Math 156 Review Sheet for 2nd Midterm Exam Fall 2009

1. True or False? Justify your answer.

a) $\lim_{n\to\infty}(1+\frac{1}{2n})^n = \frac{1}{2}$

b) $\overline{x} = M_y/m = \int_a^b xf(x)dx \,/ \int_a^b f(x)dx = \int_a^b xdx = (b^2-a^2)/2$

c) If a region R is symmetric about a line l, then the center of mass of R lies on l.

d) If $f(x)$ is the pdf of a random variable X with mean μ, then $\text{prob}(X \le \mu) = \text{prob}(X \ge \mu)$.

e) If X is a normally distributed random variable with mean μ and standard deviation σ, then the smaller the value of σ, the more likely it is that X is far from μ.

f) A radioactive material has a half-life of 100 years. If a sample has initial mass 1 kg, then there will be $\frac{1}{2}$ kg remaining after 50 years.

g) The function $f(x) = \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & \text{for } -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$ defines a valid pdf.

h) $\tanh x$ is an odd function i) $\sinh^2 x + \cosh^2 x = 1$ j) $\cosh x > \sinh x$ for all x

k) The Taylor polynomial of degree 1 for $f(x) = e^x$ at $x=0$ is $T_1(x)=1$.

l) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is a divergent geometric series

m) A convergent sequence is bounded.

n) A convergent sequence is either increasing or decreasing.

o) If $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \infty$, then $\lim_{n\to\infty} a_n b_n = 0$.

p) If $0 \le a_n \le 1$ and $a_{n+1} < a_n$, then $\lim_{n\to\infty} a_n = 0$.

q) $1 = \lim_{n\to\infty} 1 = \lim_{n\to\infty}(n+1-n) = \lim_{n\to\infty}(n+1) - \lim_{n\to\infty} n = \infty - \infty = 0$

r) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. s) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

section 9.3 (center of mass)

2. A one-dimensional metal rod on the interval $a \le x \le b$ has variable density $\rho(x)$ in units of kg/m. Find an expression for the center of mass of the rod.

3. Sketch the region, find the center of mass, and indicate CM on the sketch.

a) $\{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1-x\}$ b) $\{(x,y) : 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\,\}$

c) $\{(x,y) : 0 \le x \le 2,\ 0 \le y \le \sqrt{x(2-x)}\,\}$ d) $\{(x,y) : -1 \le x \le 1,\ 0 \le y \le \cosh x\}$

e) $\{(x,y) : -2 \le x \le 2,\ y \ge 0,\ 1 \le x^2+y^2 \le 4\,\}$

f) $\{(x,y) : 1 \le x \le 2,\ 0 \le y \le x^{-1}\}$ - also, use this result to deduce that $\ln 2 > \frac{2}{3}$.

4. Find the volume of the shape using the theorem of Pappus.

a) a sphere of radius r b) a cone of height h and base radius r

c) the shape formed by rotating a circle of radius r about the tangent line to the circle at a point on its circumference

5. Consider a set of n particles located on the x-axis with mass m_i and position x_i, for $i = 1, \ldots, n$. Define the function $f(x) = \sum_{i=1}^{n} m_i(x - x_i)^2$. Show that $f(x)$ is minimized when x is the center of mass of the particle distribution.

section 9.5 (probability)

6. Assume that the length of a human pregnancy is normally distributed with mean 270 days and standard deviation 15 days. Express the probability that a pregnancy lasts between 255 days and 285 days as an integral.

7. The lifetime of a certain car battery is a random variable with pdf $f(t) = \frac{6}{625}t^2(5 - t)^2$ for $0 \leq t \leq 5$ and zero otherwise, where the time t is measured in years.

a) Sketch $f(t)$ and show that it defines a valid pdf. Find the mean.

b) Among 1000 batteries chosen at random, how many will still be working after 3 years?

