right-hand side matches the Taylor expansion of ω up to the 2nd order term, so ω differs $\frac{\mathrm{i}}{2}\sum \mathrm{d}w_j \wedge \mathrm{d}\overline{w}_j$ by terms vanishing at p with order $\geqslant 2$. This achieves our goal, which is $\omega = \frac{\mathrm{i}}{2}\sum \mathrm{d}w_j \wedge \mathrm{d}\overline{w}_j$+terms vanishing at p with order $\geqslant 2$. ∎

23.2. **Operators on Kähler manifolds.** We have $*, \mathrm{d}, \partial, \overline{\partial}$ and the adjoint operators $\mathrm{d}^*, \partial^*, \overline{\partial}^*$.

Given the Kähler metric h, with fundamental form ω, the *Lefschetz operator* $L := \omega \wedge - : \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p+1,q+1}$. We also define $\Lambda := *^{-1} L * : \mathcal{A}_M^{p,q} \to \mathcal{A}_M^{p-1,q-1}$.

Remark 23.5. Since ω is a real form, L is a real operator and hence it commutes with conjugation. Also, because $*$ is a real operator, Λ is a real operator.

Lemma 23.6. *Λ is the adjoint of L, i.e. $\langle L\alpha, \beta\rangle = \langle \alpha, \Lambda\beta\rangle$ for all (p,q)-form α and $(p+1,q+1)$-form β.*

Proof. Recall that $\alpha \wedge \overline{*\beta} = \langle \alpha, \beta\rangle \, \mathrm{d}V$.

$$\begin{aligned}
\langle \alpha, \Lambda\beta\rangle \, \mathrm{d}V &= \alpha \wedge \overline{*(*^{-1} L *)\beta} \\
&= \alpha \wedge \overline{\omega \wedge *\beta} \\
&= (\alpha \wedge \omega) \wedge \overline{*\beta} \\
&= L\alpha \wedge \overline{*\beta} \\
&= \langle L\alpha, \beta\rangle \, \mathrm{d}V
\end{aligned}$$

□

Theorem 23.7 (Kähler identities)**.** *Let (M,h) be a Kähler manifold, then we have*

(1) $\left[\overline{\partial}^*, L\right] = \mathrm{i}\partial$.
(2) $[\partial^*, L] = -\mathrm{i}\overline{\partial}$.
(3) $[\Lambda, \partial] = \mathrm{i}\partial^*$.
(4) $\left[\Lambda, \overline{\partial}\right] = -\mathrm{i}\overline{\partial}^*$.

Proof. We will only prove (1). (2), (3), (4) follow by conjugation or taking adjoints, see Remark 23.8. □

Remark 23.8. For any operators P, Q, we have

- $[P,Q]^* = [Q^*, P^*]$.
- $(\lambda P)^* = \overline{\lambda} P^*$.

24. November 06, 2019

Proof of Theorem 23.7. Suppose first that we deal with a (rescaling by 2) of the standard metric on $\mathbb{C}^n$ such that

$$\omega = \mathrm{i} \sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j.$$

Similar computation to that for d^* gives

$$\overline{\partial}^* \underbrace{\left(\sum_{I,J} f_{I,J} \mathrm{d}z_I \wedge \mathrm{d}z_J \right)}_{\eta} = - \sum_{I,J,k} \frac{\partial f_{I,J}}{\partial z_k} i_{\frac{\partial}{\partial \overline{z}_k}} (\mathrm{d}z_k \wedge \mathrm{d}\overline{z}_J)$$

$$\begin{aligned}
\left[\overline{\partial}^*, L\right] \eta &= \overline{\partial}^* (\omega \wedge \eta) - \omega \wedge \overline{\partial}^* \eta \\
&= - \mathrm{i} \sum_{I,J,k,j} \frac{\partial f_{I,J}}{\partial z_k} i_{\frac{\partial}{\partial \overline{z}_k}} (\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j \wedge \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J) \\
&\quad + \mathrm{i} \sum_{I,J,j,k} \frac{\partial f_{I,J}}{\partial z_k} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j \wedge i_{\frac{\partial}{\partial \overline{z}_k}} (\mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J)
\end{aligned}$$

Since $i_{\frac{\partial}{\partial \overline{z}_k}}$ is a derivation, this is

$$-\mathrm{i} \sum_{I,J,j,k} \frac{\partial f_{I,J}}{\partial z_k} \underbrace{i_{\frac{\partial}{\partial \overline{z}_k}} (\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j)}_{-\delta_{jk}\mathrm{d}z_j} \wedge \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J = \mathrm{i} \sum_{I,J,k} \frac{\partial f_{I,J}}{\partial z_k} \mathrm{d}z_k \wedge \mathrm{d}z_I \wedge \mathrm{d}\overline{z}_J = \mathrm{i}\partial\eta.$$

For general case, in order to check that $\left[\overline{\partial}^*, L\right] = \mathrm{i}\partial$ at $p \in M$, we choose coordinates $z_1, \ldots, z_n$ centered at p such that $\omega = \mathrm{i} \sum_{j,k} h_{jk} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$ and $h_{jk}(p) = \delta_{jk}$, $\frac{\partial h_{jk}}{\partial z_l}(p) = \frac{\partial h_{jk}}{\partial \overline{z}_l}(p) = 0$.

Since both $\partial, \overline{\partial}^*$ are differential operators of order 1, the difference with respect to the computation for the standard metric will only involve the $\frac{\partial h_{jk}}{\partial z_l}(p)$, $\frac{\partial h_{jk}}{\partial \overline{z}_k}(p)$. All these vanishes, so we win. ☐

We will use $[\partial, \Lambda] = -\mathrm{i}\overline{\partial}^*$.

Corollary 24.1. *If (M, h) is a Kähler manifold, then $\Delta' = \Delta'' = \frac{1}{2}\Delta$.*

Proof. We have

$$\begin{aligned}
\Delta'' =& \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \\
=& \mathrm{i}\overline{\partial}(\partial\Lambda - \Lambda\partial) + \mathrm{i}(\partial\Lambda - \Lambda\partial)\overline{\partial} \\
=& \underbrace{\mathrm{i}\overline{\partial}\partial\Lambda}_{\text{real}} - \underbrace{\mathrm{i}\Lambda\partial\overline{\partial}}_{\text{real}} + \mathrm{i}\left(\partial\Lambda\overline{\partial} - \overline{\partial}\Lambda\partial\right)
\end{aligned}$$

Note that $\mathrm{i}\partial\overline{\partial}$ is a real operator. So $\overline{\mathrm{i}\partial\overline{\partial}} = -\mathrm{i}\overline{\partial}\partial = \mathrm{i}\partial\overline{\partial}$. Both L, Λ are real operators, hence $\mathrm{i}\left(\partial\Lambda\overline{\partial} - \overline{\partial\Lambda\overline{\partial}}\right)$ is a real operator.

So we know that Δ'' is a real operator. But $\overline{\Delta''} = \Delta'$. So $\Delta' = \Delta''$. Let us compute Δ.

$$\begin{aligned}
\Delta =& \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} = \left(\partial + \overline{\partial}\right)\left(\partial^* + \overline{\partial}^*\right) + \left(\partial^* + \overline{\partial}^*\right)\left(\partial + \overline{\partial}\right) \\
=& (\partial\partial^* + \partial^*\partial) + \left(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}\right) + \left(\partial\overline{\partial}^* + \partial^*\overline{\partial} + \overline{\partial}\partial^* + \overline{\partial}^*\partial\right)
\end{aligned}$$

Let us show that $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$. This is because it is $\mathrm{i}\partial(\partial\Lambda - \Lambda\partial) + \mathrm{i}(\partial\Lambda - \Lambda\partial)\partial = -\mathrm{i}\partial\Lambda\partial + \mathrm{i}\partial\Lambda\partial = 0$. Then

$$\Delta = \Delta' + \Delta'' + \left(\partial\overline{\partial}^* + \overline{\partial}^*\partial\right) + \overline{\left(\partial\overline{\partial}^* + \overline{\partial}^*\partial\right)} = \Delta' + \Delta''$$

as deried. ☐

Next we discuss the consequence of the above corollary. Recall that we have

$$\begin{aligned}
\mathscr{H}^m(M, \mathbb{C}) =& \text{ space of complex } m\text{-harmonic forms on } M \\
=& \mathscr{H}^m(M) \otimes_{\mathbb{R}} \mathbb{C} \\
=& \mathrm{null}(\Delta_M)
\end{aligned}$$

Since $\Delta = 2\Delta' = 2\Delta''$, the decomposition of $\mathcal{A}^m(M) \otimes_{\mathbb{R}} \mathbb{C}$ into (p,q)-forms induces a decomposition

$$\mathscr{H}^m(M, \mathbb{C}) = \oplus_{p+q=m} \underbrace{\mathscr{H}^{p,q}(M)}_{\text{Harmonic } (p,q)\text{-forms}}$$

We saw that the inclusion $\mathscr{H}^m(M) \hookrightarrow \{\text{closed real } m\text{-forms}\}$ induces isomorphisms $\mathscr{H}^m(M) \cong \mathrm{H}^m_{\mathsf{dR}}(M, \mathbb{R}) \Rightarrow \mathscr{H}^m(M, \mathbb{C}) \cong \mathrm{H}^m_{\mathsf{dR}}(M, \mathbb{C})$. If $\mathrm{H}^{p,q}(M)$ is the image of $\mathscr{H}^{p,q}(M)$, then $\mathrm{H}^m_{\mathsf{dR}}(M, \mathbb{C}) = \oplus_{p+q=m}\mathrm{H}^{p,q}(M)$ (Hodge decomposition). Recall that $\overline{\mathscr{H}^{p,q}_{\Delta'}} = \mathscr{H}^{q,p}_{\Delta''}$. Since $\Delta' = \Delta''$, we have $\overline{\mathrm{H}^{p,q}} = \mathrm{H}^{q,p}$. We also have $\mathscr{H}^{p,q}_{\Delta''}(M) \cong \mathrm{H}^q(M, \Omega^p) \Rightarrow \mathrm{H}^{p,q} \cong \mathrm{H}^q(M, \Omega^p)$.

Numerically, if $b_m = \dim_{\mathbb{C}} \mathrm{H}^m(M,\mathbb{C})$ (Betti numbers of M) and $h^{p,q} = \dim_{\mathbb{C}} \mathrm{H}^q(M,\Omega^p)$ (Hodge numbers of M), then we have Hodge symmetry,

$$b_m = \sum_{p+q=m} h^{p,q}$$
$$h^{p,q} = h^{q,p}$$

A consequence of this that if M is a Kähler manifold, then $b_m(M)$ is even if m is odd.

Example 24.2 (Hopf surface)**.** The Hopf surface $M = \mathbb{C}^2 \setminus \{(0,0)\}/(z_1,z_2) \sim (2z_1,2z_2)$. We saw that M is diffeomorphic to $S^3 \times S^1$. Then Künneth formula tells us that $b_3(M) = 1$. So M is not Kähler.

Next, we want to show that the decomposition of $\mathrm{H}^m(M,\mathbb{C}) = \oplus_{p+q=m}\mathrm{H}^{p,q}$ is independent of the choice of metric.

We define Bott-Chern cohomology:

$$\mathrm{H}^{p,q}_{\mathsf{BC}}(M) = \frac{\{(p,q)\text{-forms } u \mid \partial u = 0, \overline{\partial} u = 0\}}{\partial\overline{\partial}\mathcal{A}^{p-1,q-1}(M)}.$$

Note that $\partial\overline{\partial} v = \mathrm{d}\left(\overline{\partial} v\right)$. So we have a canonical map $\mathrm{H}^{p,q}_{\mathsf{BC}}(M) \to \mathrm{H}^{p+q}_{\mathsf{dR}}(M,\mathbb{C})$.

Theorem 24.3. *This map gives an isomorphism* $\mathrm{H}^{p,q}_{\mathsf{BC}}(M) \cong \mathrm{H}^{p,q}(M)$.

Lemma 24.4 ($\partial\overline{\partial}$-lemma)**.** *If u is a form such that $\partial u = 0, \overline{\partial} u = 0$, then T.F.A.E*

(1) u is d*-exact.*
(2) u is ∂-exact.
(3) u is $\overline{\partial}$-exact.
(4) $u \in \mathrm{Im}(\partial\overline{\partial})$.

25. November 08, 2019

25.1. $\partial\overline{\partial}$-lemma.

Proof of Lemma 24.4. We know $\partial\overline{\partial} v = \mathrm{d}\left(\overline{\partial} v\right), \partial\overline{\partial} = -\overline{\partial}\partial$. So clearly (4) ⇒ (1), (2), (3).

(2)⇒(4): Write $u = \partial v$. By Hodge theorem for $\overline{\partial}$, we can write $v = v_1 + \overline{\partial} v_2 + \overline{\partial}^* v_3$ where v_1 is harmonic. We've shown that $\Delta = \Delta' + \Delta''$ Corollary 24.1, so $\partial\overline{\partial}^* = -\overline{\partial}^*\partial$. Hence

$$u = \partial\overline{\partial} v_2 + \partial\overline{\partial}^* v_3$$
$$\overline{\partial} u = 0 \Rightarrow \underbrace{\overline{\partial}\partial\overline{\partial} v_2}_{=-\overline{\partial}^2\partial v_2 = 0} + \overline{\partial}\partial\overline{\partial}^* v_3 = 0$$
$$\Rightarrow \overline{\partial}\partial\overline{\partial}^* v_3 = 0$$
$$\Rightarrow \overline{\partial}\overline{\partial}^*\partial v_3 = 0$$

(If $\overline{\partial}\overline{\partial}^*\eta = 0$, then $\langle\!\langle \overline{\partial}\overline{\partial}^*\eta, \eta\rangle\!\rangle = 0 \Rightarrow \langle\!\langle \overline{\partial}^*\eta, \overline{\partial}^*\eta\rangle\!\rangle = 0 \Rightarrow \overline{\partial}^*\eta = 0$.) So $\overline{\partial}^*\partial v_3 = 0 \Rightarrow u = \partial\overline{\partial} v_2$.

(3)⇒(4): If $u \in \mathrm{Im}(\overline{\partial})$, apply previous argument for $\overline{u}$ to get $u \in \mathrm{Im}(\partial\overline{\partial})$.

(1)⇒(4): If $u = \mathrm{d}(w) = \partial w + \overline{\partial} w$, then $\partial\overline{\partial} w = 0$ and $\overline{\partial}\overline{\partial} w = 0$. By the implication (3)⇒(4) for $\overline{\partial} w$, we conclude that $\overline{\partial} w \in \mathrm{Im}(\partial\overline{\partial})$.

Similarly, $\overline{\partial} u = 0 \Rightarrow \partial w \in \mathrm{Im}(\partial\overline{\partial}) \Rightarrow u \in \mathrm{Im}(\partial\overline{\partial})$. □

We have a map

$$\varphi : \mathrm{H}^{p,q}_{\mathsf{BC}}(M) \to \mathrm{H}^{p+q}_{\mathsf{dR}}(M,\mathbb{C})$$
$$u \mapsto [u]$$

where

$$\mathrm{H}^{p,q}_{\mathsf{BC}}(M) = \frac{\{\text{global } (p,q)\text{-forms } u \mid \partial u = 0, \overline{\partial} u = 0\}}{\partial\overline{\partial}\mathcal{A}^{p-1,q-1}(M)}$$

Recall that

$$\mathrm{H}^{p,q}(M) = \{\alpha \in \mathrm{H}^{p+q}_{\mathsf{dR}}(M,\mathbb{C}) \mid \alpha = [\eta], \eta = \text{harmonic } (p,q)\text{-form}\}.$$

Since $\partial u = 0$, the Hodge theorem for ∂ implies that $u = v + \partial w$ where v is some harmonic (p,q)-form. Then $\overline{\partial} u = 0 \Rightarrow \overline{\partial}\partial w = 0$. Since $\partial\partial w = 0$, by $\partial\overline{\partial}$-lemma for ∂w, we have $\partial w \in \mathrm{Im}(\mathrm{d})$. Therefore $[u] = [v]$ in $\mathrm{H}^{p+q}_{\mathsf{dR}}(M,\mathbb{C})$. Hence $\mathrm{Im}(\varphi) \subseteq \mathrm{H}^{p,q}(M)$.

Since η is a harmonic (p,q)-form satisfying $\partial\eta = 0, \overline{\partial}\eta = 0$, we conclude that φ is surjective.

Finally, φ is injective by (1)⇒(4) in the $\partial\overline{\partial}$-lemma. Hence $\mathrm{H}^{p,q}_{\mathsf{BC}}(M) \xrightarrow{\sim} \mathrm{H}^{p,q}(M)$. This completes the proof of Theorem 24.3.

Consequences:

(1) If $\alpha \in \mathrm{H}^{p,q}(M), \beta \in \mathrm{H}^{p',q'}(M)$, then $\alpha \cup \beta \in \mathrm{H}^{p+p',q+q'}(M)$.

Proof. Write $\alpha = [u], \partial u = 0, \overline{\partial} u = 0$ where u is a (p,q)-form. Write $\beta = [v], \partial v = 0, \overline{\partial} v = 0$ where v is a (p',q')-form. Therefore $\partial(u \wedge v) = \partial u \wedge v \pm u \wedge \partial v = 0$ and similarily we have $\overline{\partial}(u \wedge v) = 0$. Hence $\alpha \cup \beta = [u \wedge v] \in \mathrm{H}^{p+p',q+q'}(M)$ by Theorem 24.3. ☐

(2) If $f : M' \to M$ is a holomorphic map of complex manifolds of Kähler type, then $f^* : \mathrm{H}^m_{\mathsf{dR}}(M,\mathbb{C}) \to \mathrm{H}^m_{\mathsf{dR}}(M',\mathbb{C})$ maps each $\mathrm{H}^{p,q}(M)$ to $\mathrm{H}^{p,q}(M')$.

Proof. If $\alpha \in \mathrm{H}^{p,q}(M)$, then by isomorphism Theorem 24.3, suppose that $\alpha = [u]$ where u is a (p,q)-form such that $\partial u = 0, \overline{\partial} u = 0$. Then $f^*\alpha = [f^*u]$. So $\partial(f^*u) = f^*(\partial u) = 0$ and $\overline{\partial}(f^*u) = 0$. Using isomorphism Theorem 24.3 again, we have $f^*\alpha \in \mathrm{H}^{p,q}(M')$. ☐

Remark 25.1. Suppose M is a compact manifold of Kähler type. Consider $\mathbb{C} \to \mathcal{O}_M$, the induced morphism in cohomology $\mathrm{H}^q(M,\mathbb{C}) \to \mathrm{H}^q(M,\mathcal{O}_M)$ are surjective.

