You may use these antiderivatives: $\int x^n dx = \frac{x^{n+1}}{n+1} (n \neq -1)$, $\int \frac{dx}{x} = \ln x$, $\int e^x dx = e^x$, $\int \sin x\, dx = -\cos x$, $\int \cos x\, dx = \sin x$, $\int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta)$, $\int \sec^3\theta\, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta))$, but all others should be derived.

1. **True or False.** Justify your answer.

a) $\lim\limits_{n\to\infty} \sum_{i=1}^n (1 + \frac{i}{n})\frac{1}{n} = \frac{3}{2}$

b) If the integral $\int_a^b f(x)\, dx$ is approximated using the right-hand Riemann sum and the number of intervals n is doubled, then the error in the approximation is also doubled.

c) If $f(x) = c_1 + c_2 x$, then the midpoint rule for $\int_a^b f(x)\, dx$ is exact.

d) If $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$, then $\lim\limits_{n\to\infty} \sum_{i=1}^n f'(x_i)\, \Delta x = f(b) - f(a)$.

e) $\frac{d}{dx} \int_0^{x^2} \sqrt{1+t}\, dt = \sqrt{1+x^2}$ f) $\int_0^\infty e^{-x} \cos x\ dx = \int_0^\infty e^{-x} \sin x\ dx$

g) If 2 Joules of work are needed to stretch a spring 20 cm beyond its natural length, then 1 Joule of work is needed to stretch it 10 cm beyond its natural length.

h) If $\lim\limits_{x\to\infty} f(x) = 0$, then the improper integral $\int_1^\infty f(x)\, dx$ converges.

i) If $0 \leq f(x) \leq g(x)$ for $x \geq 1$ and $\int_1^\infty g(x)\, dx$ converges, then $\int_1^\infty f(x)\, dx$ also converges.

appendix E, sections 5.1 - 5.4 (integrals)

2. Prove that $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$ using the method developed in class, i.e. $(i+1)^4 - i^4 = \ldots$

3. In each example express the integral as a limit of Riemann sums and evaluate the limit. Check your answer using the FTC. a) $\int_0^2 x\, dx$ b) $\int_0^1 x^3\, dx$ c) $\int_0^1 e^{-x}\, dx$

4. Evaluate the limit (by any means).

a) $\lim\limits_{n\to\infty} \sum_{i=1}^n \frac{1/n}{1+i/n}$ b) $\lim\limits_{x\to 0} \frac{1}{x}\int_0^x f(t)\, dt$ c) $\lim\limits_{x\to 0} \frac{e^x - 1}{x}$ d) $\lim\limits_{r\to 1} \frac{r^{10}-1}{r-1}$

5. Find the antiderivative.

a) $\int x e^{-x^2} dx$ b) $\int x^2 e^{-x} dx$ c) $\int x \sin x\, dx$ d) $\int \frac{dx}{4-x^2}$ e) $\int \frac{dx}{\sqrt{4-x^2}}$ f) $\int \sqrt{4-x^2}\, dx$

6. Prove. a) $\frac{1}{20} \leq \int_0^1 \frac{x^9}{1+x}\, dx \leq \frac{1}{10}$ b) $\int_0^1 x\,(1-x)^{11}\, dx = \frac{1}{156}$

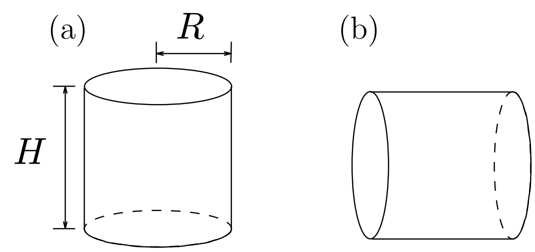
section 6.4 (work)

7. A cable of length L meters (m) is hanging from the top of a tall building. The cable has cross-sectional area A m^2 and density ρ kg/m^3. Find the work needed to raise the cable to the top of the building. If the length of the cable is doubled, is the work also doubled?