8. The length of time spent waiting in line to vote in a certain district is modeled by an exponential density function with mean 20 minutes.

a) Find the probability that a voter waits in line less than 10 minutes. (take $\sqrt{e} = 1.6$)

b) Find the probability that a voter waits in line more than 30 minutes.

c) Find the median waiting time. (take $\ln 2 = 0.7$)

section 10.1 (differential equations)

9. Which of the following functions satisfies the differential equation $y'' + 2y' + y = 0$?

a) $y = e^{-t}$ b) $y = te^{-t}$ c) $y = t^2e^{-t}$

10. Find the constant solutions, sketch the phase plane, and determine which of the constant solutions are stable or unstable.

a) $y' = y - 1$ b) $y' = y^2 - 1$ c) $y' = y^2 - 2y + 1$ d) $y' = y^2 - 3y + 2$ e) $y' = \sin y$

11. Consider a particle of mass m and time-dependent position $x(t)$ moving under the influence of a force $f(x)$. Newton's 2nd law states that $x(t)$ satisfies the differential equation $mx'' = f(x)$. Suppose that $f(x) = -V'(x)$, where $V(x)$ is the potential energy function. The total energy of the particle (kinetic + potential) is $E(t) = \frac{1}{2}mv^2 + V(x)$, where $x = x(t)$ is the particle position and $v = x'(t)$ is the particle velocity. Show that under these assumptions, the total energy is constant in time.

section 10.3 (separation of variables)

12. Solve for $y(t)$ subject to initial condition $y(0) = 1$. Sketch the solution for $t \geq 0$.

a) $y' = y$ b) $y' = ty$ c) $y' = y^2$ d) $y' = y(1 - y)$

13. A tank initially contains 1000 L of pure water. Sea water containing 0.05 kg of salt per liter enters the tank at a rate of 5 L/min. The solution is kept well mixed and drains from the tank at the same rate. Find the concentration of salt in the tank after one hour.

14. A certain country has \$10 billion in paper currency in circulation at any given time. Each day \$50 million flows into the country's banks and the same amount flows out due to normal business transactions. The paper currency is getting worn out and the government decides to replace the old bills with new bills whenever old bills come into a bank. Let $x(t)$ denote the value of new currency in circulation at time t. Assume that $x(0) = 0$.

a) Write down a differential equation for $x(t)$. Use \$1 billion as the unit of currency and 1 day as the unit of time. You may assume that when the new bills are released each day, they are instantaneously mixed with the currency in circulation.

b) Solve for $x(t)$.

c) How long will it take for the new currency to reach 90% of the currency in circulation?

15. In a certain chemical reaction, one molecule of type A and one molecule of type B combine to form one molecule of type C, A + B $\rightarrow$ C. Let a and b denote the initial concentrations of the reactants A and B, and let $c(t)$ be the concentration of the product C at time t. The law of mass action states that the reaction rate is proportional to the product of the concentrations of the reactants, $c' = k(a - c)(b - c)$, where $k > 0$. Take $a = 1$ mole/L, $b = 2$ mole/L. Assuming that no product is present at the start of the reaction, find the product concentration $c(t)$ and sketch the graph for $t \geq 0$. What is the asymptotic value of the product concentration in the limit $t \rightarrow \infty$? Would the asymptotic value change if the initial product concentration was 1.5 mole/L?

16. If a non-combustible object is heated to a high temperature T_0 and is then removed from the heat source, the heat radiates and the object's temperature $T(t)$ decays according to the differential equation $mc_PT' = -e\sigma T^4$, where m is the mass of the object, c_P is the specific heat capacity, e is the emissivity, and σ is Stefan's constant. Assume that m, c_P, e, σ are all positive. Find the temperature $T(t)$ and sketch the graph for $t \geq 0$. In the limit $t \rightarrow \infty$, the temperature decays like $t^{-\alpha}$; find the value of α.

section 10.4 (exponential growth and decay)

17. The mass of a radioactive sample is 128 kg after two hours and 2 kg after five hours. What was the initial mass of the sample? How long will it take for the sample to decay from 2 kg to 1 kg?