Proof. Compute the maps in cohomology using the soft resolution

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & (\mathcal{A}^\bullet_M \otimes_{\mathbb{R}} \mathbb{C}, \mathrm{d}) \\ \downarrow & & \downarrow \scriptstyle{\text{proj}} \\ \mathcal{O}_M & \longrightarrow & (\mathcal{A}^{0,\bullet}_M, \overline{\partial}) \end{array}$$

For every $\alpha \in \mathrm{H}^q(M,\mathcal{O}_M)$, there exists a $(0,q)$-harmonic form u such that $\alpha = [u]$. In particular, $\mathrm{d}u = 0 \Rightarrow \alpha \in \mathrm{Im}(\mathrm{H}^q(M,\mathbb{C}) \to \mathrm{H}^q(M,\mathcal{O}_M))$ ☐

25.2. **Lefschetz decomposition.** Let M be a compact Kähler manifold, with ω the fundamental form of the Kähler metric. We have two operators

$$L : \mathcal{A}^\bullet_{M,\mathbb{C}} \to \mathcal{A}^{\bullet+2}_{M,\mathbb{C}}$$
$$\eta \mapsto \omega \wedge \eta$$

where L is real and takes (p,q)-forms to $(p+1,q+1)$-forms and $\Lambda = *^{-1}L* : \mathcal{A}^\bullet_{M,\mathbb{C}} \to \mathcal{A}^{\bullet-2}_{M,\mathbb{C}}$ is the adjoint of L.

Proposition 25.2. *We have* $[L,\Lambda] = H$ *where* $H : \mathcal{A}^k_{M,\mathbb{C}} \to \mathcal{A}^k_{M,\mathbb{C}}$ *is* $(k-n)\,\mathrm{id}$ *where* $n = \dim M$.

Remark 25.3. Both L, Λ are linear operators and we only need to check this pointwise. It is enough to check it when we have a complex vector space V with Hermitian metric h and $\omega = -\operatorname{Im}(h)$. Moreover, since these are real operators, it will be enough to consider their effection on $\bigwedge^m V^*$.

Idea:

(1) We'll check this for $\dim_{\mathbb{C}} V = 1$
(2) We'll show that if $V = V' \overset{\perp}{\oplus} V''$ and if we know assertion for V', V'', we get it for V.

26. November 11, 2019

26.1. **Lefschetz decomposition (continued).** This time we discuss Lefschetz decomposition for differential forms. Next time we will discuss for cohomology classes.

Let M be a complex manifold and h a Kähler metric on M with fundamental forms ω. We have two operators

$$\begin{aligned} L : \mathcal{A}^k_{M,\mathbb{C}} &\to \mathcal{A}^{k+2}_{M,\mathbb{C}} \\ L(-) &= \omega \wedge - \\ \Lambda : \mathcal{A}^{k+2}_{M,\mathbb{C}} &\to \mathcal{A}^k_{M,\mathbb{C}} \\ \Lambda &= *^{-1} L * \end{aligned}$$

Proposition 26.1. *If $\alpha \in \mathcal{A}^k_{M,\mathbb{C}}$, then $[L, \Lambda]\alpha = (k-n)\alpha$ where $n = \dim M$.*

Proof. Since both L and Λ are linear operators, it is enough to prove it for a complex vector space V with Hermitian form h. Because both operators are real, we may assume that $\alpha \in \bigwedge^k V^*$. We prove by induction on n.

For the case $n = 1$, let x be a basis of $V/\mathbb{C}$ such that $h(x,x) = 1$. Let $y = Jx$. Write x', y' for the dual basis of $V^* = \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$. Then the volume element is $x' \wedge y' = \omega$. Hence $*(1) = \omega, *(\omega) = 1, *(x') = y'$ and $*(y') = -x'$. So $\Lambda : \mathcal{A}^2 \to \mathcal{A}^0, \omega \mapsto 1$. Therefore

$$\begin{aligned} [L, \Lambda] \quad \mathcal{A}^0 &: 1 \mapsto -1 \\ \mathcal{A}^1 &: x', y' \mapsto 0 \\ \mathcal{A}^2 &: \omega \mapsto \omega \end{aligned}$$

So $[L, \Lambda]$ acts on $\mathcal{A}^*$ as $(k-1)\,\mathrm{id}$.

Now assume that this is true for any smaller n. Write $V = V_1 \oplus V_2$ where V_1, V_2 are orthogonal with respect to h. Let $n_i = \dim V_i$ where $i = 1, 2$. Then $\bigwedge^k V^* = \oplus_{k_1+k_2=k} \left(\bigwedge^{k_1} V_1^* \otimes \bigwedge^{k_2} V_2^* \right)$. In particular, the volumen form in $\bigwedge^{2n} V^*$ is the tensor product of the volume forms in $\bigwedge^{2n_1} V_1^*$ and $\bigwedge^{2n_2} V_2^*$ respectively. We also have

$$\begin{aligned} * : \bigwedge^{k_1} V_1^* \otimes \bigwedge^{k_2} V_2^* &\to \bigwedge^{2n_1-k_1} V_1^* \otimes \bigwedge^{2n_2-k_2} V_2^* \\ \eta_1 \otimes \eta_2 &\mapsto \underbrace{(-1)^{k_2(2n_1-k_1)}}_{(-1)^{k_1k_2}} *(\eta_1) \otimes *(\eta_2). \end{aligned}$$

and $\omega = \omega_1 + \omega_2$, $L(\eta_1 \otimes \eta_2) = L_1(\eta_1) \otimes \eta_2 + \eta_1 \otimes L_2(\eta_2)$. Use the formula for $L, *$, we then have $\Lambda(\eta_1 \otimes \eta_2) = \Lambda_1(\eta_1) \otimes \eta_2 + \eta_1 \otimes \Lambda_2(\eta_2)$. So

$$\begin{aligned}
[L,\Lambda](\eta_1 \otimes \eta_2) =& L\left(\Lambda_1\eta_1 \otimes \eta_2 + \eta_1 \otimes \Lambda_2\eta_2\right) - \Lambda\left(L_1\eta_1 \otimes \eta_2 + \eta_1 \otimes L_2\eta_2\right)\\
=& L_1\Lambda_1\eta_1 \otimes \eta_2 + \cancel{\Lambda_1\eta_1 \otimes L_2\eta_2} + \cancel{L_1\eta_1 \otimes \Lambda_2\eta_2} + \eta_1 \otimes L_2\Lambda_2\eta_2\\
&- \Lambda_1 L_1\eta_1 \otimes \eta_2 \cancel{- L_1\eta_1 \otimes \Lambda_2\eta_2} \cancel{- \Lambda_1\eta_1 \otimes L_2\eta_2} - \eta_1 \otimes \Lambda_2 L_2\eta_2\\
=& [L_1,\Lambda_1]\eta_1 \otimes \eta_2 + \eta_1 \otimes [L_2,\Lambda_2]\eta_2\\
\text{By induction } =& (k_1 - n_1)\eta_1 \otimes \eta_2 + \eta_1 \otimes (k_2 - n_2)\eta_2\\
=& (k-n)\eta_1 \otimes \eta_2
\end{aligned}$$

where η_1 is a k_1-form and η_2 is a k_2-form, and $k_1 + k_2 = k$. □

Corollary 26.2. *We have* $[L^r,\Lambda] = r(k-n+r-1)L^{r-1}$ *on* $\mathcal{A}^k_{M,\mathbb{C}}$.

Proof. The proof is done by induction on $r \geqslant 1$. The case $r = 1$ is done by Proposition 26.1. For larger r, we have

$$\begin{aligned}
[L^r,\Lambda] =& L^r\Lambda - \Lambda L^r\\
=& L\left(L^{r-1}\Lambda - \Lambda L^{r-1}\right) + (L\Lambda - \Lambda L)L^{r-1}\\
=& L\left[L^{r-1},\Lambda\right] + [L,\Lambda]L^{r-1}\\
=& (r-1)(k-n+r-2)L^r + \underbrace{(k+2(r-1)-n)}_{[L,\Lambda]L^{r-1}\eta=(k+2(r-1))L^{r-1}\eta \text{ where } \eta \text{ is a } k\text{-form}} L^r\\
=& r(k-n+r-1)L^r,
\end{aligned}$$

as desired. □

Proposition 26.3. *If* $n = \dim M$, $0 \leqslant k \leqslant n$, *then* $L^{n-k} : \mathcal{A}^k_{M,\mathbb{C}} \to \mathcal{A}^{2n-k}_{M,\mathbb{C}}$ *is an isomorphism. It induces an isomorphism*

$$\begin{array}{ccc}
\mathcal{A}^k_{M,\mathbb{C}} & \xrightarrow{L^{n-k}} & \mathcal{A}^{2n-k}_{M,\mathbb{C}}\\
\uparrow & & \uparrow\\
\mathcal{A}^{p,q}_M & \xrightarrow{\sim} & \mathcal{A}^{p+n-k,q+n-k}_{M,\mathbb{C}}
\end{array}$$

if $p + q = k$.

Proof. L^{n-k} is a morphism of the vector bundles of the same rank. First we check that the bottom sets have the same rank. LHS = $\binom{n}{p}\binom{n}{q}$ and RHS = $\binom{n}{p+n-k}\binom{n}{q+n-k} = \binom{n}{q}\binom{n}{p}$. So both have the same rank.

It is enough to prove injectivity on fibers. In particular, the statement about k-forms reduces to (p,q)-forms. We prove by induction on $k \geqslant 0$.

For $k = 0$, it is OK since ω^n =constant×volume form$\neq 0$.

Fix $k \geqslant 1$, show following by induction on r: If $L^r\alpha = 0$ and α is a k-form where $0 \leqslant r \leqslant n-k$, then $\alpha = 0$.

The case $r = 0$ is trivial. For larger r, suppose $L^r\alpha = 0$. Then on the one hand, we have $[L^r,\Lambda]\alpha = L^r\Lambda\alpha - \Lambda L^r\alpha = L^r\Lambda\alpha$, on the other hand, by Corollary 26.2 $[L^r,\Lambda]\alpha = r(k-n+r-1)L^{r-1}\alpha$. So we have $L^{r-1}(L\Lambda\alpha - r(k-n+r-1)\alpha) = 0$. The induction hypothesis on $r-1$ shows that $L\Lambda\alpha - r(k-n+r-1)\alpha = 0$. Since $k-n+r-1 < 0$, we have $\alpha = L\beta$ for some $(k-2)$-form β.

Then $L^{n-k}\alpha = 0 \Rightarrow L^{n-k+1}\beta = 0$. Again by induction hypothesis on k, we know that $L^{n-(k-2)}$ is injective on $(k-2)$-forms. Hence $\beta = 0 \Rightarrow \alpha = 0$. □

Definition 26.4. Let α be a k-form, and $0 \leqslant k \leqslant n = \dim M$. If $L^{n-k+1}\alpha = 0$, then α is *primitive*.

Note that

(1) Since L is real, α is primitive if and only if $\overline{\alpha}$ is primitive.
(2) L has degree $(1,1)$. α is primitive if and only if all its (p,q)-components are primitive.

Proposition 26.5 (Lefschetz decomposition for forms)**.** *For $0 \leqslant k \leqslant n$, we have a decomposition:*

$$\mathcal{A}^k_{M,\mathbb{C}} = \oplus_{i\geqslant 0} L^i \mathsf{Prim}\left(\mathcal{A}^{k-2i}_{M,\mathbb{C}}\right)$$

where $\mathsf{Prim}\left(\mathcal{A}^k_{M,\mathbb{C}}\right)$ is the set of primitive k-forms.

Note that L^i is injective on $\mathcal{A}^{k-2i}$ since by Proposition 26.3, $L^{n-(k-2i)}$ is injective and $n-k+2i \geqslant i$.

Proof of Proposition 26.5. We prove by induction on k. If $k=0$, then this si clear since $L^{n+1}=0 \Rightarrow$ every 0-form is primitive.

Given k-form α, $\alpha - L\beta$ is primitive for some $(k-2)$-form β if and only if $L^{n-k+1}(\alpha - L\beta) = 0 \Leftrightarrow L^{n-k+1}\alpha = L^{n-k+2}\beta$. Proposition 26.3 shows that there exists a unique $(k-2)$-form β such that this holds, i.e. $\alpha = \alpha_0 + L\beta$ with α_0 primitive. Then we conclude by applying the induction hypothesis to β. □

27. November 13, 2019

27.1. **The hard Lefschetz theorem.** Recall that (M,h) is a compact n-dimensional Kähler manifold and ω is the fundamental form. We have the operator $L = \omega \wedge -$. And we have

- $L^{n-k} : \mathcal{A}^k_{M,\mathbb{C}} \xrightarrow{\sim} \mathcal{A}^{2n-k}_{M,\mathbb{C}}$ for any $k \leqslant n$.
- For $k \leqslant n$, a k-form η is primitive if $L^{n-k+1}\eta = 0$.
- Lefschetz decomposition: for $k \leqslant n$, we have $\mathcal{A}^k_{M,\mathbb{C}} = \oplus_{i\geqslant 0} L^i\left(\mathsf{Prim}(\mathcal{A}^{k-2i}_{M,\mathbb{C}})\right)$

Remark 27.1. If $k > n$, then we have $L^{k-n} : \mathcal{A}^{2n-k}_{M,\mathbb{C}} \xrightarrow{\sim} \mathcal{A}^k_{M,\mathbb{C}}$. Lefschetz decomposition on LHS gives us that $\mathcal{A}^k_{M,\mathbb{C}} = \oplus_{i\geqslant 0} L^{k-n+i}\mathsf{Prim}(\mathcal{A}^{2n-k-2i}_{M,\mathbb{C}})$. Note that L^{k-n+i} is injective on $\mathcal{A}^{2n-k-2i}_{M,\mathbb{C}}$ since $k-n+i \leqslant n-(2n-k-2i) = k-n+2i$.

Lemma 27.2. *A k-form α is primitive if and only if $\Lambda\alpha = 0$.*

Proof. Note that L is injective on $\mathcal{A}^{<n}_{M,\mathbb{C}}$. So $\Lambda = \star^{-1}L\star$ is injective on $\mathcal{A}_{M,\mathbb{C}}$. Hence we may assume that $k \leqslant n$. Then Corollary 26.2 implies that $\left[L^{n-k+1}, \Lambda\right]\alpha = 0 \Rightarrow L^{n-k=1}\Lambda\alpha = \Lambda \underbrace{L^{n-k+1}\alpha}_{(2n-k+2)\text{-form}}$. Since Λ is injective on $\mathcal{A}^{>n}$, the right hand side is zero if and only if α is primitive. By the same reason, $\Lambda\alpha$ is a $(k-2)$-form and L^{n-k+2} is injective on $(k-2)$-forms. Left-hand side is zero if and only if $\Lambda\alpha = 0$. □

For $k \leqslant n$ and a k-form α, write $\alpha = L^i\beta$ where β is a primitive $(k-2i)$-form. Then

$$\Lambda\alpha = \Lambda L^i\beta = \underbrace{L^i\Lambda\beta}_{=0 \text{ by Lemma 27.2}} - \underbrace{\left[L^i, \Lambda\right]\beta}_{i(k-2i-n+i-1)L^{i-1}\beta}$$
$$\Rightarrow \Lambda\left(L^i\beta\right) = i\left(n-k+i+1\right)L^{i-1}\beta$$

For example, $\Lambda\alpha$ is primitive if and only if $\alpha = \alpha_0 + L\alpha_1$ with α_0, α_1 primitive. In this case: $\Lambda\alpha = (n-k+2)\,\alpha_1$.

Lemma 27.3. *We have $[\Delta, L] = 0$.*

Proof. Recall that $\Delta = 2\Delta''$ on a Kähler manifold and $\Delta'' = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. We know that $\left[\overline{\partial}^*, L\right] = \mathrm{i}\partial$.

Since $\mathrm{d}\omega = 0$, we have $\partial\omega = 0, \overline{\partial}\omega = 0$. For any η, we have $\left[\overline{\partial}, L\right]\eta = \overline{\partial}\left(\omega \wedge \eta\right) - \omega \wedge \overline{\partial}\eta = \overline{\partial}\omega \wedge \eta = 0$. So $\left[\overline{\partial}, L\right] = 0 \Leftrightarrow L\overline{\partial} = \overline{\partial}L$. Next we compute

$$\begin{aligned}[\Delta'', L] &= \left(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}\right)L - L\left(\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}\right) \\ &= \underbrace{\overline{\partial}\left(\overline{\partial}^* L - L\overline{\partial}^*\right)}_{\overline{\partial}\left[\overline{\partial}^*, L\right]} + \underbrace{\overline{\partial}L\overline{\partial}^* - L\overline{\partial}\overline{\partial}^*}_{\overline{\partial}L - L\overline{\partial} = 0} + \underbrace{\overline{\partial}^*\overline{\partial}L - L\overline{\partial}^*\overline{\partial}}_{\left[\overline{\partial}^*, L\right]\overline{\partial}}\end{aligned}$$

Here we used $\overline{\partial}L = L\overline{\partial}$. So $[\Delta'', L] = \overline{\partial}\mathrm{i}\partial + \mathrm{i}\partial\overline{\partial} = 0$, as desired. □

Consequences:

(1) α harmonic $\Rightarrow L\alpha$ harmonic. Because $\Delta\alpha = 0 \Rightarrow \Delta L\alpha = L\Delta\alpha = 0$.
(2) If $k \leqslant n$ and $i \leqslant n - k$, then $L^i\alpha$ harmonic $\Rightarrow \alpha$ harmonic. Because $\Delta L^i\alpha = L^i\Delta\alpha = 0$ and that L^i is injective on $\mathcal{A}^k_{M,\mathbb{C}}$ imply that $\Delta\alpha = 0$.

Hence for $k \leqslant n$, we have isomorphisms

$$\begin{array}{ccc} \mathcal{A}^k_{M,\mathbb{C}} & \xrightarrow{L^{n-k}} & \mathcal{A}^{2n-k}_{M,\mathbb{C}} \\ \uparrow & & \uparrow \\ \mathscr{H}^k(M, \mathbb{C}) & \xrightarrow{\sim} & \mathscr{H}^{2n-k}(M, \mathbb{C}) \end{array}$$

So we get

Theorem 27.4 (Hard Lefschetz)**.** *For $k \leqslant n$,*

$$[\omega]^k \cup - : \mathrm{H}^k_{\mathsf{dR}}(M, \mathbb{C}) \to \mathrm{H}^{2n-k}_{\mathsf{dR}}(M, \mathbb{C})$$

is an isomorphism.

Remark 27.5. The fact that the spaces are isomorphic follows from Poincaré duality, but this is a much stronger statement, e.g. it implies for $k \leqslant n - 2$, $b_k(M) \leqslant b_{k+2}(M)$ because $[\omega]^{n-k} \cup -$ injective $\Rightarrow [\omega] \wedge -$ injective.

Definition 27.6. If $k \leqslant n$, $\alpha \in \mathrm{H}^k_{\mathsf{dR}}(M, \mathbb{C})$ is *primitive* if $[\omega]^{n-k+1} \cup \alpha = 0$.

If we choose a harmonic k-form η such that $\alpha = [\eta]$. Then $[\omega]^{n-k+1} \cup \alpha = [\underbrace{L^{n-k+1}\eta}_{\text{harmonic}}]$. Hence α is primitive if and only if η is primitive.

Lemma 27.7. *Let $k \leqslant n$ and α be a harmonic k-form with Lefschetz decomposition $\alpha = \sum_{i \geqslant 0} L^i\alpha_i$ where α_i are primitive. Then all α_i are harmonic.*

Proof. Argue by induction on k. The case $k = 0$ is trivial.