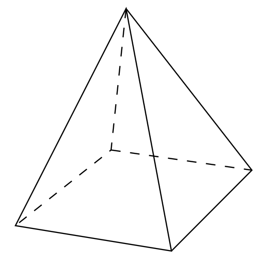
8. A force of 30 N is needed to maintain a spring when it is stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

9. Two ions repel each other with a force $f(r) = -\frac{q^2}{r^2}$, where q is the ionic charge and r is the distance between the ions. a) One ion is held fixed at $x = 0$. Find the work done in moving a second ion from $x = 3$ to $x = 2$. b) One ion is held fixed at $x = 1$. Find the work done in moving a second ion from $x = 3$ to $x = 2$.

11. A gas tank has the shape of a cylinder with height H and radius R, as shown in the figure. The tank is full of gas with density ρ kg/m^3. Consider two cases, the tank is (a) standing up, (b) lying on its side. Find the work needed in each case to pump the gas out the top of the tank.



12. A pyramid (sketched) is built of stone with mass density ρ kg/m^3. The base of the pyramid is a square and the sides are triangles. The vertex of the pyramid is directly above the center of the base. The length of a side of the base is L meters and the height of the vertex above the base is H meters. Derive a formula for the work done in building the pyramid in terms of L, H, and ρ (consider just the work done in raising the stone from ground level).



section 8.8 (improper integrals)

13. Determine whether the integral converges or diverges. If it converges, evaluate it.

a) $\int_1^\infty \frac{dx}{x^4}$ b) $\int_0^\infty xe^{-x}dx$ c) $\int_0^\infty e^{-x}\sin x\,dx$ d) $\int_1^\infty (\frac{1}{x} - \frac{1}{x+1})\,dx$

e) $\int_1^\infty \frac{dx}{1+x^2}$ f) $\int_1^\infty \frac{dx}{\sqrt{1+x^2}}$ g) $\int_1^\infty \frac{dx}{1-x^2}$ h) $\int_0^1 \frac{dx}{1-x}$ i) $\int_0^1 \frac{dx}{1-x^2}$ j) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

14. Prove: $\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$ (hint: substitute $u = \frac{1}{x}$)

section 9.1 (arclength)

15. Find the arclength of the curve on the interval $0 \le x \le 1$.

a) $y = \sqrt{1-x^2}$ b) $y = \int_0^x \sqrt{1-t^2}\,dt$ c) $y = \frac{e^x+e^{-x}}{2}$ d) $y = \sqrt{x^3}$

16. Sketch the curve $y = \sqrt{2x - x^2}$ for $0 \le x \le 2$ and find its arclength.

section 9.2 (surface area)

17. Find the surface area obtained by rotating the curve about the x-axis.

a) $y = x^3$, $0 \le x \le 1$ b) $y = 2\sqrt{1-x}$, $0 \le x \le 1$ c) $y = \frac{e^x+e^{-x}}{2}$, $0 \le x \le 1$

d) $y = (\frac{r_2-r_1}{x_2-x_1})(x - x_1) + r_1$, $x_1 \le x \le x_2$ (this gives the surface area of a conical slice)

18. Consider the curve $y = e^x$ between $x = 0$ and $x = 1$. Find an expression for the surface area formed by rotating the curve about the (a) x-axis, (b) y-axis. Leave the answers in the form of integrals (do not evaluate). Which surface area is larger?

19. Let S be the surface area of a zone on a sphere between two parallel planes. Show that $S = 2\pi rd$, where r is the radius of the sphere and d is the distance between the planes. (Note that S depends only on the distance between the planes, not on their location.)

20. An ellipse is defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (i) Sketch the ellipse in the xy-plane (assume $a > b > 0$). (ii) Set up integrals giving the area A of the ellipse, the arclength L of the ellipse, and the surface area S obtained by rotating the ellipse about the x-axis. (iii) Show that $A = \pi ab$ and $S = 2\pi b(b + a(\sin^{-1} c)/c)$, where $c = \sqrt{a^2 - b^2}/a$. Check your answers in the limit $a \to b$. (note: L cannot be evaluated in terms of elementary functions.)