18. A thermometer is taken outside from an air-conditioned room where the temperature is 21°C. It reads 27°C after one minute and 30°C after two minutes. Find the outdoor temperature.

19. Fallen leaves in a forest accumulate on the ground at a rate of r kg/year and they decompose at a rate proportional to the total mass of leaves on the ground, with proportionality factor α. Assume that $r > 0$, $\alpha > 0$.

a) Set up a differential equation for $y(t)$, the total mass of leaves on the ground at time t.

b) Let M be the constant solution of the equation. Find M in terms of r and α. Sketch the phase plane of the differential equation. Is the constant solution stable or unstable?

c) A flood washes away all the leaves. How long will it take for the total leaf mass to recover to $\frac{1}{2}M$?

20. Bob and Ray order 8 oz cups of coffee which are served steaming hot at temperature T_h. They use different strategies to cool the coffee. Bob immediately adds 1 oz of cold milk with temperature $T_c < T_h$ and waits 2 minutes before drinking, while Ray waits 2 minutes and then adds 1 oz of cold milk. Assume that the ambient temperature T_a in the cafe satisfies $T_c < T_a < T_h$. Who ends up drinking colder coffee, Bob or Ray? You can explain your answer intuitively, but for full credit justify your answer by finding $T_{\text{Bob}}(t), T_{\text{Ray}}(t)$ in terms of T_c, T_a, T_h and k (the temperature rate constant for a cup of coffee).

section 10.5 (logistic equation)

21. Assume that the rate at which a rumor spreads is proportional to the product of two terms, the fraction of the population who have already heard the rumor and the fraction of the population who have not yet heard the rumor. Consider a town with 1000 inhabitants. Suppose that 10 people have heard a certain rumor at 8am and 20 people have heard the rumor at 9am. At what time will half the population have heard the rumor?

sections 12.1 (sequences), 12.2 (series)

22. Determine whether the sequence $\{a_n\}$ converges or diverges as $n \to \infty$. If the sequence converges, give the limit.

a) $a_n = \dfrac{(-1)^n}{n}$ b) $a_n = \dfrac{1}{2^n}$ c) $a_n = \dfrac{n}{2^n}$ d) $a_n = \dfrac{n!}{2^n}$

23. Find the sum of the series. a) $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots$ b) $\dfrac{2}{3} - \dfrac{4}{9} + \dfrac{8}{27} - \dfrac{16}{81} + \cdots$

hyperbolic functions

24. Find the antiderivative. a) $\int \cosh 2x\, dx$ b) $\int \tanh 2x\, dx$ c) $\int \cosh^2 x\, dx$

25. Consider the curve $y = \cosh x$ on the interval $-1 \le x \le 1$. Find (a) the arclength of the curve and (b) the surface area obtained by rotating the curve about the x-axis.

26. Derive the following addition formulas for sinh and cosh. (Hint: first derive (a); then (b), (c), (d) can be derived with almost no extra work.)

a) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

b) $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

c) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

d) $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

Taylor polynomials

27. Consider $f(x)$ and $x = a$ given below. In each case find $T_1(x)$ and $T_2(x)$, the linear and quadratic Taylor approximations at $x = a$. Sketch $f(x)$, $T_1(x)$, $T_2(x)$ on the same graph.

(i) $f(x) = x^{-1},\ a = 1$ (ii) $f(x) = x^{-1},\ a = 2$ (iii) $f(x) = \sqrt{x},\ a = 4$

Note that setting $x = 5$ in case (iii) yields $f(5) = \sqrt{5}$, so we can approximate $\sqrt{5}$ by $T_1(5)$ or $T_2(5)$. Find these two values. Which is a more accurate approximation for $\sqrt{5}$? Why?