We saw that $[\Delta, L] = 0 \Rightarrow [\Delta, \Lambda] = 0$. So if α is harmonic, then $\Delta\alpha = 0 \Rightarrow \Lambda\alpha$ is also harmonic.

Since

$$\Lambda\alpha = \sum_{i \geqslant 1} \underbrace{i(n - k + i + 1)}_{>0} L^{i-1}\alpha_i$$

By induction, we have each α_i is harmonic for $i \geqslant 1$. So $L^i\alpha_i$ are harmonic for $i \geqslant 1$. Hence α_0 is also harmonic. □

Theorem 27.8 (Lefschetz decomposition for cohomology)**.** *If $k \leqslant n$, we have a decomposition*

$$\mathrm{H}^k_{\mathsf{dR}}(M, \mathbb{C}) = \oplus_{i \geqslant 0} L^i \mathrm{H}^{k-2i}_{\mathsf{Prim}}(M, \mathbb{C})$$

where we still denote by L the map on cohomology given by $[\omega] \cup -$ and $\mathrm{H}^j_{\mathsf{Prim}}(M,\mathbb{C}) \subseteq \mathrm{H}^j_{\mathsf{dR}}(M,\mathbb{C})$ is the subset of primitive cohomology classes.

Remark 27.9. Since ω is a real $(1,1)$-form, the Lefschetz decomposition is compatible with real structure and the decomposition into (p,q)-subspaces.

28. November 15, 2019

28.1. **The Hodge-Riemann bilinear relations.** Note that the primitive part of $\mathrm{H}^k(M)$ is the piece accounted for by lower degree cohomology, $\dim \mathrm{H}^k_{\mathsf{Prim}}(M) = b_k(M) - b_{k-2}(M)$.

We want to look at the "sign" of some natural bilinear pairings on cohomology. Let (M,h) be a compact complex manifold of dimension n with Kähler structure and fundamental form ω. Given two k-forms α,β with $k \leqslant n$, may consider $\int_M \omega^{n-k} \wedge \alpha \wedge \beta$. The goal is to understand positive properties of the pairing.

Definition 28.1. Given $d \leqslant n$, define for $\alpha,\beta \in \Gamma(M, \mathcal{A}^k_{M,\mathbb{C}})$

$$H_k(\alpha,\beta) = \mathrm{i}^k \int_M \omega^{n-k} \wedge \alpha \wedge \overline{\beta}$$

- clearly bilinear in α
- Hermitian form: $\underbrace{H_k(\beta,\alpha)}_{\mathrm{i}^k \int_M \omega^{n-k}\wedge\beta\wedge\overline{\alpha}} = \underbrace{\overline{H_k(\alpha,\beta)}}_{(-\mathrm{i})^k \int_M \omega^{n-k}\wedge\overline{\alpha}\wedge\beta}$.

Remark 28.2. This induces a Hermitian form on $\mathrm{H}^k_{\mathsf{dR}}(M,\mathbb{C})$, also denoted by H_k. Need to check: if $\mathrm{d}\alpha = 0, \mathrm{d}\beta = 0$, and one of α or β exact, then

$$\int_M \omega^{n-k} \wedge \alpha \wedge \overline{\beta} = 0$$

OK by [See photo]

Remark 28.3. The Lefschetz decomposition of $\mathrm{H}^k_{\mathsf{dR}}(M,\mathbb{C})$ is an orthogonal decomposition with respect to H_k.

Proof. We need to show that if $\alpha = L^i\gamma, \beta = L^j\delta, i < j$, γ,δ primitive forms. then $H_k(\alpha,\beta) = 0$. (assume that γ,δ are harmonic representatives).

$$\omega^{n-k} \wedge L^i\gamma \wedge L^j\overline{\delta} = L^{n-k+i+j}\gamma \wedge \overline{\delta}$$

Since δ is a primitive $(k-2i)$-form, $L^{n-(k-2i)+1}\gamma = 0$. Since $j \geqslant i+1 \Rightarrow n-k+i+j \geqslant n-(k-2i)+1 \Rightarrow \omega^{n-k} \wedge \alpha \wedge \overline{\beta} = 0$ ☐

Remark 28.4.

$$\mathrm{H}^{k-2j}_{\mathsf{Prim}}(M,\mathbb{C}) \xrightarrow{L^j} \mathrm{H}^k_{\mathsf{dR}}(M,\mathbb{C})$$

On $\mathrm{H}^k_{\mathsf{dR}}(M,\mathbb{C})$, we have $H_k(L^j\alpha, L^j\beta) = \mathrm{i}^k \int_M \omega^{n-k+2j} \wedge\alpha\wedge\overline{\beta}$ and on H^{k-2j} we have $H_{k-2j}(\alpha,\beta) = \mathrm{i}^{k-2j} \int_M \omega^{n-k+2j} \wedge \alpha \wedge \overline{\beta}$. They agree up to a $(-1)^j$ factor. Hence to understand the "sign" of all H_k on $\mathrm{H}^k(M,\mathbb{C})$, it is enough to understand them on the primitive cohomology.

Remark 28.5. The Hodge decomposition $\mathrm{H}^k_{\mathsf{dR}}(M,\mathbb{C}) = \oplus_{p+q=k}\mathrm{H}^{p,q}(M)$ is orthogonal with respect to H_k.

Proof. If α is a (p,q)-form and β is a (p',q')-form. In order to get a nonzero integral, need $\omega^{n-k} \wedge \alpha \wedge \overline{\beta}$ has type (n,n). Since $k = p+q = p'+q'$.

$$\begin{cases} n-k+p+q' = n \\ n-k+q+p' = n \end{cases} \Rightarrow \begin{cases} p = p' \\ q = q' \end{cases}$$

☐

Theorem 28.6 (Hodge-Riemann bilinear relation)**.** *Given $k \leqslant n$ and (p,q) with $p+q=k$. The Hermitian form $(-1)^{\frac{k(k-1)}{2}} \mathrm{i}^{p-q-k} H_k$ is positive-definite on $\mathrm{H}^{p,q}_{\mathsf{Prim}}(M)$.*

The key ingredient is a formula for $*\alpha$ when α is primitive.

More generally,

Proposition 28.7. *If α is a primitive (p,q)-form and $k = p+q$, then*

$$*L^j \alpha = (-1)^{\frac{k(k+1)}{2}} \mathrm{i}^{p-q} \frac{j!}{(n-k-j)!} L^{n-k-j} \alpha$$

for $j \leqslant n-k$. In particular, $*\alpha = (-1)^{\frac{k(k+1)}{2}} \frac{\mathrm{i}^{p-q}}{(n-k)!} L^{n-k} \alpha$.

Proof of Theorem 28.6. Let α be a (p,q) primitive form. Then

$$\begin{aligned}
(-1)^{\frac{k(k-1)}{2}} \mathrm{i}^{p-q-k} H_k(\alpha,\alpha) &= (-1)^{\frac{k(k-1)}{2}} \mathrm{i}^{p-q} \int_M \underbrace{\omega^{n-k} \wedge \alpha}_{L^{n-k}\alpha} \wedge \overline{\alpha} \\
&= (n-k)!(-1)^k \int_M (*\alpha) \wedge \overline{\alpha} \\
&= (n-k)! \int_M \overline{\alpha} \wedge \overline{(*\overline{\alpha})} \\
&= (n-k)! \int_M \langle \overline{\alpha}, \overline{\alpha} \rangle \,\mathsf{dvol} \\
&= (n-k)! \langle\!\langle \overline{\alpha}, \overline{\alpha} \rangle\!\rangle > 0
\end{aligned}$$

unless $\alpha = 0$. □

Sketch Proof of Proposition 28.7. This is a linear algebra statement. So we can just consider complex vector space V of dimension n, with Hermitian metric h. Prove by induction on n.

The base case $n = 1$: V has a basis x over $\mathbb{C}$ such that $h(x,x) = 1$, let $y = Jx$. Then $V^*_{\mathbb{C}}$ has dual basis x', y', and $\omega = x' \wedge y'$. We know that $*1 = \omega$, $*\omega = 1$, $*x' = y'$ and $*y' = -x'$.

$$\begin{aligned}
k = 0, j = 0 : &\quad \alpha = 1, *\alpha = \omega, \mathrm{RHS} = L\alpha = \omega \\
k = 0, j = 1 : &\quad \alpha = 1, *L\alpha = 1, \mathrm{RHS} = L^0\alpha = 1 \\
k = 1, j = 0 : &\quad (p,q) = (1,0), \alpha = x' + \mathrm{i}y' \Rightarrow *\alpha = y' - \mathrm{i}x' = -\mathrm{i}\alpha, \mathrm{RHS} = (-1)\mathrm{i}\alpha
\end{aligned}$$

and the case $(p,q) = (0,1)$ is similiar.

Inductive step: suppose $V = V_1 \oplus V_2$ where V_2 is of dimension 1 with x, y as before.

$$(\bigwedge^k V^*)_{\mathbb{C}} = (\bigwedge^k V_1^*)_{\mathbb{C}} \oplus (\bigwedge^{k-1} V_1^* \otimes_{\mathbb{R}} V_2^*)_{\mathbb{C}} \oplus (\bigwedge^{k-2} V_2^* \otimes \bigwedge^2 V_2^*)_{\mathbb{C}}$$

So

$$\alpha = \alpha_k + \alpha'_{k-1} \otimes x' + \alpha''_{k-1} \otimes y' + \alpha_{k-2} \otimes \omega$$

Recall that $\Lambda(\eta_1 \otimes \eta_2) = \Lambda_1 \eta_1 \otimes \eta_2 + \eta_1 \otimes \Lambda_2 \eta_2$. Then

$$\Lambda\alpha = \Lambda_1\alpha_k + \Lambda_1\alpha'_{k-1} \otimes x' + \Lambda_1\alpha''_{k-1} \otimes y' + \Lambda_1\alpha_{k-2} \otimes \omega + \alpha_{k-2}$$

α is primitive iff $\Lambda\alpha = 0$ iff

$$\begin{aligned}
&\Lambda_1\alpha'_{k-1} = 0, \Lambda_1\alpha''_{k-1} = 0 \Leftrightarrow \alpha'_{k-1}, \alpha''_{k-1} \text{ is primitive} \\
&\Lambda_1\alpha_{k-2} = 0, \Lambda_1\alpha_k = -\alpha_{k-2} \Leftrightarrow \alpha_k = \gamma_k + L\gamma_{k-2} \text{ with } \gamma_k, \gamma_{k-2} \text{ primitive}, \alpha_{k-2} = -(n-k+1)\gamma_{k-2}
\end{aligned}$$

The equivalence on the second line uses the formula for Λ with respect to the Lefschetz decomposition. The compute $*L^j\alpha$ using induction and the formula for $*, L$ with respect to $*_1, L_1, *_2, L_2$. □

Kähler package: Poincaré duality, hard Lefschetz, Hodge-Riemann bilinear relation. For example, intersection cohomology (for projective singular algebraic variety) also satisfies these conditions.

29. November 18, 2019

Recall that (M,h) is a compact Kähler manifold with fundamental form ω. The Hodge-Riemann bilinear relation is $(-1)^{\frac{k(k-1)}{2}} \mathrm{i}^{p-q-k} H_k$ on $\mathrm{H}^{p,q}_{\mathsf{Prim}}(M)$. If $\alpha \in \mathrm{H}^{p,q}_{\mathsf{Prim}}(M)$, $k=p+q$, then $(-1)^{\frac{k(k-1)}{2}} \mathrm{i}^{p-q} \int\limits_M \omega^{n-k} \wedge \alpha \wedge \overline{\alpha} > 0$.

Theorem 29.1 (Hodge index theorem)**.** *If* $\dim M = n = 2m$ *and on* $\mathrm{H}^n(M,\mathbb{R})$ *the form* $(\alpha,\beta) \mapsto \int\limits_M \alpha \wedge \beta$ *has signature* $\sum_{p,q=0}^n (-1)^p h^{p,q}$.

Note that the signature is a topological invariant. So this alternating sum of Hodge numbers is also a topological invariant.

Proof. Consider the Lefschetz Hodge decomposition:

$$\mathrm{H}^n(M,\mathbb{R}) = \oplus_{j\geq 0} L^j \mathrm{H}^{m-j,m-j}_{\mathsf{Prim}}(M)_{\mathbb{R}} \oplus \oplus_{p>q,p+q=2m} \oplus_{j\geqslant 0} L^j \left(\mathrm{H}^{p-j,q-j}_{\mathsf{Prim}}(M) \oplus \mathrm{H}^{q,p}_{\mathsf{Prim}}(M)\right)_{\mathbb{R}}$$

If $\alpha \in \mathrm{H}^{m-j,m-j}_{\mathsf{Prim}}(M,\mathbb{R})$, then Hodge-Riemann (Theorem 28.6) tells us that $(-1)^{m-j} \int\limits_M \omega^{n-j} \wedge \alpha \wedge \alpha > 0$.

If $\alpha \in \mathrm{H}^{p-j,q-j}_{\mathsf{Prim}}(M,\mathbb{C})$ where $p+q=2m$, then Hodge-Riemann (Theorem 28.6) tells us that $\underbrace{(-1)^{m-j}\mathrm{i}^{p-q}}_{(-1)^{q-j}} \int\limits_M \omega^{n-j} \wedge \alpha \wedge \overline{\alpha} > 0$.

If $\beta \in \mathrm{H}^{q-j,p-j}_{\mathsf{Prim}}(M,\mathbb{C})$, then $\underbrace{(-1)^{p-j}}_{(-1)^{q-j}} \int\limits_M \omega^{n-j} \wedge \beta \wedge \overline{\beta} > 0$.

$$\alpha+\beta \in \left(\mathrm{H}^{p,q}_{\mathsf{Prim}} \oplus \mathrm{H}^{q,p}_{\mathsf{Prim}}\right)_{\mathbb{R}} \Leftrightarrow \alpha+\beta = \overline{\beta}+\overline{\alpha} \Leftrightarrow \beta = \overline{\alpha}$$

So

$$\begin{aligned}(-1)^{q-j} \int\limits_M \omega^{n-j} \wedge (\alpha+\beta) \wedge (\alpha+\beta) &= (-1)^{q-j} \int\limits_M \omega^{n-j} \wedge (\alpha\wedge\beta + \beta\wedge\alpha + \alpha\wedge\alpha + \beta\wedge\beta) \\ &= (-1)^{q-j} \int\limits_M \omega^{n-j} \wedge (\alpha\wedge\overline{\alpha} + \beta\wedge\overline{\beta}) > 0\end{aligned}$$

So the signature of Q is given by

$$\begin{aligned}\text{signature}(Q) &= \sum_{j\geqslant 0} (-1)^{m-j} h^{m-j,m-j}_{\mathsf{Prim}} + 2 \sum_{p+q=n,p>q} \sum_{j\geqslant 0} (-1)^{q-j} h^{p-j,q-j}_{\mathsf{Prim}} \\ &= \sum_{p+q=n} \sum_{j\geqslant 0} (-1)^{q-j} h^{p-j,q-j}_{\mathsf{Prim}} \\ &= \sum_{p+q=n} \sum_{j\geqslant 0} (-1)^{q-j} (h^{p-j,q-j} - h^{p-j-1,q-j-1}) \\ &= \sum_{a+b=n-\text{nonzero even}} (-1)^b h^{a,b} + \sum_{a+b=n-2-\text{nonzero even}} (-1)^b h^{a,b}\end{aligned}$$

Since $h^{q,b} = h^{n-a,n-b}$, and $a+b = n-2-\text{nonzero even} \Leftrightarrow (n-a)+(n-b) = n+2+\text{nonzero even}$. We have

$$\text{signature}(Q) = \sum_{a+b \text{ even}} (-1)^b h^{a.b}$$

Finally, $\sum_{a+b \text{ odd}} (-1)^b h^{a,b} = 0$ since $h^{a,b} = h^{b,a}$ and $(-1)^a = -(-1)^b$ if $a+b$ is odd. □

Suppose that $n = 2$. Then the signature of

$$\begin{aligned} \mathrm{H}^2(M,\mathbb{R}) \times \mathrm{H}^2(M,\mathbb{R}) &\to \mathbb{R} \\ (\alpha,\beta) &\mapsto \int_M \alpha\wedge\beta \end{aligned}$$

is $\sum_{a+b \text{ even}} (-1)^b h^{a,b} = h^{0,0} + h^{0,2} - h^{1,1} + h^{2,0} + h^{2,2} = 2 + 2h^{2,0} - h^{1,1}$.

Claim is that on $\left(\mathrm{H}^{2,0}(M) \oplus \mathrm{H}^{0,2}(M)\right)_{\mathbb{R}}$, the form $(\alpha,\beta) \mapsto \int \alpha\wedge\beta$ is positive-definite by Hodge-Riemann Theorem 28.6 (See computation in proof the theorem).

Conclusion is that the signature of the integral form on $\mathrm{H}^{1,1}(M)_{\mathbb{R}}$ is $(1, h^{1,1} - 1)$.

29.1. **Chern classes of line bundles.** Let M be a complex manifold (more generally, analytic space). We have $\mathcal{O}_M$ the sheaf of holomorphic functions and $\mathcal{O}_M^*$ the sheaf of invertible holomorphic functions with multiplication. We also have

$$\begin{aligned} \mathcal{O}_M &\to \mathcal{O}_M^* \\ f &\mapsto \exp(f) \end{aligned}$$

morphisms of abelian groups.

Locally on $\mathbb{C}^*$, we have a holomorphic inverse of exp. So $\mathcal{O}_M \to \mathcal{O}_M^*$ is surjection. If $U \subseteq M$ is open and connected, then $f \in \mathcal{O}(U)$ is such that $\exp(f) = 1$, then $f = 2\pi i k$ for some $k \in \mathbb{Z}$. So we have a short exact sequence

$$0 \to \mathbb{Z} \overset{n\mapsto 2\pi i n}{\longrightarrow} \mathcal{O}_M \overset{f\mapsto \exp(f)}{\longrightarrow} \mathcal{O}_M^* \to 0$$

We have long exact sequence in cohomology:

$$\mathrm{H}^1(M,\mathbb{Z}) \to \mathrm{H}^1(M,\mathcal{O}_M) \to \mathrm{H}^1(M,\mathcal{O}_M^*) \to \mathrm{H}^2(M,\mathbb{Z}) \to \mathrm{H}^2(M,\mathcal{O}_M) \to \cdots$$

The *Picard group* $\mathrm{Pic}(M)$ is the group generated by all holomorphic line bundles under tensor product modulo isomorphisms. So we have isomorphisms $\mathrm{Pic}(M) \cong \check{\mathrm{H}}^1(M,\mathcal{O}_M^*)$.

On any topological space, for any sheaf $\mathscr{F}$ of abelian groups, there exists a canonical isomorphism $\mathrm{H}^1(M,\mathscr{F}) \to \check{\mathrm{H}}^1(M,\mathscr{F})$.

Definition 29.2. The *Chern class* of L is defined to be the image of L under the map $c_1 : \mathrm{Pic}(M) \to \mathrm{H}^2(M,\mathbb{Z})$.

Exercise 29.3. Check functoriality of Chern classes. If $f : M' \to M$ is holomorphic, and $L \in \mathrm{Pic}(M)$, then $c_1(f^*L) = f^*(c_1(L))$.

Next time, given L, if we choose metric on L, then we get a closed, real $(1,1)$-form ω on X such that $c_1(L) = [\omega]$. If X is algebraic, L ample, then ω is positive.

30. November 20, 2019

30.1. **Chern classes of line bundles (continued).** Let M be a complex manifold, L an line bundle on M. Given a cover $M = \cup_j U_j$, we have isomorphisms $\varphi_j : L|_{U_j} \xrightarrow{\sim} \mathcal{O}_{U_j}$ such that $1 \in \mathcal{O}_{U_j}$ maps to $\sigma_j \in \Gamma(U_j, L)$, and $g_{jk} \in \mathcal{O}(U_j \cap U_k)^*$ such that $\varphi_k \circ \varphi_j^{-1}$ is multiplication by g_{jk}, and $\sigma_j = \sigma_k g_{jk}$ on $U_j \cap U_k$ for any j,k. A global section of L is the the collections of $f_j \in \mathcal{O}(U_j)$ such that $f_k = f_j g_{jk}$.

A Hermitian metric on L is a smoothly varying family of Hermitian metrics on the fibers of L. Given a cover as above, put $h_j = h(\sigma_j,\sigma_j) : U_j \to \mathbb{R}_{\geqslant 0}$. This is supposed to satisfy $h_j = |g_{jk}|^2 h_k$. And conversely, a family of such functions with this compatibility gives a Hermitian metric on L.

Suppose that h is a metric on L and given a cover as above. On each U_j, consider $\omega_j = \frac{1}{2\pi i}\partial\overline{\partial}\log h_j$ a $(1,1)$-form on U_j. On $U_j \cap U_k$, we have $\log h_j = \log|g_{jk}|^2 + \log h_k$. Since $\partial\overline{\partial}\log|g_{jk}|^2 = \partial\overline{\partial}(\log g_{jk} + \log \overline{g}_{jk}) = 0$, we know that all ω_j glue together to a global $(1,1)$-form ω.

Note that

(1) Since $\omega_j = \partial\overline{\partial}(\cdots)$, $\mathrm{d}\omega_j = 0$. Hence ω is closed.
(2) $\mathrm{i}\partial\overline{\partial}$ is a real operator and each h_j is a real form. So ω is a real form
(3) ω is Kähler (foundamental form of a Kähler metric) $\Leftrightarrow -\mathrm{i}\omega(v,v) > 0$ for any $v \neq 0$.

Proposition 30.1. *Given a Hermitian metric h on L with corresponding form ω_h, the image of $\mathrm{c}^1(L)$ in $\mathrm{H}^2(M,\mathbb{R})$ is $[\omega_h]$.*

Proof. Since M is a smooth manifold, there exists an open cover $M = \cup_j U_j$ such that all intersections $U_{j_0} \cap \cdots \cap U_{j_k}$ are contractible. Moreover, we can choose this as fine as needed. So we may assume that $L|_{U_j} \simeq \mathcal{O}_{U_j}$ for any j.

$$0 \to \mathbb{Z} \to \mathcal{O}_M \to \mathcal{O}_M^* \to 0$$

So $[L] \in \mathrm{Pic}(M)$ corresponds to $(g_{jk}) \in \mathrm{H}^1(M, \mathcal{O}_M^*)$. Since all U_j are contractible, we have $\mathrm{H}^p(U_j,\mathbb{Z}) = 0$ for any $p > 0$. Hence $g_{jk} = \exp(\widetilde{g}_{jk})$ for some holomorphic $\widetilde{g}_{jk}$ on $U_j \cap U_k$.

So $\frac{1}{2\pi\mathrm{i}}(\widetilde{g}_{kl} - \widetilde{g}_{jl} + \widetilde{g}_{jk})$ on U_{jkl} is an integer, denote as n_{jkl}. We have $n_{jkl} \in \check{\mathrm{H}}^2(\mathcal{U};\mathbb{Z})$.

We need to understand the isomorphism $\check{\mathrm{H}}^2(\mathcal{U},\mathbb{R}) \cong \mathrm{H}^2_{\mathsf{dR}}(M)$. Consider the double complex $C^{\bullet,\bullet}$ where

$$C^{p,q} = \oplus_{i_0<\cdots<i_q} \Gamma(U_{i_0} \cap \cdots \cap U_{i_q}; \mathcal{A}^p_M)$$

If we fix $(i_0,\cdots,i_q)$, then the corresponding de Rham complex of $U_{i_0} \cap \cdots \cap U_{i_q}$ is acyclic, with cohomology in degree 0 being $\Gamma(U_{i_0} \cap \cdots \cap U_{i_q}, \mathbb{R})$. So we get isomorphism $\check{\mathrm{H}}^p(\mathcal{U},\mathbb{R}) \cong \mathrm{H}^p(\mathrm{Tot}(C^{\bullet,\bullet}))$.

Similarily, if we fix p, then we get Čech complex with respect to $\mathcal{U}$ for $\mathcal{A}^p$. Since $\mathcal{A}^p$ is soft, this is again acyclic, with cohomology in degree 0 being $\Gamma(M,\mathcal{A}^p)$. Hence we get isomorphisms $\mathrm{H}^p_{\mathsf{dR}}(M) \cong \mathrm{H}^p(\mathrm{Tot}(C^{\bullet,\bullet}))$.

$$\begin{array}{ccccc}
\oplus_j \Gamma(U_j,\mathcal{A}^0) & \longrightarrow & \oplus_j \Gamma(U_j,\mathcal{A}^1) & \longrightarrow & \oplus_j \Gamma(U_j,\mathcal{A}^2) \\
\downarrow & & \downarrow & & \downarrow \\
\oplus_{j<k} \Gamma(U_j \cap U_k,\mathcal{A}^0) & \longrightarrow & \oplus_{j<k} \Gamma(U_j \cap U_k,\mathcal{A}^1) & \longrightarrow & \oplus_{j<k} \Gamma(U_j \cap U_k,\mathcal{A}^2) \\
\downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots
\end{array}$$

We have $\omega_j \in \Gamma(U_j,\mathcal{A}^2)$ and $\omega_j = \mathrm{d}\beta_j$ where $\beta_j = \frac{1}{2\pi\mathrm{i}}\overline{\partial}\log h_j \in \Gamma(U_j,\mathcal{A}^1)$. We have

$$\begin{aligned}
\beta_k - \beta_j &= \frac{1}{2\pi\mathrm{i}}\overline{\partial}(\log h_k - \log h_j) = -\frac{1}{2\pi\mathrm{i}}\overline{\partial}\left(\widetilde{g}_{jk} + \overline{\widetilde{g}_{jk}}\right) \\
&= -\frac{1}{2\pi\mathrm{i}}\overline{\partial\widetilde{g}_{jk}} = \partial\overline{\frac{1}{2\pi\mathrm{i}}\widetilde{g}_{jk}} \\
&= \mathrm{d}\left(\overline{\frac{1}{2\pi\mathrm{i}}\widetilde{g}_{jk}}\right)
\end{aligned}$$

since $g_{jk} = \exp(\widetilde{g}_{jk})$. Sign error?. So $[\omega] \in \mathrm{H}^2_{\mathsf{dR}}(M)$ maps to $\frac{1}{2\pi\mathrm{i}}\left(\overline{\widetilde{g}}_{kl} - \overline{\widetilde{g}}_{jl} + \overline{\widetilde{g}}_{jk}\right) = \overline{n}_{jkl} = n_{jkl}$. □

Recall that on $\mathbb{P}^n$ we have the Fubini-Study metric with corresponding form ω_{FS} given by following construction. Let $z_0,\cdots,z_n$ be homogeneous coordinates, and U_j is the set where $z_j \neq 0$. Then

$$\omega_{\mathrm{FS}}|_{U_j} = \frac{2\pi}{\mathrm{i}}\partial\overline{\partial}\log\sum_{k=0}^{n}\left|\frac{z_k}{z_j}\right|^2$$

Claim. *There is a metric h on $\mathcal{O}(1)$ such that $\omega_h = \omega_{\mathrm{FS}}$.*

This is because we can choose $h_j = \left(\sum_k |\frac{z_k}{z_j}|^2\right)^{-1}$. Recall that $g_{jk} = \frac{z_j}{z_k}$, then $h_j = |g_{jk}|^2 h_k$. So we win.

On $\mathbb{P}^n$ we have a metric h on $\mathcal{O}(1)$ such that ω_h is positive.

Remark 30.2. We have following

(1) If $f : M' \to M$ is holomorphic and L is a line bundle on M with Hermitian metric h, then we get Hermitian metric h' on $L' = f^*L$ such that $\omega_{h'} = f^*\omega_h$.
(2) If X is a smooth projective variety with ample line bundle L, then there exists $N > 0$ and an embedding $X \overset{i}{\hookrightarrow} \mathbb{P}^m$ such that $L^N \cong i^*\mathcal{O}(1)$. So the metric we had on $\mathcal{O}_{\mathbb{P}^m}(1)$ induces a metric on L^N such that the corresponding form is Kähler. By replace (h_j) with $(h_j^{1/N})$, we get a metric on L with positive $(1,1)$-form.

So we can apply HL or Lefschetz for $c_1(L)$ when L is an ample line bundle on a smooth projective algebraic variety.

30.2. **Kodaira's embedding theorem.** If X be a compact Kähler manifold whose foundamental class is the Chern class of a line bundle $L \in \mathrm{Pic}(M)$, then X is algebraic and L is ample.

31. November 22, 2019

31.1. **Kodaira's embedding theorem (continued).**

Remark 31.1 (Lefschetz theorem on $(1,1)$-classes). We have the expontential sequence
$$0 \to \mathbb{Z} \to \mathcal{O}_M \xrightarrow{\exp} \mathcal{O}_M^* \to 0$$
where M is the complex manifold. The Chern class is the map $c_1 : \mathrm{Pic}(M) = \mathrm{H}^1(M, \mathcal{O}_M^*) \to \mathrm{H}^2(M, \mathbb{Z})$.

Suppose that M is compact and Kähler. If $L \in \mathrm{Pic}(M)$, then the image of $c_1(L)$ in $\mathrm{H}^2(M,\mathbb{C})$ lies in $\mathrm{H}^{1,1}(M)$.

Conversely, if $\alpha \in \mathrm{H}^2(M,\mathbb{Z})$ whose image in $\mathrm{H}^2(M,\mathbb{C})$ lies in $\mathrm{H}^{1,1}(M)$, then there exists $L \in \mathrm{Pic}(M)$ such that $\alpha = c_1(L)$.

It is enough to show that the image of α in $\mathrm{H}^2(M,\mathcal{O}_M)$ is zero. We have the map induced by $\mathbb{C} \to \mathcal{O}_M$ from $\mathrm{H}^2(M,\mathbb{C})$ to $\mathrm{H}^2(M,\mathcal{O}_M) = \mathrm{H}^{0,2}(M)$ which is the projection with respect to Hodge decomposition. So the image of α is zero.

Example 31.2 (Example 1 of Hodge decomposition)**.** Let V be a complex vector space of dimension n. Let $L \subseteq V$ be a lattice, i.e. $L \cong \mathbb{Z}^{2n}$ such that $L \otimes_{\mathbb{Z}} \mathbb{R} \cong V$. Then $M = V/L$ is a compact Kähler manifold of dimension n.

The projection map $V \overset{\pi}{\to} M$ is a covering map and V is simply connected. So $\pi_1(M) \cong L$. Hence $\mathrm{H}_1(M,\mathbb{Z}) \cong L \Rightarrow \mathrm{H}^1(M,\mathbb{Z}) = L^* = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$. We have canonical map $\overset{\bullet}{\bigwedge}\mathrm{H}^1(M,\mathbb{Z}) \overset{\cup}{\to} \mathrm{H}^*(M,\mathbb{Z})$. This is an isomorphism by Künneth since M is homeomorphic to $(S^1)^n$.

We have
$$\mathrm{H}^1(M,\mathbb{R}) = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z}) \otimes \mathbb{R} = V_{\mathbb{R}}^* = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R})$$
$$\mathrm{H}^1(M,\mathbb{C}) = V_{\mathbb{R}}^* \otimes_{\mathbb{R}} \mathbb{C} = V^* \oplus \overline{V^*}$$
By the group action on M, we have $\mathscr{T}_M = \mathrm{T}_0 M \otimes_{\mathbb{R}} \mathscr{C}_M^\infty = V \otimes_{\mathbb{R}} \mathscr{C}_M^\infty$. So
$$\mathcal{A}_M^m \cong \bigwedge^m V^* \otimes_{\mathbb{R}} \mathscr{C}_M^\infty$$
$$\mathcal{A}_M^{p,q} \cong \bigwedge^p V^* \otimes_{\mathbb{C}} \bigwedge^q \overline{V^*} \otimes_{\mathbb{C}} \mathscr{C}_{M,\mathbb{C}}^\infty$$
Choose an isomorphism $V \cong \mathbb{C}^n$ and the standard metric on $\mathbb{C}^n$, which is translation-invariant under the group action, we get a metric on M.

Hodge decomposition for H^1: Clearly we have $V^* \subseteq \mathrm{H}^{1,0} \subseteq \Gamma(M, \mathcal{A}_M^{1,0})$. Since $\dim \mathrm{H}^{1,0} = \dim \mathrm{H}^0(M,\Omega_M) = \dim_{\mathbb{C}} V^*$. We have $V^* = \mathrm{H}^{1,0}$ and $\overline{V^*} = \mathrm{H}^{0,1}$. So the Hodge decomposition for $\mathrm{H}^m(M,\mathbb{Z})$ has $\mathrm{H}^{p,q} = \bigwedge^p V^* \otimes \bigwedge^q \overline{V^*}$. The containment $\supseteq$ follows from the one for H^1 and in fact we have equality.

Remark 31.3. If L is general and $n \geqslant 2$, then M is not algebraic.

Sketch of Proof. Let z_1, z_2 be coordinates on $\mathbb{C}^2$ and ω be the form induced by $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ on M. Then ω is a holomorphic 2-form. If C is a smooth complex algebraic curve and $f : C \to M$ is a non-constant holomorphic map, then $f^*(\omega) = 0$ since $\dim C = 1$.

Equivalently, since $f_*([C]) \in \mathrm{H}_2(M, \mathbb{Z})$ where $[C] \in \mathrm{H}_2(C)$, we have

$$[\omega] \cap f_*([C]) = 0 \Rightarrow f_*(f^*[\omega] \cap [C]) = f_*(\underbrace{[f^*\omega]}_{=0}) \tag{31.1}$$

There is a general fact. If M is algebraic, then $f_*([C]) \neq 0$.

If we write it in terms of the basis of $\mathrm{H}_2(M, \mathbb{Z}) = \bigwedge^2 L$ and if L is described by a matrix $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$.

Using the fact that $[\omega] \cap$ generators of $\mathrm{H}_2(M, \mathbb{Z}) \cong \bigwedge^2 L$ is given by the 2×2 minors of the matrix, we have nontrivial relations with $\mathbb{C}$-coefficients between these 6 minors. However, for $a_1, \ldots, a_n, b_1, \ldots, b_n$ general in $\mathbb{C}$, these 6 minors are linearly independent. Hence M is not algebraic. ☐

31.2. **Two theorems in algebraic topology.** Recall that $\pi : M_1 \to M_2$ is locally trivial with fiber F if there exists an open cover $M_2 = \cup_i U_i$ such that

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\sim} & U_i \times F \\ & \searrow \quad \swarrow & \\ & U_i & \end{array}$$

In our setup, the cover will always be finite.

Theorem 31.4 (Leray-Hirsch)**.** *Suppose $\pi : X \to Y$ is locally trivial with fiber F. Suppose all $\mathrm{H}^i(F, \mathbb{Z})$ are free and finitely generated, and we have $\alpha_1, \ldots, \alpha_n \in \mathrm{H}^*(X, \mathbb{Z})$ such that $\forall y \in Y$, their restriction to $\mathrm{H}^*(\pi^{-1}(y), \mathbb{Z})$ is a basis. Then we have isomorphism*

$$\mathrm{H}^*(Y) \times \mathbb{Z}^n \to \mathrm{H}^*(X)$$
$$(\beta, (m_1, \ldots, m_n)) \mapsto \pi^*(\beta) \cup (m_1\alpha_1 + \cdots + m_n\alpha_n)$$

Proof. If $X = Y \times F$, then this follows from Künneth. If we have a *finite* trivialization cover, argue by induction on the number of sets in the cover using Mayer-Vietoris. ☐

Next, suppose $E \xrightarrow{\pi} X$ is oriented, real vector bundle of rank r. Write $E_x = \pi^{-1}(x)$. Then an orientation is a compatible system of generators for each $\mathrm{H}^r(E_x, E \smallsetminus \{0\}, \mathbb{Z}) \cong \mathbb{Z}$.

Theorem 31.5. *We have*

(1) We can view $X \to E$ as a submanifold by the 0-section map. There exists a unique cohomology class $\eta_E \in \mathrm{H}^r(E, E \smallsetminus X, \mathbb{Z})$ such that for any $x \in X$, the restriction of η_E to $\mathrm{H}^r(E_x, E_x \smallsetminus \{0\}, \mathbb{Z})$ is our orientation. η_E is called Thom class *of E.*
(2) For every $W \subseteq X$ we have isomorphism

$$\mathrm{H}^{j-r}(X, X \smallsetminus W, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^j(E, E \smallsetminus W, \mathbb{Z})$$
$$\beta \mapsto \pi^*(\beta) \cup \eta_E$$

where the convention is that LHS= 0 if $j < r$.

Proof. First treat the case $E = M \times \mathbb{R}^n$ using Künneth, then by induction on the number of elements in a trivial cover for E using M-V sequence. ☐

32.1. **Two theorems in algebraic topology (continued).** Last time: E is an oriented rank r vector bundle on X. Then there exists $\eta_E \in \mathrm{H}^r(E, E \smallsetminus X, \mathbb{Z})$ such that $\forall x \in X$, η_E restricts to $\mathrm{H}^r(E_x, E_x \smallsetminus \{0\}, \mathbb{Z})$ to the generators corresponding to the orientation of E_x.

For any $W \subseteq X$ closed, we have isomorphisms $\mathrm{H}^j(X, X \smallsetminus W, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^{j+r}(E \smallsetminus W, \mathbb{Z})$ for all $j \in \mathbb{Z}$. This is the *Thom class.*

For any $W \subseteq X$ closed, we have isomorphisms

$$\mathrm{H}^j(X, X \smallsetminus W, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^{j+r}(E, E \smallsetminus W, \mathbb{Z})$$
$$\alpha \mapsto \pi^*(\alpha) \cup \eta_E$$

for any $j \in \mathbb{Z}$. These are *Thom isomorphisms.*

Suppose that Y is a smooth real manifold of dimension n and X is a smooth real submanifold of dimension m, both are oriented. Since $\det(\mathrm{N}_{X/Y}) = \det(\mathrm{T}_Y)|_X \otimes \det(\mathrm{T}_Y)^{-1}$. $\mathrm{N}_{X/Y}$ is oriented of rank $n - m$.

Theorem 32.1 (Tubular neighbourhood theorem)**.** *There exists an open neighbourhood U of X in Y and a restriction $r : U \to X$ such that*

$$\begin{array}{ccccc} X & \hookrightarrow & U & \longrightarrow & X \\ \downarrow = & & \downarrow \sim & & \downarrow = \\ X & \hookrightarrow & \mathrm{N}_{X/Y} & \xrightarrow{\pi} & X \end{array}$$

where $r = n - m$.

If $W \subseteq X$ is closed, then

$$\mathrm{H}^j(X, X \smallsetminus W, \mathbb{Z}) \overset{\text{Thom}}{\simeq} \mathrm{H}^{j+r}(\mathrm{N}_{X/Y}, \mathrm{N}_{X/Y} \smallsetminus W, \mathbb{Z}) \simeq \mathrm{H}^{j+r}(U, U \smallsetminus W, \mathbb{Z}) \overset{\text{excision}}{\simeq} \mathrm{H}^{j+r}(Y, Y \smallsetminus W, \mathbb{Z})$$

In particular, for $W \subseteq X$, we have

$$\mathrm{H}^j(X, \mathbb{Z}) \simeq \mathrm{H}^{j+r}(Y, Y \smallsetminus X, \mathbb{Z})$$

Then the long exact sequence in cohomology becomes

$$\mathrm{H}^{p-1}(Y \smallsetminus X, \mathbb{Z}) \to \mathrm{H}^{p-r}(X, \mathbb{Z}) \to \mathrm{H}^p(Y, \mathbb{Z}) \to \mathrm{H}^p(Y \smallsetminus X, \mathbb{Z}) \to \cdots$$

where the map in the middle ($mH^{p-r}(X, \mathbb{Z}) \to \mathrm{H}^p(Y, \mathbb{Z})$) is denoted by i_* if $i : X \hookrightarrow Y$ is the inclusion map. This is Gysin homomorphism.

Recall, if X is compact, oriented and smooth manifold of dimension n, then there exists a fundamental class $\mu_x \in \mathrm{H}_n(X, \mathbb{Z})$ and the map

$$\mathrm{H}^p(X, \mathbb{Z}) \to \mathrm{H}_{n-p}(X, \mathbb{Z})$$
$$\alpha \mapsto \alpha \cap \mu_x$$

is an isomorphism.

Let $f : X \to Y$ be a smooth map between compact oriented smooth manifold. Write $n = \dim(Y), m = \dim(X), r = n - m$, then we have

$$\begin{array}{ccc} \mathrm{H}^p(X, \mathbb{Z}) & \xrightarrow{f_*} & \mathrm{H}^{p+r}(Y, \mathbb{Z}) \\ \mathrm{PD} \downarrow \sim & & \mathrm{PD} \downarrow \sim \\ \mathrm{H}_{m-p}(X, \mathbb{Z}) & \xrightarrow{f_*} & \mathrm{H}_{n-p-r}(Y, \mathbb{Z}) \end{array}$$

Fact: if f is the inclusion of a submanifold, then this agrees with the previous definition.

Special features for algebraic varieties.

(1) If X is a complete complex algebraic variety of dimension n, then X^{an} has a triangulation with simplicies of dimension at most $2n$. Moreover, if $Y \subseteq X$ is a closed subvariety, then we can find a compatible triangulation for $X^{\mathsf{an}}, Y^{\mathsf{an}}$.
(2) If W is any complex algebraic variety, then by Nagata, there is an open immersion $W \hookrightarrow X$ where X is a complete algebraic variety. Using (1), one can show that all $\mathrm{H}^p(W,\mathbb{Z}), \mathrm{H}_p(W,\mathbb{Z})$ are finitely generated.

Given any *complete irreducible* algebraic variety X, of dimension n, we have a fundamental class $\mu_x \in \mathrm{H}_{2n}(X,\mathbb{Z})$. If X is smooth, then it is clear. In general, use the resolution of singularity to construct $\pi : Y \to X$ proper, birational with Y smooth and $\mu_x = \pi_* \mu_Y$.

Claim. *This does not depend on π.*

Proof. Easy. Enough to show that if $f : W \to Y$ is proper and birational where both W and Y are smooth and complete, then $f_* \mu_W = \mu_Y$.

Since $\mathrm{H}_{2n}(Y) = \mathbb{Z}\mu_Y$ where $n = \dim(Y)$, $f_*\mu_W = d\mu_Y$. Need to check $d = 1$. To check this, tensor with $\mathbb{R}$ and use de Rham cohomology.

$$\begin{aligned} \mathrm{H}_{2n}(Y,\mathbb{R}) &\simeq \mathrm{H}^{2n}_{dR}(Y)^* \\ \mu_Y &\mapsto ([\omega] \mapsto \int_Y \omega) \end{aligned}$$

To check that $f_*\mu_W = \mu_Y$ modulo these identites, it is enough to check that if ω is $2n$-form on Y, then $\int_Y \omega = \int_W f^*(\omega)$. This is OK since f is a diffeomorphism outside measure 0 subset. □

One can argue similarly to show that if $f : W \to X$ is a surjective generically finite map with W, X smooth, complete, then $f_*\mu_W = \deg(f)\mu_X$.

Exercise 32.2. Use the projection formula to show $f_* f^* \alpha = \deg(f)\alpha$ on $\mathrm{H}^*(X)$. In particular, f^* is injective with $\mathbb{Q}$-coefficient and f^* is injective with $\mathbb{Z}$-coefficient if f is birational.

More general result. If $f : X \to Y$ is surjective holomorphic map of compact complex manifolds, with X Kähler, then $f^* : \mathrm{H}^i(Y,\mathbb{Q}) \to \mathrm{H}^i(X,\mathbb{Q})$ is injective for any i.

Remark 32.3. If X is smooth complete variety, $Y \subseteq X$ an irreducible closed subvariety, write $i : Y \hookrightarrow X$, $\dim X = n, \dim Y = m, r = n - m$, then $i_*\mu_Y \in \mathrm{H}_{2m}(X,\mathbb{Z}) \overset{\mathrm{PD}}{\simeq} \mathrm{H}^{2r}(Y,\mathbb{Z})$. This is the cohomology class of Y, denoted $[Y]$. We can extend this to cycles. Write $\alpha = \sum_{i=1}^r n_i Y_i$ where $\operatorname{codim}(Y_i) = r$, then $[\alpha] := \sum_{i=1}^r n_i [Y_i] \in \mathrm{H}^{2r}(X,\mathbb{Z})$.

Example 32.4. Suppose X is a smooth complete variety with D a smooth divisor on X. Then $[D] = c^1(\mathcal{O}_X(D))$. This is true even if D is just a prime divisor.

More generally, suppose X is a smooth, complete variety and $D_1, \cdots, D_r$ are smooth divisors, intersecting tranversely. Let $Y = D_1 \cap \cdots \cap D_r$. Induction on r gives us $[Y] = c_1(\mathcal{O}(D_1)) \cup \cdots \cup c_1(\mathcal{O}(D_r))$.

The special case $r = n = \dim(X)$: under the isomorphism $\mathrm{H}^{2n}(X) \cong \mathbb{Z}$, we have $c_1(\mathcal{O}(D_1)) \cup \cdots \cup c_1(\mathcal{O}(D_r)) = \#Y$.

Example 32.5. If $X = \mathbb{P}^n$ and $h = c_1(\mathcal{O}(1))$, then $h^r = [L_r]$ where $L_r \subseteq \mathbb{P}^n$ is a linear subspace of codimension r. In particular, $h^n = 1$ via $\mathrm{H}^{2n}(\mathbb{P}^n,\mathbb{Z}) \cong \mathbb{Z}$.

Exercise 33.1. Let X be a smooth complete complex algebraic variety, $Y \subseteq X$ a smooth irreducible closed subvariety of codimension r. Write $j : Y \hookrightarrow X$. Then we have induced maps

$$\begin{array}{ccc} \mathrm{H}^p(X) & \xrightarrow{j^*} & \mathrm{H}^p(Y) \\ & & \downarrow{\scriptstyle j_*} \\ & & \mathrm{H}^{p+2n}(X) \end{array}$$

Show that

(1) $j_* j^* \alpha = \alpha \cup [Y]$ for $\alpha \in \mathrm{H}^*(X)$.
(2) $j^* j_* \beta = \beta \cup j^*[Y]$.

For (1), use description of j_* via Poincaré duality and projection formula.

For (2), use description of j_* via Thom isomorphism.

Remark 33.2. If $f : Y \to X$ is holomorphic map of compact Kähler manifold, we defined

$$f_* : \mathrm{H}^p(Y) \to \mathrm{H}^{p+2r}(X)$$

where $r = \dim X - \dim Y$ via Poincaré duality. Using the behavior of Poincaré duality with respect to Hodge decomposition: after tensoring with $\mathbb{C}$, $f_*(\mathrm{H}^{i,j}(Y)) \subseteq \mathrm{H}^{i+r,j+r}(X)$.

In particular, if X is a smooth complete algebraic variety, $Y \subseteq X$ an irreducible closed subvariety of codimension r, so $[Y] \in \mathrm{H}^{2r}(X)$, then the image of $[Y]$ in $\mathrm{H}^{2r}(X, \mathbb{C})$ lies in $\mathrm{H}^{r,r}(X)$. Since if $\widetilde{Y} \to Y$ is a resolution of singularity and $m = \dim Y$, then $\mu_{\widetilde{Y}} \in \mathrm{H}_{2m}(\widetilde{Y}) \cong \mathrm{H}^0(\widetilde{Y})$ of type $(0,0)$.

If X is a smooth, projective complex agebraic variety, then

$$\mathrm{Hdg}^p(X) = \{\alpha \in \mathrm{H}^{2p}(X, \mathbb{Q}) \mid \text{the image of } \alpha \text{ in } \mathrm{H}^{2p}(X, \mathbb{C}) \text{ lies in } \mathrm{H}^{p,p}\}.$$

We have a linear map

$$Z^p(X)_{\mathbb{Q}} \to \mathrm{Hdg}^p(X)$$
$$\sum n_i Y_i \mapsto \sum n_i [Y_i]$$

where $Z^p(X)_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space with basis given by codimension p irreducible subvariety.

Conjecture 33.3 (Hodge conjecture)**.** *This is a surjective map.*

Remark 33.4. This is true for $p = 1$, even with $\mathbb{Z}$-coefficients. We showed that if $\alpha \in \mathrm{H}^2(X, \mathbb{Z})$, with image in $\mathrm{H}^{1,1}$, then $\alpha = c_1(L)$ for some $L \in \mathrm{Pic}(X)$. If $L = \mathcal{O}(D)$ where $D = \sum n_i D_i$, then $\alpha = \sum n_i [D_i]$.

Theorem 33.5 (Lefschetz hyperplane theorem)**.** *Let X be a projective smooth complex algebraic variety, $D \subseteq X$ a smooth divisor such that $L = \mathcal{O}(D)$ is ample, and $j : D \hookrightarrow X$ the inclusion map. Then $j^* : \mathrm{H}^p(X, \mathbb{Z}) \to \mathrm{H}^p(D, \mathbb{Z})$ is the isomorphism for $p \leqslant n-2$ and injective for $p \leqslant n-1$, where $n = \dim X$.*

Remark 33.6. $j_* j^* \alpha = [D] \cup \alpha = c_1(L) \cup \alpha$. Hard Lefschetz $\Rightarrow$ this is surjective on $\mathrm{H}^p(X, \mathbb{Q})$ for $p \leqslant n \Rightarrow j^*$ injective on $\mathrm{H}^p(X, \mathbb{Q})$ for $p \leqslant n-1$.

Remark 33.7. Key ingradient for the proof. (due to Andreatti-Frankel). If Y is an affine complex algebraic variety of dimension n, Y has the homology type of a CW complex of dimension $\leqslant n$.

In particular, $\mathrm{H}_p(Y, \mathbb{Z}) = 0, \mathrm{H}^p(Y, \mathbb{Z}) = 0$ for $p > n$. In our setting, L ample implies that $X \smallsetminus D$ is affine. mD such that L^m very ample, $X \hookrightarrow \mathbb{P}^N$. Long exact sequence in cohomology gives

$$\mathrm{H}^{p-1}(X \smallsetminus D, \mathbb{Z}) \to \mathrm{H}^{p-2}(D, \mathbb{Z}) \xrightarrow{j_*} \mathrm{H}^p(X, \mathbb{Z}) \to \mathrm{H}^p(X \smallsetminus D, \mathbb{Z})$$

implies that $j_* : \mathrm{H}^{p-2}(D) \to \mathrm{H}^p(X)$ isomorphism if $p - 1 > n$ and surjective if $p > n$.

Some cohomology computations: (1) Case of $\mathbb{P}^n$:

$$\mathrm{H}^i(\mathbb{P}^n,\mathbb{Z}) = \begin{cases}\mathbb{Z} & 0 \leqslant i \leqslant 2n, i \text{ even}\\ 0 & \text{otherwise}\end{cases}$$

use induction on n, with $n = 0$ trivial. Indeductive step: have $\mathbb{P}^{n-1} \cong H \subseteq \mathbb{P}^n$ such that $\mathbb{P}^n \smallsetminus H \cong \mathbb{C}^n$ contractible.

$$0 \to \mathrm{H}^{i-1}(\mathbb{P}^n \smallsetminus H) \to \mathrm{H}^{i-2}(\mathbb{P}^{n-1}) \to \mathrm{H}^i(\mathbb{P}^n) \to \mathrm{H}^i(\mathbb{P}^n \smallsetminus H) = 0 \text{ if } i > 0$$

If $i \geqslant 2$, then we have

$$\begin{aligned}&\mathrm{H}^i(\mathbb{P}^n) \cong \mathrm{H}^{i-2}(\mathbb{P}^{n-1})\\ &\mathrm{H}^1(\mathbb{P}^n) = 0\\ &\mathrm{H}^0(\mathbb{P}^n) = \mathrm{H}^0(\mathbb{P}^n \smallsetminus H) = \mathbb{Z}\end{aligned}$$

clear: HD is such that $\mathrm{H}^{2k}(\mathbb{P}^n,\mathbb{C}) = \mathrm{H}^{k,k}(\mathbb{P}^n)$ (by Hodge symmetry). In particular, $\mathrm{H}^q(\mathbb{P}^n,\mathcal{O}^{\mathsf{an}}_{\mathbb{P}^n}) = 0$ for $q > 0$.

In fact, $\mathrm{H}^*(\mathbb{P}^n,\mathbb{Z}) \cong \mathbb{Z}[h]/(h^{n+1})$ where $h = c_1(\mathcal{O}(1))$.

Why: note that

$$\begin{aligned}\mathrm{H}^{2n}(\mathbb{P}^n,\mathbb{Z}) &\overset{\deg}{\to} \mathrm{H}_0(\mathbb{P}^n,\mathbb{Z})\\ h^n &\mapsto 1\end{aligned}$$

since we have n hyperplane intersecting transversely in 1 point.

Since $\mathrm{H}^{2k}(\mathbb{P}^{2n},\mathbb{Z}) \cong \mathbb{Z}$ for $0 \leqslant k \leqslant n$, $h^k \in \mathrm{H}^{2k}(\mathbb{P}^n,\mathbb{Z})$ nonzero. For every $\alpha \in \mathrm{H}^{2k}(\mathbb{P}^n,\mathbb{Z})$, can write $\alpha = uh^k$ for some $u \in \mathbb{Q}$.

$$\alpha \cup h^{n-k} \mapsto \deg(\alpha \cup h^{n-k}) \in \mathbb{Z}$$

So $uh^n \mapsto u \Rightarrow u \in \mathbb{Z}$.

34. December 02, 2019

Recall that last time we showed

$$\begin{aligned}\mathbb{Z}[x]/(x^{n+1}) &\to \mathrm{H}^*(\mathbb{P}^n,\mathbb{Z})\\ x^i &\mapsto c_1(\mathcal{O}(1))^i\end{aligned}$$

Leray-Hirsch Theorem 31.4 $\Rightarrow$ if E is a rank $r+1$ vector bundle on X and $\pi : \mathbb{P}(E) \to X$, then

$$\begin{aligned}\mathrm{H}^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x^{r+1}) &\overset{\sim}{\to} \mathrm{H}^*(\mathbb{P}(E))\\ \alpha x^i &\mapsto \pi^*(\alpha)c_1(O(1))^i\end{aligned}$$

In particular, $\mathrm{H}^{p,q}(\mathbb{P}(E)) \simeq \oplus_{i=0}^r \mathrm{H}^{p-i,q-i}(X)$.

Exercise 34.1. Suppose X compact. Show that $\pi_*(\pi^*(\alpha)c_1(O(1))^i) = \begin{cases}0 & i < r\\ \alpha & i = r\end{cases}$ where $0 \leqslant i \leqslant r$.

34.1. **Cohomology of the blow-up.** Let X be a smooth projective complex variety, $Y \subseteq X$ a smooth closed subvariety of codimension r. Let

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\pi} & X \\ {\scriptstyle i}\uparrow & & \uparrow \\ E & \xrightarrow{\varphi} & Y \end{array}$$

be the blow-up along Y where E is the exceptional divisor. Then for any p,

$$\mathrm{H}^p(X,\mathbb{Z}) \oplus \oplus_{i=1}^{r} \mathrm{H}^{p-2i}(Y,\mathbb{Z}) \simeq \mathrm{H}^p(\widetilde{X},\mathbb{Z})$$
$$(\alpha_0,\alpha_1,\ldots,\alpha_{r-1}) \mapsto \pi^*\alpha_0 + i_* \sum_{j=1}^{r-1} c_1(O_E(1))^{j-1}\varphi^*\alpha_j$$

Sketch of proof. We have 2 long exact sequence:

$$\begin{array}{ccccccccc} \mathrm{H}^{p-1}(X \smallsetminus Y) & \longrightarrow & \mathrm{H}^{p-2r}(Y) & \longrightarrow & \mathrm{H}^p(X) & \longrightarrow & \mathrm{H}^p(X \smallsetminus Y) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \varphi_p} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \sim} & & \\ \mathrm{H}^{p-1}(\widetilde{X} \smallsetminus E) & \longrightarrow & \mathrm{H}^{p-2r}(E) & \longrightarrow & \mathrm{H}^p(\widetilde{X}) & \longrightarrow & \mathrm{H}^p(\widetilde{X} \smallsetminus E) & \longrightarrow & \cdots \end{array}$$

where E is the projective bundle over Y of rank r. Then $\mathrm{H}^{p-2}(E) \simeq \oplus_{i=1}^{r}\mathrm{H}^{p-2i}(Y)$.

Exercise 34.2. $\varphi_* \circ \varphi_p = \mathrm{id}_{\mathrm{H}^{p-2r}(Y)}$.

Equivalently (via previous exercise), (projection onto last component) $\circ\varphi_p = \mathrm{id}_{\mathrm{H}^{p-2r}(Y)}$.

Then the conclusion follows by a diagram chasing. (check this !) □

34.2. **Pure Hodge structures.**

Definition 34.3. A pure (integral) Hodge structure of weight m is given by a finitely generated free abelian group H (also written $H_{\mathbb{Z}}$) together with a decomposition $H_{\mathbb{C}} \coloneqq H \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=m} H^{p,q}$ such that for any p,q, $\overline{H}^{p,q} = H^{q,p}$.

Some variants: a rational (resp. real) pure Hodge structure starts with a finite dimensional vector space over $\mathbb{Q}$ (resp. $\mathbb{R}$).

Example 34.4 (Main Example I)**.** If X is a compact Kähler manifold, then $\mathrm{H}^i(X,\mathbb{Z})/\text{torsion}$ is endowed with a canonical pure Hodge structure of weight i.

Example 34.5 (Main Example II)**.** Tate Hodge structure: $\mathbb{Z}(1) = 2\pi\mathrm{i}\,\mathbb{Z} \subseteq \mathbb{C}$ pure of type $(-1,-1)$ (weight -2) and $\mathbb{Z}(-1) = \frac{1}{2\pi\mathrm{i}}\mathbb{Z} \subseteq \mathbb{C}$ type $(1,1)$.

Proposition 34.6. *A pure Hodge structure of weight m is equivalent to a free finitely generated abelian group H together with a finite decreasing filtration $(\mathrm{F}^p H_{\mathbb{C}})_{p\in\mathbb{Z}}$ on $H_{\mathbb{C}}$ such that*

$$\mathrm{F}^p \oplus \overline{\mathrm{F}}^{m+1-p} = H_{\mathbb{C}}. \tag{34.1}$$

Definition 34.7. A filtration $\mathrm{F}^\bullet H_{\mathbb{C}}$ is the *Hodge filtration* on $H_{\mathbb{C}}$ and $\overline{\mathrm{F}}^\bullet H_{\mathbb{C}}$ is the *conjugate Hodge filtration.*

Proof of Proposition 34.6. Suppose we have a decomposition $H_{\mathbb{C}} = \oplus_{p+q} H^{p,q}$ as in the definition of Hodge structure. Then $\mathrm{F}^p H_{\mathbb{C}} = \oplus_{i+j=m, i\geqslant p} H^{i,j}$ is clearly a decreasing, finite filtration and

$$\overline{\mathrm{F}}^{m+1-p} = \bigoplus_{\substack{i+j=m \\ i\geqslant m+1-p}} \overline{H}^{i,j} = \bigoplus_{\substack{i+j=m \\ j\leqslant p-1}} H^{j,i}$$

Hence (34.1) holds.

A similar computation shows $H^{p,q} = \mathrm{F}^p \cap \overline{\mathrm{F}}^{m-p}$. Conversely, suppose $\mathrm{F}^\bullet H$ = finite decreasing filtration satisfies (34.1). Define $H^{p,q} = \mathrm{F}^p \cap \overline{\mathrm{F}}^q$.

- Suppose $\sum_{p+q=m} \alpha_{p,q} = 0$, $\alpha_{p,q} \in \mathrm{H}^{p,q}$. If not all $\alpha_{p,q}$ are zero, then choose minimal p such that $\alpha_{p,q} \neq 0$. Then $\alpha_{p,q} = -\sum_{p'>p} \alpha_{p',q'} \in \mathrm{F}^{p+1} \cap \overline{\mathrm{F}}^q = (0)$ by (34.1).
- Let us show that every $u \in H$ lies in $\sum_{p+q=m} H^{p,q}$. Suppose $u \in \mathrm{F}^p$. We argue by a decreasing induction on p and note that if $p >> 0$, then $u = 0$. Since $H_{\mathbb{C}} = \mathrm{F}^{p+1} \oplus \overline{\mathrm{F}}^{m-p} \Rightarrow u = v_1 + v_2$ where $v_1 \in \mathrm{F}^{p+1}, v_2 \in \overline{\mathrm{F}}^{m-p}$. By induction v_1 is in $\sum H^{p',q'}$. Then $v_2 = u - v_1 \in \mathrm{F}^p$. But $v_2 \in \overline{\mathrm{F}}^{m-p}$. So $v_2 \in \mathrm{H}^{p,q} \subseteq \sum_{p'+q'=m} \mathrm{H}^{p',q'} \Rightarrow v \in \sum_{p'+q'=m} \mathrm{H}^{p',q'}$.

In fact, we see $\mathrm{F}^p \subseteq \oplus_{p'+q'=m, p' \geqslant p} H^{p',q'}$ and the reverse inclusion is clear. So $\mathrm{F}^p = \oplus_{p' \geqslant p} H^{p',q'}$. Hence these are inverse constructions. □

Definition 34.8. If A, B are pure Hodge structures of weights $m, m+2r$, then a morphism of Hodge structures of type (r,r) is a morphism of abelian groups $\varphi : A \to B$ such that $\varphi_{\mathbb{C}}(A^{p,q}) \subseteq B^{p+r,q+r}$ for every p, q. Equivalently, $\varphi_{\mathbb{C}}(\mathrm{F}^p(A)) \subseteq \mathrm{F}^{p+r}(B)$ for every p.

35. December 04, 2019

Let E be a rank $r+1$ vector bundle on X and $\mathbb{P}(E) \xrightarrow{\pi} X$. Then we have a map

$$\mathrm{H}^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(x^{r+1}) \simeq \mathrm{H}^*(\mathbb{P}(E))$$
$$\alpha \otimes x^i \mapsto \pi^* \alpha \cup c_1(O(1))^i$$

is a group homomorphism, *not* a ring homomorphism. Because $c_1(O(1))^{r+1} \neq 0$ in general. By looking at its coefficients in $\mathrm{H}^*(X)$, get Chern classes of $c_i(E)$.

In Definition 34.8, if $r = 0$, then it is called a morphism of pure Hodge structure.

Example 35.1. If $f : X \to Y$ is a morphism of compact Kähler manifolds and $f^* : \mathrm{H}^i(Y,\mathbb{Z})/\text{torsions} \to \mathrm{H}^i(X,\mathbb{Z})/\text{torsions}$ is a morphism of pure Hodge structures for any i.

35.1. Operations with pure Hodge structure.

(1) Finite direct sums of pure Hodge structure of the same weight.
(2) Kernels and cokernels: $f : A \to B$ is a morphism of Hodge structures of weight (r,r) where A is of weight m and B is of weight $m+2r$, then $\mathrm{Ker}(f) \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=m} \mathrm{Ker}(A^{p,q} \to A^{p+r,q+r})$ pure Hodge structure of weight m. $\mathrm{Coker}(f)/\text{torsions} \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Coker}(A^{p,q} \to B^{p+r,q+r})$ pure Hodge structure of weight $m+2r$. These satisfy usual universal property. Therefore the category of pure Hodge structure of weight m is an abelian category.
(3) Tensor product: A is of weight m and B of weight n, then $A_{\mathbb{Z}} \otimes_{\mathbb{Z}} B_{\mathbb{Z}}$ is a finitely generated abelian group. And $A_{\mathbb{C}} \otimes_{\mathbb{C}} B_{\mathbb{C}} = (\oplus_{p+q=m} A^{p,q}) \otimes \left(\oplus_{p'+q'=n} B^{p',q'}\right)$ has weight $m+n$ and $(A \otimes B)^{p,q} = \oplus_{\substack{i+i'=p \\ j+j'=q}} A^{i,j} \otimes B^{i',j'}$.

Example 35.2. $\mathbb{Z}(m) = (2\pi \mathrm{i})^m \mathbb{Z}$ is of type $(-m,-m)$ (weight $-2m$). We have $\mathbb{Z}(i) \otimes \mathbb{Z}(j) \simeq \mathbb{Z}(i+j)$. In general, if A is of pure Hodge structure weight m, then $A(i) := A \otimes_{\mathbb{Z}} \mathbb{Z}(i)$ of weight $m - 2i$.

Example 35.3. If X is a compact Kähler manifold, and $\mathrm{H}^i(X,\mathbb{Z})/\text{torsions} \otimes \mathrm{H}^j(X,\mathbb{Z})/\text{torsions} \to \mathrm{H}^{i+j}(X,\mathbb{Z})/\text{torsions}$ is a morphism of pure Hodge structure of weight $i+j$.

(4) The dual of a pure Hodge structure A of weight m: $A^* = \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$ and $A^*_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(A_{\mathbb{C}},\mathbb{C}) = \oplus_{p+q=m} \underbrace{\mathrm{Hom}(A^{p,q},\mathbb{C})}_{(A^*)^{-p,-q}}$ pure Hodge structure of weight $-m$.

Example 35.4. $\mathbb{Z}(-m) \simeq \mathbb{Z}(m)^*$

Example 35.5. Let X be a compact Kähler manifold. Then $\mathrm{H}_i(X,\mathbb{Z})/\text{torsion} \simeq \left(\mathrm{H}^i(X,\mathbb{Z})/\text{torsion}\right)^*$ by the universal coefficient theorem. So $\mathrm{H}_i(X,\mathbb{Z})/\text{torsion}$ carries a pure Hodge structure of weight $-i$.

Let X be a compact Kähler manifold of dimension n.

(1) $\mathrm{H}^{2n}(X,\mathbb{Z}) \simeq \mathbb{Z}(-n)$ such that elements that caps with μ_X to give $1 \in \mathrm{H}_0$ maps to $(2\pi \mathrm{i})^{-n}$.
(2) $(\mathrm{H}^k(X,\mathbb{Z})/\text{torsion})^* \simeq (\mathrm{H}^{2n-k}(X,\mathbb{Z})/\text{torsions})(n)$ via Poincaré duality.
(3) Similarly, $\mathrm{H}^k(X,\mathbb{Z})/\text{torsion} \simeq (\mathrm{H}_{2n-k}(X,\mathbb{Z})/\text{torsions})(n)$ via Poincaré duality is an isomorphism of Hodge structure.
(4) Gysin maps. $f : X^n \to Y^m$ holomorphic maps of compact Kähler manifolds. Then $\mathrm{H}^p(X,\mathbb{Z})/\text{torsion} \xrightarrow{f_*} (\mathrm{H}^{p+2d}(Y,\mathbb{Z})/\text{torsion})(d)$ where $d = m - n$.

35.2. **Polarized Hodge structures.** Suppose A is a pure Hodge structure of weight m. A polarization on A is given by a bilinear form $A_{\mathbb{Z}} \times A_{\mathbb{Z}} \xrightarrow{Q} \mathbb{Z}$ which is symmetric if m is even and skew-symmetric if m is odd. (Equivalently, $(u,v) \mapsto \mathrm{i}^m Q(u,\overline{v})$ is a Hermitian form.) Such that after tensor with $\mathbb{C}$,

(1) $Q(A^{p,q}, A^{p',q'}) = 0$ unless $p = p', q = q'$
(2) The Herimitian form $(u,v) \mapsto \mathrm{i}^{p-q}(-1)^{\frac{m(m-1)}{2}} Q(u,\overline{v})$ is positive definite on $A^{p,q}$.

In particular, Q is non-degenerate. We get via Q an isomorphism $A(m) \to A^*$.

Example 35.6 (Main example)**.** Let X be a smooth projective variety and $L \in \mathrm{Pic}(X)$ an ample line bundle. For any $k \leqslant n = \dim(X)$, we have

$$\mathrm{P}\overline{\mathrm{H}}^k(X,\mathbb{Z}) = \{\alpha \in \mathrm{H}^k(X,\mathbb{Z})/\text{torsion} | c_1(L)^{n-k+1} \cup \alpha = 0\}$$

is a polarized Hodge structure of weight k by Hodge-Riemann bilinear relation.

Moreover, we get a polarization on $\overline{\mathrm{H}}^k(X,\mathbb{Z})$ using the Lefschetz decomposition:

$$(\mathrm{H}^k(X,\mathbb{Z})/\text{torsion} =:)\overline{\mathrm{H}}^k(X,\mathbb{Z}) = \oplus_{i \geqslant 0} \mathrm{P}\overline{\mathrm{H}}^{k-2i}(X,\mathbb{Z})(-i)$$

Proposition 35.7. *If V is a polarized pure rational Hodge structure of weight m and $W \subseteq V$ is a pure sub-Hodge structure of V, then the restriction of Q (polarization on V) to W gives a polarization of W and there is a polarized pure sub-Hodge structure W' of V such that $V \simeq W \oplus W'$.*

Proof. Take $W' = W^{\perp}$ with respect to Q and verify the statement. □

36. December 06, 2019

36.1. De Rham cohomology.

36.1.1. *Introduction to spectral sequence.* Let $\mathscr{C}$ be an abelian category, $A^\bullet$ a complex in $\mathscr{C}$. Let $(\mathrm{F}^p A^\bullet)_{p\in\mathbb{Z}}$ decreasing filtration such that

(1) there exists m such that $\mathrm{F}^m A^\bullet = A^\bullet$
(2) For any i, there exists p such that $\mathrm{F}^p A^i = 0$.

We get an induced filtration on $\mathrm{H}^*(A^\bullet)$ by $\mathrm{F}^p H^i(A^\bullet) = \mathrm{Im}(\mathrm{H}^i(\mathrm{F}^p A^\bullet) \to \mathrm{H}^i(A^\bullet))$. The goal is to compute $\mathrm{F}^p\mathrm{H}^i(A^\bullet)/\mathrm{F}^{p+1}\mathrm{H}^i(A^\bullet)$. This is done by the spectral sequence associated to this filtered complex $(E_r^{p,q}, d_r^{p,q})_{r\geqslant 1}$ where $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^2 = 0$, together with isomorphism $E_{r+1}^{p,q} \simeq \mathrm{Ker}(d_r^{p,q})/\,\mathrm{Im}(d_r^{p-r,q+r-1})$.

For example, $E_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{F}^p A^\bullet/\mathrm{F}^{p+1}A^\bullet)$ for any p,q with $d_1^{p,q} : \mathrm{H}^{p+q}(\mathrm{F}^p A^\bullet/\mathrm{F}^{p+1}A^\bullet) \to \mathrm{H}^{p+q+1}(\mathrm{F}^{p+1}A^\bullet/\mathrm{F}^{p+2}A^\bullet)$ the boundary homorphism corresponding to the exact sequence of complex

$$0 \to \mathrm{F}^{p+1}A^\bullet/\mathrm{F}^{p+2}A^\bullet \to \mathrm{F}^p A^\bullet/\mathrm{F}^{p+2}A^\bullet \to \mathrm{F}^p A^\bullet/\mathrm{F}^{p+1}A^\bullet \to 0$$

We have $E_1^{p,q} \Rightarrow \mathrm{H}^* A^\bullet$ means

(1) For every p,q, $E_r^{p,q}$ stabilizes for r large enough.
(2) There exists an isomorphism $\mathrm{F}^p\mathrm{H}^{p+q}(A^\bullet)/\mathrm{F}^{p+1}\mathrm{H}^{p+q}(A^\bullet) \simeq E_\infty^{p,q}$.

36.1.2. *Hypercohomology.* Let $\mathscr{C}, \mathscr{D}$ be two abelian categories such that $\mathscr{C}$ has enough injectives. Let $T : \mathscr{C} \to \mathscr{D}$ be a left exact functor, $A^\bullet$ a complex in $\mathscr{C}$, bounded to the left. Then there exists a complex $I^\bullet$ consisting of injective objects, bounded to the left and a morphism of complex $A^\bullet \to I^\bullet$ which is a quasi-isomorphism.

Then $\mathbb{R}^q T(A^\bullet) := \mathscr{H}^q(T(I^\bullet)) \in \mathrm{Obj}(\mathscr{D})$. This is independent of the choice of the resolution and functorial. Suppose now we have a filtration $(\mathrm{F}^p A^\bullet)_{p\in\mathbb{Z}}$ on $A^\bullet$ as before. One can construct $A^\bullet \to I^\bullet$ as before such that we have a decreasing filtration $\mathrm{F}^p I^\bullet$ on $I^\bullet$ with $\mathrm{F}^p I^m / \mathrm{F}^p I^{m+1}$ (hence also for $\mathrm{F}^p I^m$) injective. For any p, $\mathrm{F}^p A^\bullet \to \mathrm{F}^p I^\bullet$ is a quasi-isomorphism and $\mathrm{F}^p A^\bullet / \mathrm{F}^{p+1} A^\bullet \to \mathrm{F}^p I^\bullet / \mathrm{F}^{p+1} I^\bullet$ quasi-isomorphism.

Apply T to $I^\bullet$ to get a filtration on $T(I^\bullet)$ by $\mathrm{F}^p T(I^\bullet) = T(\mathrm{F}^p I^\bullet) \hookrightarrow T(I^\bullet)$. Hence we get a spectral sequence

$$\begin{aligned} E_1^{p,q} &= \mathrm{H}^{p+q}(\underbrace{T(\mathrm{F}^p I^\bullet)/T(\mathrm{F}^{p+1} I^\bullet)}_{T(\mathrm{F}^p I^\bullet / \mathrm{F}^{p+1} I^\bullet)}) \\ &= \mathbb{R}^{p+q} T(\mathrm{F}^p A^\bullet / \mathrm{F}^{p+1} A^\bullet). \end{aligned}$$

And $E_1^{p,q} \Rightarrow \mathbb{R}^{p+q} T(A^\bullet)$.

Example 36.1 (Key example)**.** Consider on $A^\bullet$ the "naïve" filtration $\mathrm{F}^p A^\bullet = 0 \to A^p \to A^{p+1} \to \cdots$, then $\mathrm{F}^p A^\bullet / \mathrm{F}^{p+1} A^\bullet = A^p[-p]$. Then the spectral sequence shows that $E_1^{p,q} = \mathbb{R}^q T(A^p) \Rightarrow \mathbb{R}^{p+q} T(A^\bullet)$.

36.1.3. *De Rham cohomology.* Let $k = \overline{k}$ be an algebraically closed field, let X be the smooth algebraic variety over k. Then we have the de Rham complex

$$0 \to \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega_X^n \to 0$$

where $\Omega_X^i = \bigwedge^i (\Omega_X)$ and $n = \dim X$. Note $\Omega_X^\bullet$ is coherent for any i, but d is *not* $\mathcal{O}_X$-linear.

Definition 36.2. The de Rham cohomology of X is $\mathrm{H}^i_{\mathsf{dR}}(X) := \mathbb{H}^i(X, \Omega_X^\bullet) = \mathbb{R}^i \Gamma(X, \Omega_X^\bullet)$ k-vector space.

Theorem 36.3 (Grothendieck)**.** *If X is a complete smooth variety over $\mathbb{C}$, then there exists a canonical isomorphism* $\mathrm{H}^i_{\mathsf{dR}}(X) \simeq \mathrm{H}^i(X^{\mathsf{an}}, \mathbb{C})$.

Remark 36.4. The result also holds if X is not complete by Deligne.

Proof. Consider the corresponding analytic de Rham complex

$$0 \to \mathcal{O}_{X^{\mathsf{an}}} \xrightarrow{\mathrm{d}} \Omega_{X^{\mathsf{an}}} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n_{X^{\mathsf{an}}} \to 0$$

obtained via the analytification functor from $\Omega_X^\bullet$.

The key fact is that this is acyclic, with 0-cohomology $\underline{\mathbb{C}}_{X^{\mathsf{an}}}$.

Proof. Recall that we have a double complex $\left(\mathcal{A}^{\bullet,\bullet}_{X^{\mathsf{an}}}, \partial, \overline{\partial}\right)$. By $\partial\overline{\partial}$-lemma, $(\mathcal{A}^{p,\bullet}_{X^{\mathsf{an}}}, \overline{\partial})$ is acyclic complex with 0-cohomology $\Omega^p_{X^{\mathsf{an}}}$. So the inclusion $\Omega^\bullet_{X^{\mathsf{an}}} \hookrightarrow \mathrm{Tot}(\mathcal{A}^{\bullet,\bullet}_{X^{\mathsf{an}}})$ is a quasi-isomorphism. So $\mathrm{Tot}(\mathcal{A}^{\bullet,\bullet}_{X^{\mathsf{an}}})$ is the de Rham complex of X^{an} of smooth $\mathbb{C}$-forms. This is acyclic with 0-cohomology $\underline{\mathbb{C}}_{X^{\mathsf{an}}}$. □

So $\mathbb{H}^i(X^{\mathsf{an}}, \Omega^\bullet_{X^{\mathsf{an}}}) \simeq \mathrm{H}^i(X, \underline{\mathbb{C}}_{X^{\mathsf{an}}}) = \mathrm{H}^i(X^{\mathsf{an}}, \mathbb{C})$. Hence it is enough to show that the canonical map $\mathbb{H}^i(X, \Omega_X^\bullet) \to \mathbb{H}^i(X^{\mathsf{an}}, \Omega^\bullet_{X^{\mathsf{an}}})$ is an isomorphism.

Hypercohomology spectral sequence for $\Omega_X^\bullet$: $E_1^{p,q} = \mathrm{H}^q(X, \Omega_X^p) \Rightarrow \mathrm{H}^{p+q}_{\mathsf{dR}}(X)$.

Have a corresponding spectral sequence for $\Omega^\bullet_{X^{\mathsf{an}}}$: $\widetilde{E}_1^{p,q} = \mathrm{H}^q(X^{\mathsf{an}}, \Omega^p_{X^{\mathsf{an}}}) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^\bullet_{X^{\mathsf{an}}})$.

Functorality of the hypercohomology spectral sequence gives morphism of spectral sequence: $E_r^{p,q} \to \widetilde{E}_r^{p,q}$. This is an isomorphism for $r = 1$ by GAGA.

This implies that this is isomorphism for all r. By convention of spectral sequence, we have

$$\mathrm{F}^p \mathbb{H}^{p+q}(X, \Omega_X^\bullet) / \mathrm{F}^{p+1} \mathbb{H}^{p+q}(X, \Omega_X^\bullet) \simeq \mathrm{F}^p \mathbb{H}^{p+q}(X^{\mathsf{an}}, \Omega^\bullet_{X^{\mathsf{an}}}) / \mathrm{F}^{p+1} \mathbb{H}^{p+q}(X^{\mathsf{an}}, \Omega^\bullet_{X^{\mathsf{an}}})$$

for all p, q.

These are finite filtrations, so it induces maps $\mathbb{H}^i(X, \Omega_X^\bullet) \simeq \mathbb{H}^i(X^{\mathsf{an}}, \Omega^\bullet_{X^{\mathsf{an}}})$ is an isomorphism. □

37. December 09, 2019

37.1. **Hodge-to-de Rham spectral sequence.** Let $k = \overline{k}$ be algebraically closed, X a smooth projective variety over k. Let $\Omega_X^\bullet$ be the algebraic de Rham complex on X with the naive filtration on $\Omega_X^\bullet$, we get the "Hodge-to-de Rham spectral sequence": $E_1^{p,q} = \mathrm{H}^q(X, \Omega_X^p) \Rightarrow \mathrm{H}^{p+q}(X, \Omega_X^\bullet)$.

Theorem 37.1. *If* $\mathrm{char}(k) = 0$*, then this collapes on its* 1*st page.*

Proof. Enough to show this when $k = \mathbb{C}$. We showed: in this case, we have $\mathbb{H}^n(X, \Omega_X^\bullet) \simeq \mathrm{H}^n(X^{\mathsf{an}}, \mathbb{C})$.

Wnat to show $d_r^{p,q} = 0$ for any p, q and $r \geqslant 1$. In general, $E_{r+1}^{p,q}$ is a subquotient of $E_r^{p,q}$ and $E_{r+1}^{p,q} = E_r^{p,q}, \forall p, q$ if and only if $d_r^{p,q} = 0, \forall p, q$.

Hence the spectral sequence collapses on the first page if and only if $\dim \mathrm{F}^p\mathrm{H}^n/\mathrm{F}^{p+1}\mathrm{H}^n = \dim E_1^{p,n-p}, \forall p$.

$$\Leftrightarrow \underbrace{\dim \mathbb{H}^n(X, \Omega_X^\bullet)}_{\dim \mathrm{H}^n(X^{\mathsf{an}}, \mathbb{C})} = \sum_{p+q=n} \underbrace{\dim E_1^{p,q}}_{h^{p,q}(X)}$$

Hence the assertion follows from the Hodge decomposition. □

Remark 37.2. The naïve filtration on $\Omega_X^\bullet$ gives a filtration on $\mathbb{H}^*(X, \Omega_X^\bullet)$, which via the identifying $\mathbb{H}^*(X, \Omega_X^\bullet) \simeq \mathrm{H}^*(X^{\mathsf{an}}, \mathbb{C})$ is the *Hodge filtration*.

Step 1: It is enough to prove the corresponding statement for $\omega_{X^{\mathsf{an}}}^\bullet$.

Step 2: We saw that the inclusion $\Omega_X^\bullet \hookrightarrow \mathcal{A}_{X^{\mathsf{an}},\mathbb{C}}^\bullet$ is an quasi-isomorphism, i.e. "naïve filtration" $\mathrm{F}^p\Omega_{X^{\mathsf{an}}}^\bullet \hookrightarrow \underbrace{\mathrm{F}^p\mathcal{A}_{X^{\mathsf{an}},\mathbb{C}}^\bullet}_{\oplus_{i+j=\bullet, i\geqslant p}\mathcal{A}^{i,j}}$ is quasi-isomorphism again by $\partial\overline{\partial}$-lemma.

So $\Rightarrow \mathrm{F}^p\mathbb{H}(X^{\mathsf{an}}, \Omega_{X^{\mathsf{an}}}^\bullet) = \mathrm{Im}(\mathscr{H}^\bullet\Gamma(\mathrm{F}^p\mathcal{A}_{X^{\mathsf{an}},\mathbb{C}}^\bullet) \to \mathscr{H}^\bullet\Gamma(\mathcal{A}_{X^{\mathsf{an}},\mathbb{C}}^\bullet))$

Choose a Kähler metric on X^{an} such that we have harmonic forms.

$$\begin{array}{ccc} \mathscr{H}^m\Gamma(\mathrm{F}^p\mathcal{A}_{X^{\mathsf{an}},\mathbb{C}}^\bullet) & \longrightarrow & \mathscr{H}^m\Gamma(\mathcal{A}_{X^{\mathsf{an}},\mathbb{C}}^\bullet) \\ & \nwarrow \qquad \nearrow & \\ & \oplus_{i+j=m, i\geqslant p}\mathscr{H}^{i,j}(X^{\mathsf{an}}) & \end{array}$$

where $\mathscr{H}^{i,j}(X^{\mathsf{an}})$ is the space of (i,j) harmonic forms. The right-up map is an isomorphism by the Hodge decomposition onto $\mathrm{F}^p(\cdots)$. Hence it is enough to show the natural map (left-up) in the diagonal is an isomorphism.

Argue by decreasing induction on p: write $\mathscr{C}_p = \Gamma(\mathrm{F}^p\mathcal{A}_{X^{\mathsf{an}},\mathbb{C}}^\bullet)$, then

$$0 \to \mathscr{C}_{p+1} \to \mathscr{C}_p \to \mathscr{C}_p/\mathscr{C}_{p+1} = \Gamma(X^{\mathsf{an}}, \mathcal{A}_{X^{\mathsf{an}}}^{p,\bullet}) \to 0$$

$$\begin{array}{ccccccccc} \Rightarrow \mathbb{H}^m(\mathscr{C}_{p+1}) & \longrightarrow & \mathbb{H}^m(\mathscr{C}_p) & \longrightarrow & \mathbb{H}^m\Gamma(X^{\mathsf{an}}, \mathcal{A}_{X^{\mathsf{an}}}^{p,\bullet}) & \xrightarrow{\delta} & \mathbb{H}^{m+1}(\mathscr{C}_{p+1}) & & \\ & & \uparrow{\scriptstyle \varphi_{p+1}} & & \uparrow{\scriptstyle \varphi_p} & & \uparrow{\scriptstyle \simeq} & & \\ 0 & \longrightarrow & \oplus_{i+j=m, i\geqslant p}\mathscr{H}^{i,j} & \longrightarrow & \oplus_{i+j=m, i\geqslant p+1}\mathscr{H}^{i,j} & \longrightarrow & \mathscr{H}^{p,m-p} & \longrightarrow & 0 \end{array}$$

Note $\delta = 0$ since we can lift a harmonic representative by a harmonic form, which is both ∂ and $\overline{\partial}$ closed. Hence we have a morphism of short exact sequences. φ_{p+1} isomorphism by induction $\Rightarrow \varphi_p$ isomorphism. QED

Note that one can ask whether the degree of Hodge-to-de Rham also holds in char $p > 0$ (false). However, Deligne and Illusie gave an algebraic proof in char 0 by reduction mod p. However, this does *not* impliy that $\mathrm{H}^m(X, \mathbb{C}) = \mathrm{F}^p \oplus \overline{\mathrm{F}}^{m+1-p}$.

37.2. **Introduction to variations of Hodge structure.**

Theorem 37.3 (Ehresman). *If $\pi : X \to B$ is a smooth map between smooth (real) manifolds which is* proper and submersion*, then for any $b_0 \in B$, there exists an open neighbourhood U of b_0 and a diffeomorphism $\pi^{-1}(U) \simeq U \times X_{b_0}$ over U.*

Proof.

$$\begin{array}{ccc} X_0 = X_{b_0} & \hookrightarrow & X^n \\ \downarrow & & \downarrow \\ b_0 & \xrightarrow{\in} & B^m \end{array}$$

π submersion implies that $X_0 \hookrightarrow X$ is a submanifold of dimension $n - m$. Tubular neighbourhood theorem (Theorem 32.1) shows that there exists an open neighbourhood W of X_0 and a retraction $r : W \to X_0$ of the inclusion $X_0 \hookrightarrow W$.

Define $\varphi : W \to B \times X$ by $\varphi(x) = (\pi(x), r(x))$. If $p \in X_0$, then $\mathrm{Ker}(\mathrm{d}\pi)_p = \mathrm{T}_p X_0$ and $(\mathrm{d}r)_p|_{\mathrm{T}_p X} = \mathrm{id}$. So $\mathrm{d}\varphi_p$ is an isomorphism and φ is a local diffeomorphism at every point of X_0. Moreover, $\varphi|_{X_0}$ is injective. X_0 is compact since π is proper. All of these implies that there exists W' an open neighbourhood of X_0 such that φ is injective and locally diffeomorphism on W'. So it is an open embedding on W'.

So there exists U' an open neighbourhood of b_0 such that $\pi^{-1}(U') \subseteq \varphi(W')$. Replace B by U', W' by $W' \cap \pi^{-1}(U')$ to assume that φ is surjective.

Take $U = B \smallsetminus \underbrace{\pi(X \smallsetminus W')}_{\text{closed in } B \text{ since } \pi \text{ is proper}}$. Then U is an open neighbourhood of b_0 such that φ gives diffeomorphism on $\pi^{-1}(U) \simeq U \times X_{b_0}$. ☐

38. December 11, 2019

Let $f : X \to B$ be a smooth proper submersion between smooth manifolds. For any $b \in B$, there exists an open neighbourhood $U \ni b$ such that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\text{diffeo}} & U \times X_b \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

Corollary 38.1. *Let $f : X \to B$ be a smooth, projective morphism of complex algebraic variety. If B is connected, then this map $(B \ni t \mapsto h^{p,q}(X_t))$ is constant.*

Proof. Theorem 37.3 (Ehresman) implies that $(B \ni t \mapsto b_i(X_t))$ is locally constant, hence constant as B is connected.

Note that if we take a resolution of singularities $B' \to B$, we may replace f by f'

$$\begin{array}{ccc} X' & \xrightarrow{f'} & B' \\ \downarrow & \square & \downarrow \\ X & \xrightarrow{f} & B \end{array}$$

and thus assume that B (hence also) X smooth.

Since each fiber is smooth and projective, Hodge decomposition shows that $b^i(X_t) = \sum_{p+q=i} h^{p,q}(X_t)$. Apply semicontinuity to $\Omega^p_{X/B}$ (locally free $\mathcal{O}_X$-module, faithfully flat $\Rightarrow$ they are flat over B) to see that this function $(t \mapsto h^{p,q}(X_t))$ is upper-semicontinuous. Therefore it can only go up under specialization. Since the sums of $h^{p,q}$ for $p+q$ constant are continuous, each of these functions must be constant. □

Remark 38.2. We get stronger results by using Grauert's theorem: Let B be reduced. Since all $h^q(X_t, \Omega^p_{X_t})$ are constant, all $\mathbb{R}^q f_* \Omega^p_{X/B}$ are locally free and commute with base-change. In particular, for any $t \in B$, the canonical map $(\mathbb{R}^q f_* \Omega^p_{X/B})_{(t)} \to \mathrm{H}^q(X_t, \Omega^p_{X_t})$ is an isomorphism.

Definition 38.3. Let X be a topological space, and $\mathscr{F}$ be a sheaf of abelian groups on X. Then $\mathscr{F}$ is *locally constant* if for any $x \in X$, there exists an open neighbourhood $U \ni x$ such that $\mathscr{F}|_U \simeq \underline{A}_U$ for some abelian group A ($A \simeq \mathscr{F}_x$). If $\mathscr{F}$ is a sheaf of vector space (over $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$) which is locally constant, and with finite dimensional stalks over corresponding field, then it is a *local system*.

Corollary 38.4. *Suppose that $f : X \to B$ is a smooth map between smooth real manifolds. If f is a proper submersion, then $\mathbb{R}^* f_* \underline{A}_X$ is locally constant for any abelian group A.*

Proof. For any $x \in B$, if $U \ni x$ is an open neighbourhood of x such that

$$\begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & U \times F \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

then $\mathbb{R}^k f_* \underline{A}_X|_U$ is isomorphic to the sheaf associated to the presheaf $\big(V \mapsto \mathrm{H}^k(V \times F; A)\big)$. By taking V to be a basis of contractible open neighbourhood of x, we have $\mathbb{R}^k f_* \underline{A}_X|_U \simeq \underline{\mathrm{H}^k(F, A)}_U$. □

38.1. **Overviews of Riemann-Hilbert correspondence.** Let X be a compact manifold and $\mathscr{E}$ a locally free sheaf on X.

Definition 38.5. A *connection* ∇ on $\mathscr{E}$ is a $\mathbb{C}$-linear map $\nabla : \mathscr{E} \to \Omega_X \otimes \mathscr{E}$ which satisfies Leibniz rule: $\nabla(fs) = f\nabla(s) + \mathrm{d}f \otimes s$ where f is a local section of $\mathcal{O}_X$ and s is a local section of $\mathscr{E}$.

Given any such ∇, we get induced $\mathbb{C}$-linear maps $\nabla : \Omega^p_X \otimes \mathscr{E} \to \Omega^{p+1}_X \otimes \mathscr{E}$ by $\nabla(\eta \otimes s) = \eta \wedge \nabla(s) + \mathrm{d}\eta \otimes s$.

Definition 38.6. A connection ∇ is *flat* if $\nabla \circ \nabla = 0 : \mathscr{E} \to \Omega^2_X \otimes \mathscr{E}$. Equivalently, $\nabla \circ \nabla = 0 : \Omega^p \otimes \mathscr{E} \to \Omega^{p+2} \otimes \mathscr{E}$.

Hence given such vector bundles with flat connection $(\mathscr{E}, \nabla)$, we get de Rham complex $\mathsf{dR}_X(\mathscr{E}, \nabla)$:

$$0 \to \mathscr{E} \xrightarrow{\nabla} \Omega_X \otimes \mathscr{E} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^{\dim X}_X \otimes \mathscr{E} \to 0$$

Example 38.7 (Basic example)**.** Let $\mathscr{E} = \mathcal{O}_X$, $\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X$ is a flat connection and the corresponding complex is the holomorphic de Rham complex of X.

More generally, suppose $\mathscr{L}$ is a local system of $\mathbb{C}$-vector spaces on X. Take $\mathscr{E} = \mathscr{L} \otimes_{\mathbb{C}} \mathcal{O}_X$ locally free. Then

$$\underbrace{\mathscr{E}}_{\mathscr{L} \otimes \mathcal{O}_X} \xrightarrow{1 \otimes \mathrm{d}} \underbrace{\mathcal{O}_X \otimes \mathscr{E}}_{\mathscr{L} \otimes \Omega_X} \Omega_X \otimes \mathscr{E}?$$

are integrable connections on $\mathscr{E}$.

The de Rham complex

$$0 \to \mathscr{L} \otimes \mathcal{O}_X \to \mathscr{L} \otimes \Omega^1_X \to \cdots \to \mathscr{L} \otimes \Omega^{\dim X}_X \to 0$$

is quasi-isomorphic to $\mathscr{L}$. Note $\mathscr{L} = \mathscr{E}^{\nabla} = \mathrm{Ker}\big(\nabla : \mathscr{E} \to \Omega^1 \otimes \mathscr{E}\big)$.

Theorem 38.8 (Riemann-Hilbert correspondence)**.** *This is an equivalence of categories*

$$\{\textit{local systems of } \mathbb{C}\textit{-vector spaces}\} \simeq \{\textit{vector bundles with integrable connections}\}$$

with inverse given by $(\mathscr{E}, \nabla) \mapsto \mathscr{E}^{\nabla}$.

Idea of the proof. To show $\mathscr{E}^\nabla$ is a local system, for any $x \in M, u \in \mathscr{E}_{(x)}$, there exists locally a unique flat section s of $\mathscr{E}$ such that $s(x) = u$. This is a local statement, we may assume that we have sections $e_1, \dots, e_r$ traivialize $\mathscr{E}$ coordinates $x_1, \dots, x_r$ on X, then

$$\nabla : \mathscr{E} \to \Omega \otimes \mathscr{E}$$
$$e_j \mapsto \sum_{i=1}^n \sum_{k=1}^r \Gamma_{i,j}^k \mathrm{d}x_i \otimes e_k$$

A section is of the form $s = \Sigma_{j=1}^r s_j e_j$. So

$$\nabla(s) = \sum_{j=1}^r s_j \sum_{i=1}^n \sum_{k=1}^r \Gamma_{i,j}^k \mathrm{d}x_i \otimes e_k + \sum_{k=1}^r \sum_{i=1}^n \frac{\partial s_k}{\partial x_i} \mathrm{d}x_i \otimes e_k$$

Then s is flat if and only if

$$\frac{\partial s_k}{\partial x_i} = -\sum_{j=1}^r \Gamma_{i,j}^k s_j \quad \forall i, k$$

Key point is that ∇ is integrable if and only if integrability of this sytem of linear PDE. In this case you get existence and uniqueness (locally) with initial condition. □

39. December 13, 2019

39.1. **Introduction to variations of Hodge structures (continued).** Let $\pi : X \to B$ be a projective smooth morphism of smooth complex varieties. We know that $\mathbb{R}^k \pi_* \underline{\mathbb{C}}_X$ is a local system with stalk at b given by $(\mathbb{R}^k \pi_*)_b \simeq \mathrm{H}^k(X_b, \mathbb{C})$. This corresponds to an analytical vector bundle with integrable connnection (Gauss-Manin connection)

First, describe the vector bundle. Do a relative version of the de Rham cohomology point of view. On X we have $\Omega_X^\bullet$, but also $\Omega_{X/B}^\bullet$ where $\Omega_{X/B}^p = \bigwedge^p \Omega_{X/B}$. Note that $\Omega_{X/B}^\bullet$ is a complex of $\pi^{-1}\mathcal{O}_B$-modules.

Define $\mathscr{H}^k := \mathbb{R}^k \pi_*(\Omega_{X/B}^\bullet)$. This is an $\mathcal{O}_B$-module. The "naïve" filtration $\mathrm{F}^\bullet \Omega_{X/B}^\bullet$ induces a filtration on $\mathscr{H}^k$:

$$\mathrm{F}^p \mathscr{H}^k = \mathrm{Im}(\mathbb{R}^k \pi_* \mathrm{F}^p \Omega_{X/B}^\bullet \to \mathbb{R}^k \pi_* \Omega_{X/B}^\bullet).$$

We have a spectral sequence

$$E_1^{p,q} = \mathbb{R}^q \pi_* \Omega_{X/B}^p \Rightarrow \mathscr{H}^{p+q} \tag{39.1}$$

Each $E_1^{p,q}$ is locally free and we have isomorphisms $(E_1^{p,q})_{(b)} \simeq \mathrm{H}^q(X_b, \Omega_{X_b}^p)$. Recall that we always have $(\Omega_{X/B})_{(b)} \simeq \Omega_{X_b}$.

For every $b \in B$, the Hodge-to-de Rham spectral sequence degenerates on 1st page, i.e. (39.1)$_{(b)}$ degenerates on 1st page. Hence (39.1) degenerates on 1st page. So $\mathrm{F}^p \mathscr{H}^k / \mathrm{F}^{p+1} \mathscr{H}^k \simeq \mathbb{R}^{k-p} \pi_* \Omega_{X/B}^p$. So we deduce that each $\mathrm{F}^p \mathscr{H}^k$ is a submodule of $\mathscr{H}^k$. Moreover, for every $b \in B$, $\mathrm{F}^p \mathscr{H}_{(b)}^k = \mathrm{F}^p \mathbb{H}^k(\Omega_{X_b}^\bullet) \simeq \mathrm{F}^p \mathrm{H}^k(X_b, \mathbb{C})$.

Remark 39.1. We can run similar arguments for $\mathbb{R}^k \pi_* \underbrace{\Omega_{X^{\mathsf{an}}/B^{\mathsf{an}}}^\bullet}_{(\Omega_{X/B}^\bullet)^{\mathsf{an}}}$. So we have a canonical map

$$(\mathscr{H}^k)^{\mathsf{an}} \to \mathbb{R}^k \pi_* \Omega_{X^{\mathsf{an}}/B^{\mathsf{an}}}^\bullet$$

This is an isomorphism using the spectral sequence associated to the 2 "naïve" filtration which degenerates on 1st page, it is enough to show

$$\left(\mathbb{R}^q \pi_* \Omega_{X/B}^p\right)^{\mathsf{an}} \xrightarrow{\sim} \mathbb{R}^q \pi_* \Omega_{X^{\mathsf{an}}/B^{\mathsf{an}}}^p \quad \forall p, q$$

This is a general fact by relative GAGA. But in our setting, we can see this by taking fibers at each $b \in B$, and $\mathrm{H}^p(X_b, \Omega_{X_b}) \simeq \mathrm{H}^p(X_b^{\mathsf{an}}, \Omega_{X_b}^{\mathsf{an}})$ by usual GAGA.

Note that we have a canonical morphism

$$\begin{aligned} \underline{\mathbb{C}}_X &\to \Omega^\bullet_{X^{\mathsf{an}}/B^{\mathsf{an}}} \\ \Rightarrow \mathbb{R}^k\pi_*\underline{\mathbb{C}}_X &\to \mathbb{R}^k\pi_*\left(\Omega^\bullet_{X^{\mathsf{an}}/B^{\mathsf{an}}}\right) \\ \Rightarrow \left(\mathbb{R}^k\pi_*\mathbb{C}_X\right)\otimes_{\mathbb{C}}\mathcal{O}_{B^{\mathsf{an}}} &\to \mathbb{R}^k\pi_*\left(\Omega^\bullet_{X^{\mathsf{an}}/B^{\mathsf{an}}}\right) \end{aligned}$$

This is an isomorphism because it is an isomorphism on fibers.

The conclusion is that the analytic vector bundle associated to $\mathbb{R}^k\pi_*\underline{\mathbb{C}}_X$ is $\left(\mathscr{H}^k\right)^{\mathsf{an}}$.

39.2. **The Gauss-Manin connection.** As we will see, the connection on $(\mathscr{H}^k)^{\mathsf{an}}$ comes from a connection on $\mathscr{H}^k$. Its main property is *Griffiths transversality*:

$$\nabla : \mathscr{H}^k \to \Omega_B \otimes \mathscr{H}^k$$
$$\nabla(\mathrm{F}^p\mathscr{H}^k) \subseteq \Omega_B \otimes \mathrm{F}^{p-1}\mathscr{H}^k$$

Therefore ∇ induces a map $\mathrm{F}^p\mathscr{H}^k/\mathrm{F}^{p+1}\mathscr{H}^k \xrightarrow{\overline{\nabla}} \Omega_B \otimes \mathrm{F}^{p-1}\mathscr{H}^k/\mathrm{F}^p\mathscr{H}^k$. This is $\mathcal{O}_B$-linear: $\nabla(fs) = f\nabla(s) + \mathrm{d}f\otimes s \Rightarrow \overline{\nabla}(f\overline{s}) = f\overline{\nabla}(\overline{s})$.

We will describe $\overline{\nabla}$ in the level of fibers. For $b \in B$, we have

$$\mathrm{H}^q(X_b, \Omega^p_{X_b}) \to \mathrm{T}^*_bB \otimes \mathrm{H}^{q+1}(X_b, \Omega^{p-1}_{X_b})$$

where $q = k - p$

Theorem 39.2 (Griffiths)**.** *For every $b \in B, u \in \mathrm{T}_bB$, the map*

$$\mathrm{H}^q(X_b, \Omega^p_{X_b}) \to \mathrm{H}^{q+1}(X_b, \Omega^{p-1}_{X_b}),$$

induced by $\overline{\nabla}_b$ and u, is the cup product with the Kodaira-Spencer class of u, which is an element of $\mathrm{H}^1(X, \mathrm{T}_X)$. ($\mathrm{H}^q(\Omega^p_{X_b}) \to \mathrm{H}^{q+1}(\Omega^p_{X_b}\otimes \mathrm{T}_{X_b}) \to \mathrm{H}^{q+1}(\Omega^{p-1}_{X_b})$.)

Recall that

$$\begin{array}{ccc} X & \hookleftarrow & X_b \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ B & \xleftarrow[\in]{} & b \end{array}$$

So we have

$$0 \to \mathrm{T}_{X_b} \to \mathrm{T}_X|_{X_b} \to \underbrace{\mathrm{N}_{X_b/X}}_{\mathrm{T}_bB\otimes\mathcal{O}_{X_b}} \to 0$$

The long exact sequence in cohomology gives us

$$\underbrace{\mathrm{H}^0(\mathrm{N}_{X_b/X})}_{\mathrm{T}_bB} \to \mathrm{H}^1(X_b, \mathrm{T}_{X_b})$$

which is the *Kodaira-Spencer* map.

The significance is that if Y is a smooth projective variety, then $\mathrm{H}^1(Y, \mathrm{T}_Y)$ parameterizes order 1 deformation of Y, that is

$$\begin{array}{ccc} Y & \longrightarrow & y \\ \downarrow & & \downarrow{\scriptstyle \text{flat}} \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}[\varepsilon]) \end{array}$$

Given $\pi : X \to B$ and $u \in \mathrm{T}_b B$, we have

$$\begin{array}{ccc} \widetilde{X}_b & \longrightarrow & X \\ \downarrow & \square & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(\mathbb{C}[\varepsilon]) & \xrightarrow[u]{} & B \end{array}$$

where u maps the closed point to b. The image of u by Kodaira-Spencer maps onto $\widetilde{X}_b$. Now we describe the Gauss-Manin connection after Katz-Oda. Let $\pi : X \to B$ as before. We have

$$0 \to \pi^* \Omega_B \to \Omega_X \to \Omega_{X/B} \to 0.$$

Consider $\Omega_X^\bullet$ with the filtration given by $\mathrm{G}^i \Omega_X^p = \mathrm{Im}(\pi^* \Omega_B^i \otimes \Omega_X^{p-i} \to \Omega_X^p)$, then $\mathrm{G}^i \Omega_X^p / \mathrm{G}^{i+1} \Omega_X^p \simeq \pi^* \Omega_B^i \otimes \Omega_{X/B}^{p-i}$ and $(\mathrm{G}^i \Omega_X^\bullet)_{i \geqslant 0}$ induces a spectral sequence with respect to π_*.

$$\begin{aligned} E_1^{p,q} &= \mathbb{R}^{p+q} \pi_* (\underbrace{\mathrm{G}^P \Omega_X^\bullet / \mathrm{G}^{p+1} \Omega_X^\bullet}_{\pi^* \Omega_B^p \otimes \Omega_{X/B}^{\bullet - p}}) \\ &= \mathbb{R}^q \pi_* \left(\pi^* \Omega_B^p \otimes \Omega_{X/B}^\bullet \right) \\ &= \Omega_B^p \otimes \mathbb{R}^q \pi_* \Omega_{X/B}^\bullet \end{aligned}$$

In particular, we have a map

$$\begin{array}{ccc} E_1^{0,q} & \longrightarrow & E_1^{1,q} \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ \mathscr{H}^q & \longrightarrow & \Omega_B \otimes \mathscr{H}^q \end{array}$$

This is the Gauss-Manin connection.

Let us deduce Griffiths tranversalty from this description. First we describe $d_1^{0,q}$

$$0 \longrightarrow \underbrace{\mathrm{G}^1 \Omega_X^\bullet / \mathrm{G}^2 \Omega_X^\bullet}_{\pi^*(\Omega_B) \otimes \Omega_X^{\bullet - 1}} \longrightarrow \Omega_X^\bullet / \mathrm{G}^2 \Omega_X^\bullet \longrightarrow \underbrace{\Omega_X^\bullet / \mathrm{G}^1 \Omega_X^\bullet}_{\simeq \Omega_{X/B}^\bullet} \longrightarrow 0$$

$d_1^{0,q}$ is the connecting homomorphism $\mathbb{R}^q \pi_* \Omega_{X/B}^\bullet \to \underbrace{\mathbb{R}^{q+1} \pi_* (\pi^* \Omega_B \otimes \Omega_{X/B}^{\bullet - 1})}_{\Omega_B \otimes \mathbb{R}^q \pi_* \Omega_{X/B}^\bullet}$. Consider the naïve filtration, we have

$$0 \to \pi^* \Omega_B \otimes \mathrm{F}^{p-1} \Omega_{X/B}^\bullet \to \mathrm{F}^p \left(\Omega_X^\bullet / \mathrm{G}^2 \Omega_X^\bullet \right) \to \mathrm{F}^p \Omega_{X/B}^\bullet \to 0$$

$$\Rightarrow \begin{array}{ccc} \mathbb{R}^q \pi_* \mathrm{F}^p \Omega_{X/B}^\bullet & \longrightarrow & \Omega_B \otimes \mathbb{R}^q \pi_* \mathrm{F}^{p-1} \Omega_{X/B}^\bullet \\ \downarrow & & \downarrow \\ \mathbb{R}^q \pi_* \Omega_{X/B}^\bullet & \xrightarrow{\nabla} & \Omega_B \otimes \mathbb{R}^q \pi_* \Omega_{X/B}^\bullet \end{array}$$

$\Rightarrow \nabla \left(\mathrm{F}^p \mathscr{H}^k \right) \subseteq \Omega_B \otimes \mathrm{F}^{p-1} \mathscr{H}^k$. Similarily, one can prove the other theorem of Griffiths.

39.3. **General definition of VHS.**

Definition 39.3. An (analytic) VHS (variation of rational Hodge structure) on a complex manifold B is given by the following data:

(1) A vector bundle with integrable connections $(\mathscr{E}, \nabla)$.
(2) ($\mathbb{Q}$-structure) A local system $\mathscr{L}_{\mathbb{Q}}$ of $\mathbb{Q}$-vector spaces, together with an isomorphism $(\mathscr{L}_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathscr{E}^{\nabla}$.
(3) (Hodge filtration) A finite decreasing filtration $\mathrm{F}^\bullet \mathscr{E}$ on $\mathscr{E}$ by sub-bundles, which satisfy Griffiths tranversality $\nabla(\mathrm{F}^p \mathscr{E}) \subseteq \Omega \otimes \mathrm{F}^{p-1} \mathscr{E}$, such that for all $b \in B$, $(\mathrm{F}^p \mathscr{E}_{(b)})$ gives a Hodge structure on $\mathscr{L}_b$.

The VHS that come from smooth morphisms are called *geometric*.

39.3.1. *Period maps.* Let $\pi : X \to B$ be as before. Fix $b_0 \in B$, to understand behavior near b_0, we choose a contractible open neighbourhood U of b_0 such that $\pi^{-1}(U) \simeq U \times X_{b_0}$. Given any $b \in U$, we have the canonical isomorphism

$$\begin{array}{ccc} & \mathrm{H}^k(\pi^{-1}(U);\mathbb{C}) & \\ \swarrow^{\text{rest.}} & & \searrow^{\text{rest.}} \\ \mathrm{H}^k(X_b,\mathbb{C}) & \xrightarrow{\qquad\simeq\qquad} & \mathrm{H}^k(X_{b_0},\mathbb{C}) \end{array}$$

Fix p, for every $b \in U$, we get a linear subspace

$$\mathrm{F}^p\mathrm{H}^k(X_b,\mathbb{C}) \subseteq \mathrm{H}^k(X_b,\mathbb{C}) \simeq \mathrm{H}^k(X_{b_0},\mathbb{C}).$$

Get a period

$$\begin{aligned} \mathscr{P}^{k,p} : U &\to G(r,V) \\ b &\mapsto [\mathrm{F}^p\mathrm{H}^k(X_b,\mathbb{C})] \end{aligned}$$

where $V = \mathrm{H}^k(X_{b_0},\mathbb{C})$ and $r = \sum_{p'+q'=k, p'\geqslant p} h^{p',q'}$.

Proposition 39.4. *$\mathscr{P}^{k,p}$ is a holomorphic map.*

Proof. We have a subbundle $\mathrm{F}^p\mathscr{H}^k|_U \subseteq \mathscr{H}^k|_U = V \otimes \mathcal{O}_U$. This defines a holomorphic map $U \to G(r,V)$, which is precisely $\mathscr{P}^{k,p}$. □

The goal is to relate $\mathrm{d}\mathscr{P}^{k,p} : \mathrm{T}_bB \to \mathrm{T}_{[W]}G$ to the Gauss-Manin connection. Let $[W] \in G(r,V) = G$. Then $\mathrm{T}_{[W]}G = \{\mathcal{W} \subseteq V \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$ free $\mathbb{C}[\varepsilon]$-module of rank r with free coker$|W = \overline{\mathcal{W}} = \mathcal{W}/\varepsilon\mathcal{W} \subseteq V\}$. Given such $\mathcal{W}$, for each $w \in W$, there exists $w + \varepsilon\widetilde{w} \in \mathcal{W}$. Define $\varphi : W \to V/W$ by $w \mapsto \widetilde{w} \mod W$. This gives an isomorphism $\mathrm{T}_{[W]}G \simeq \mathrm{Hom}_{\mathbb{C}}(W, V/W)$.

Explicitly, choose a basis $e_1, \dots, e_n$ for V. Suppose that we are in the chart of G with subspaces generated by vectors of the form $e_i + \sum_{j=r+1}^n a_{i,j}e_j$. This chart is isomorphic to an affine space with coordinates $A_{i,j}$. Suppose W is generated by $\{e_i + \sum_{j=r+1}^n a_{i,j}e_j | i\}$ and $u = \sum_{i,j} b_{i,j} \frac{\partial}{\partial A_{i,j}}|_{[W]}$. The corresponding map $W \to V/W$ maps $e_i + \sum a_{i,j}e_j$ to $\overline{\sum b_{i,j}e_j}$.

Consider now $\mathscr{T} \hookrightarrow V \otimes \mathcal{O}_G$ sheaf of sections of the universal subbundles. Given $[W] \in G$, consider the trivial vector bundle $V \otimes \mathcal{O}_G$ with the connection $1 \otimes \mathrm{d}$. On our chart, $\mathscr{T}$ is generated by the sections $s_i = e_i + \sum_{j=r+1}^n A_{i,j}e_j$. Then $(1 \otimes \mathrm{d})(s_i) = \sum_{j=r+1}^n \mathrm{d}A_{i,j} \otimes e_j$, which implies that $\overline{\nabla_u(s_i)} = \varphi(s_i([W]))$ where $\overline{(-)}$ stands for the image in V/W and $u \in \mathrm{T}_{[W]}G$ corresponding to $\varphi \in \mathrm{Hom}_{\mathbb{C}}(W, V/W)$.

Using this, we see that given the period map $\mathscr{P}^{k,p} : U \to G$ associated to $\pi : X \to B$, for every $b \in U$, every $u \in \mathrm{T}_bU$, if $[W] = \mathscr{P}^{k,p}(b)$, then $\underbrace{\mathrm{d}\mathscr{P}_b^{k,p}(u)}_{\in \mathrm{Hom}(W,V/W)} = (\nabla|_{\mathrm{F}^p\mathscr{H}^k})_{b,u} : W \to V/W$.

(Note that $\nabla : \mathscr{H}^k \to \Omega \otimes \mathscr{H}^k$ and $\mathrm{F}^p \subseteq sH^k$ gives at b a map $V = \mathscr{H}^k_{(b)} \to \mathrm{T}_b^*B \otimes V \xrightarrow{u} V$

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